SHINTANI LIFTS, $p$-ADIC FAMILIES AND DERIVATIVES OF QUADRATIC TWISTS

HENRI DARMON, MCGILL UNIVERSITY

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1. INTRODUCTION

Let $N$ be an odd, square-free integer, $S_k(\Gamma_0(N))$ be the space of cusp forms of weight $k$ on $\Gamma_0(N), k \equiv 2 \text{ mod } 4$, and $S_{(k+1)/2}(\Gamma_0(4N))$ the space of forms of weight $(k + 1)/2$ in the Kohnen subspace.

**Theorem 1.1** (Shimura-Kohnen). The spaces $S_{\text{new}}^k(\Gamma_0(N))$ and $S_{\text{new}}^{k+1/2}(\Gamma_0(4N))$ are isomorphic as modules over the Hecke algebra.

Given an eigenform $f \in S_{\text{new}}^k(\Gamma_0(4N))$ there is a corresponding $g \in S_{\text{new}}^{k+1/2}(\Gamma_0(4N))$, which is unique up to multiplication by a non-zero scalar. Let $g(q) = \sum \sum_{D>0} c(D)q^D$ denote its Fourier expansion.

**Theorem 1.2** (Kohnen). We have

$$c(D)^2 = \begin{cases} \lambda_g \sqrt{D^{k-1}} L(f, \chi_D, k/2) & \text{if } -D \equiv 0, 1 \text{ mod } 4, \chi_D(\ell) = \omega_\ell \text{ if } \ell | N \\ 0 & \text{otherwise} \end{cases} \quad (1.1)$$

The question which motivates the present talk is the following. By Kohnen’s theorem, the central critical values of quadratic twists of $f$ are packaged into a modular generating series. Can we do something similar for first derivatives? Is there a way to package them into a generating series with a modular interpretation?

This is the motivation. We will discuss a partial result saying something in this direction (joint work in progress with Gonzalo Tornaria). We exploit $p$-adic families of modular forms.

2. OVERVIEW OF $p$-ADIC FAMILIES

Let $f$ be a modular form of weight 2 on $\Gamma_0(N)$ which is associated to an elliptic curve $E$. Fix a prime $p$ dividing $N$ (recall that $N$ is a square-free integer, so $p || N$). This implies $a_p(f) = \pm 1$ (the negative of the eigenvalue of the Atkin-Lehner involution), and thus $a_p(f)$ is a $p$-adic unit.

Fix a $p$-adic domain $U \subset \mathbb{Z}_p$.

**Definition 2.1.** A $p$-adic family of modular forms is a formal $q$-series $\sum a_n(k)q^n$, where the $a_n(k)$ are $p$-adic analytic functions of $k \in U$, with the property that $f_k := \sum a_n(k)q^n$ is a normalised eigenform of weight $k$ on $\Gamma_0(N)$, for all $k \in U \cap \mathbb{Z}_{\geq 2}$.

**Example 2.2.** There are the following basic examples of $p$-adic families of modular forms:
• **Eisenstein series.** Let
  \[ E^*_k = \zeta^*(1 - k) + \sum_{n=1}^{\infty} \sigma^*_k(n)q^n, \]  
  with \( \sigma^*_k(n) = \sum_{d|n,(p,d)=1} d^{k-1} \) (the \( \zeta^* \) means we remove the factor corresponding to \( p \)).

• **Binary theta series.** Let \( \psi \) be a Hecke character of an imaginary quadratic field \( K \), of \( \infty \) type \((1,0)\),
  \[ \theta_k = \sum_{a \in \mathcal{O}_K} \psi(a)q^{a^2}. \]  

The following theorem of Hida shows that these families are in some sense quite ubiquitous.

**Theorem 2.3** (Hida). There exists a unique \( p \)-adic family \( f_k \) satisfying \( f_2 = f \).

**Remark 2.4** (Important). For all \( k \in U \cap \mathbb{Z}^{>2} \), we have \( f_k \in S_k(\Gamma_0(N)) \), though these are not necessarily newforms. For \( k > 2 \), they are not new at \( p \), but are new at all other primes dividing \( N := pM \). Thus there exists a newform \( f_k^\# \in S_k^{\text{new}}(\Gamma_0(M)) \) with the same Hecke eigenvalues at \( \ell \neq p \).

Let \( g_k \in S_k^{\text{new}}(\Gamma_0(4M)) \) be the form which corresponds to \( f_k^\# \) under the Shimura-Kohnen correspondence:
  \[ g_k = \sum_{D>0} c(D,k)q^D. \]  
These forms have potentially twice as many non-vanishing Fourier coefficients. This is because the level has dropped.

