EXTEMPORANEOUS TALK ON RANDOM MATRIX THEORY AND SYMMETRIC POWERS

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ABSTRACT. Any errors should solely be attributed to the typist, Steven J. Miller.

Want to discuss symmetric powers of $L$-functions of elliptic curves. Will ignore bad primes, should have a degree two Euler product at the good primes:

$$L(E, s) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1}.$$  

The symmetric cube $L$-function is

$$L(\text{sym}^3 E, s) = \prod_p \left(1 - \frac{\alpha(p)^3}{p^s}\right)^{-1} \left(1 - \frac{\alpha(p)^2 \beta(p)}{p^s}\right)^{-1} \cdot \left(1 - \frac{\alpha(p) \beta(p)^2}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)^3}{p^s}\right)^{-1}. $$  

Note that $\beta(p) = \overline{\alpha(p)}$.

There is a functional equation for $L(\text{sym}^3 E, s)$ relating $s$ to $4 - s$. We have a Birch and Swinnerton-Dyer like conjecture:

$$\frac{L(\text{sym}^3 E, \text{center})}{\Omega_{\text{im}}^3 \Omega_{\text{re}}} \cdot (2\pi N_E) = \text{rational with small denominator}. $$

When twist we get

$$\frac{L(\text{sym}^3 E_d, \text{center})}{\Omega/d^2} \cdot (2\pi N_E) = \text{rational with small denominator}. $$

This should be an orthogonal family. We discretize:

$$\text{Prob} \left( L(\text{sym}^3 E_d, \text{center}) < t \right) \sim \alpha_E t^{1/2} (\log - \text{term}).$$

This leads to

$$\text{Prob} \left( L(\text{sym}^3 E_d, \text{center}) = 0 \right) \sim (1/d^2)^{1/2} \log - \text{term}. $$

Summing gives that the number of non-vanishing twists of even parity up to $D$ is a power of $\log D$.

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<tr>
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<th>11a</th>
<th>14a (5000)</th>
<th>15a (4000)</th>
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<tbody>
<tr>
<td>double</td>
<td>58</td>
<td>88</td>
<td>83</td>
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<tr>
<td>triple</td>
<td>1</td>
<td>3</td>
<td>2</td>
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Now $L(\text{sym}^3 E, s)$ is not primitive in the CM-case. We should have

$$\frac{L(\text{sym}^3 \psi)}{\Omega_{\text{im}}^3} (2\pi) \cdot \text{(rational with small denominator)},$$

with $L(\psi^3)$ of degree 2 (is this in the numerator?).

Twists

$$\frac{L(\text{sym}^3 \psi_d)}{\Omega/d^{3/2}} \cdot \text{(rational with small denominator)},$$

and then summing gives the number of twists should be $D^{1/4}(\log D)^{\text{power}}$.

In the non-CM case there is roughly 14000 data points. In the CM case there are around $10^5$ or $10^6$ data points, and no triple zeros.