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Random Matrix Theory and L-Functions at $s = 1/2$

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Abstract: Recent results of Katz and Sarnak [8,9] suggest that the low-lying zeros of families of L-functions display the statistics of the eigenvalues of one of the compact groups of matrices $U(N)$, $O(N)$ or $USp(2N)$. We here explore the link between the value distributions of the L-functions within these families at the central point $s = 1/2$ and those of the characteristic polynomials $Z(U, \theta)$ of matrices U with respect to averages over $SO(2N)$ and $USp(2N)$ at the corresponding point $\theta = 0$, using techniques previously developed for $U(N)$ in [10]. For any matrix size N we find exact expressions for the moments of $Z(U, 0)$ for each ensemble, and hence calculate the asymptotic (large N) value distributions for $Z(U, 0)$ and $\log Z(U, 0)$. The asymptotic results for the integer moments agree precisely with the few corresponding values known for L-functions. The value distributions suggest consequences for the non-vanishing of L-functions at the central point.

1. Introduction

For L-functions, as for the well-known Riemann zeta function, it is conjectured, and widely believed, that the non-trivial zeros lie on the line $\mathrm{Re}\, s = 1/2$; this being the generalised Riemann hypothesis (GRH). Whereas high on this critical line the zeros of any given L-function appear to have the same statistical distribution as the eigenvalues of the group $U(N)$ of (large) $N \times N$ unitary matrices endowed with Haar measure (otherwise known as the Circular Unitary Ensemble (CUE) of random matrix theory) [13,14,12], Katz and Sarnak [8,9] have proposed that the low-lying zeros of families of L-functions follow the statistics of the eigenvalues not always of $U(N)$, but in some cases of the compact groups $O(N)$ or $USp(2N)$. This is supported by the fact that for the function field analogue the equivalent of the Riemann Hypothesis is known to be true, and Katz and Sarnak have discovered that zeta functions over function fields have zero statistics which show exactly the behaviour just described.

Conrey and Farmer [3] have extended this idea that the low-lying zeros of families of L-functions show particular statistics to the study of the mean values of the L-functions $L_f(s)$ within families at the central point $s = 1/2$. They have found evidence that the symmetry type to which the low-lying zeros subscribe also determines the behaviour of these mean values. In particular, they conjecture that in general, as $\mathcal{Q} \to \infty$,

$$\frac{1}{\mathcal{Q}^*} \sum_{\substack{f \in \mathcal{F} \\ c(f) \leq \mathcal{Q}}} V(L_f(\tfrac{1}{2}))^k \sim g_k \frac{a(k)}{\Gamma(1 + B(k))} (\log \mathcal{Q}^A)^{B(k)}, \tag{1}$$

where they choose $V(x)$ depending on the symmetry type ($V(z) = |z|^2$ for unitary symmetry and $V(z) = z$ for the orthogonal or symplectic case); A is a symmetry-dependent constant; the family, $\mathcal{F}$, over which the average is performed is considered to be partially ordered by the conductor, $c(f)$, of each L-function; and the sum is over the $\mathcal{Q}^*$ elements with $c(f) \leq \mathcal{Q}$. The symmetry type of the family manifests itself in the expectation that it alone determines the values of g_k and $B(k)$. These functions are thus universal, being independent of the details of the particular family in question. $a(k)$, on the other hand, is expected to depend on the specific family involved.

Previously [10] we have studied mean values of the Riemann zeta function, which is defined by the Euler product over prime numbers or a Dirichlet sum,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2}$$

for Re$s > 1$, and by an analytical continuation in the rest of the complex plane. We conjectured that the moments of $\zeta(1/2 + it)$ high on the critical line $t \in \mathbb{R}$ factor into a part which is specific to the Riemann zeta function, and a universal component which is the corresponding moment of the characteristic polynomial $Z(U, \theta)$ of matrices in $U(N)$, defined with respect to an average over the CUE. The connection between N and the height T up the critical line corresponds to equating the mean density of eigenvalues $N/2\pi$ with the mean density of zeros $\frac{1}{2\pi} \log \frac{T}{2\pi}$. This idea has subsequently been applied by Brézin and Hikami [2] to other random matrix ensembles, and by Coram and Diaconis [4] to other statistics.