There are two types of \( D \) for which, a priori, \( c(D,k) \) could be non-zero:

- **Type I:** \( D \) such that \( \chi_{-D}(\ell) = \omega_\ell \) for all \( \ell | N \): for these \( L(f,-D,s) \) has sign 1 it its functional equation and \( c(D) \) encodes its central critical value \( L(f,-D,1) \);
- **Type II:** \( D \) such that \( \chi_{-D}(\ell) = \omega_\ell \) for all \( \ell | M \) but \( \chi_{-D}(p) = -\omega_p \): for these, \( L(f,-D,s) \) has sign \(-1\) if its functional equation and it becomes natural to consider \( L'(f,-D,1) \).

Note that the \( g_k \) are only defined up to a non-zero scalar. We will normalise them by setting a coefficient equal to 1. There is a fact (due to Glenn Stevens) which asserts that there is a \( \Delta_0 \) (of type I) such that \( c(\Delta_0, k) \neq 0 \) for all \( k \) in a \( p \)-adic neighborhood of 2, and in particular \( c(\Delta_0) \neq 0 \).

By dividing by that coefficient, we can define
  \[ \tilde{c}(D,k) = \frac{1 - \chi_{-D}(p)a_p(k)^{-1}p^{-\frac{k-2}{2}}}{1 - \chi_{-\Delta_0}(p)a_p(k)^{-1}p^{-\frac{k-2}{2}}} \cdot \frac{c(D,k)}{c(\Delta_0, k)}. \]  
These are defined for \( k \in U \cap \mathbb{Z}^{>2} \), and we can show
  \[ \tilde{c}(D,k) = \frac{c(p^2D,k)}{c(p^2\Delta_0,k)}. \]  

**Theorem 2.5** (Hida, Stevens). The functions \( k \mapsto \tilde{c}(D,k) \) extend to analytic functions in a neighborhood of \( k = 2 \).

If the discriminant \( D \) is of type II, then \( \tilde{c}(D,2) = 0 \). Our main theorem is the following:

**Theorem 2.6** (Tornaria-Darmon). Let \(-D\) be a type II discriminant. Then there exists a point \( P_D \in E^{-D}(\mathbb{Q}) \otimes Q \) (the Mordell-Weil group attached to the elliptic curve \( E^{-D} \)) such that
(1) \( \frac{d}{dk} \tilde{c}(D, k)_{k=2} = \log_p(P_D) \) (here \( \log_p : E^{-D}(\mathbb{Q}_p) \to \mathbb{Q}_p \) is the \( p \)-adic formal group logarithm).

(2) The point \( P_D \) is of infinite order if and only if \( L'(f, -D, 1) \neq 0 \).

One way of stating this result is the following. After normalising \( g_k \) (and applying a Hecke operator \( U_p^2 \) to it) we have the following “first order expansion” of \( g_k \) in a neighbourhood of weight \( 3/2 \):

\[
g_k = \sum_{D \text{ type I}} c(D) q^D + (k - 2) \sum_{D \text{ type II}} \log_p(P_D) q^D + O((k - 2)^2).
\]

(2.7)

**Question 2.7. Natural questions:**

(1) What can we say about the order of vanishing at \( k = 2 \) of \( c(D, k) \)? Presumably this is related to the rank of \( E^{-D} \).

(2) What can we say about leading terms?

(3) Is there a non-\( p \)-adic version?

(4) Is this formula useful (for the types of questions considered in this workshop)? The original formula of Kohnen and Waldspurger helps analyze a large number of non-vanishing of quadratic twists. Can we do something similar for first derivatives?

To amplify on the last question: Consider a \( k \) which is \( p \)-adically close to 2, say \( k = 2 + (p - 1)p^M \). We can look at \( \tilde{c}(D, k) \). If \( D \) is of type II, we know \( \tilde{c}(D, k) \equiv 0 \mod p^{M+1} \). If \( \tilde{c}(D, k) \not\equiv 0 \mod p^{2M} \) then \( L'(f, -D, 1) \neq 0 \). So if we are looking for non-vanishing twists of derivatives, we can search instead for coefficients of forms of higher half integral weight that are not divisible by \( p^{2M} \).

The key ingredient in the proof of the main theorem are

- a formula (due to Kohnen) relating integrals of modular forms to products of the coefficients: \( c(D, k)c(\Delta, k) \) is a combination of geodesic cycle integrals attached to \( f_k \) and binary quadratic forms of discriminant \( D\Delta \);
- The theory of Stark-Heegner points.