Our purpose here is to extend these calculations to $SO(2N)$ and $USp(2N)$, and to compare the results with what is known about the L-functions. (Only $SO(2N)$ is relevant, because a family of L-functions governed by $O(N)$ falls approximately into two halves; one displaying even symmetry about $s = 1/2$, and the other odd symmetry. This latter class contributes zero to averages at the central value, while the zero statistics of the former are expected to follow those of $SO(2N)$.) We therefore consider the value distribution of the characteristic polynomial of $2N \times 2N$ orthogonal or unitary symplectic matrices,

$$Z(U, \theta) = \prod_{n=1}^{N} \left(1 - e^{i(\theta_n - \theta)}\right) \left(1 - e^{i(-\theta_n - \theta)}\right), \tag{3}$$

averaged over these groups when $\theta = 0$, which is the symmetry point for the eigenvalues, just as $s = 1/2$ is for the L-function zeros. In each case we derive an explicit expression for $\langle Z(U, 0)^s \rangle$, valid for all N, and from this obtain the leading order $N \to \infty$ asymptotics, together with a simplified formula when s is an integer. We also derive the value distributions of $\log Z(U,0)$ and $Z(U,0)$. Comparing with results for various families of L-functions suggests that, as for $\zeta(s)$, random matrix theory determines the universal part of (1). This then provides further support for the programs of Katz and Sarnak, and Conrey and Farmer.

2. Symplectic Symmetry

2.1. Random matrices in $USp(2N)$. We are interested here in the group of symplectic unitary matrices, $USp(2N)$. These are $2N \times 2N$ matrices, U, with $UU^\dagger = 1$ and $U^t JU = J$, where $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ and I_N is the $N \times N$ identity matrix. For these matrices, the eigenvalues lie on the unit circle and come in complex conjugate pairs. Thus the characteristic polynomial related to such a matrix with eigenvalues $e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, e^{-i\theta_2}, \ldots\ e^{i\theta_N}, e^{-i\theta_N}$ takes the form (3).

Our first step will be to calculate the moments of $Z(U,0)$ defined by averaging with respect to Haar measure. The joint probability density function of the eigenvalues is thus [17]

$$\begin{aligned} N_{Sp} \prod_{1\le i<j\le N} &\left(\sin\left(\frac{\theta_i-\theta_j}{2}\right)\sin\left(\frac{\theta_i+\theta_j}{2}\right)\right)^2 \prod_{k=1}^N \sin^2\theta_k \\ &= N_{Sp} \prod_{1\le i<j\le N}\left(\frac{1}{2}(\cos\theta_j-\cos\theta_i)\right)^2 \prod_{k=1}^N \sin^2\theta_k, \qquad (4)\end{aligned}$$

with normalization constant

$$N_{Sp} = 2^{-3N}\prod_{j=1}^N \frac{\Gamma(1+N+j)}{\Gamma(1+j)(\Gamma(1/2+j))^2} = \frac{2^{2N^2-2N}}{\pi^N N!}. \qquad (5)$$

Noting that

$$\begin{aligned} Z(U,0) &= \prod_{n=1}^N \left|(1-e^{i\theta_n})\right|^2 \\ &= 2^{2N}\prod_{n=1}^N \sin^2(\theta_n/2) \\ &= 2^N \prod_{n=1}^N (1-\cos\theta_n), \qquad (6)\end{aligned}$$

we proceed to

$$\begin{aligned} \left\langle Z(U,0)^s\right\rangle_{USp(2N)} = N_{Sp}\, 2^{Ns} &\int_0^{2\pi}\cdots\int_0^{2\pi} d\theta_1\cdots d\theta_N \\ &\times \prod_{1\le i<j\le N}\left(\frac{1}{2}(\cos\theta_j-\cos\theta_i)\right)^2 \prod_{k=1}^N \sin^2\theta_k \prod_{n=1}^N (1-\cos\theta_n)^s \end{aligned}$$

$$= N_{Sp}\, 2^{Ns+2N-N^2} \int_0^{\pi} \cdots \int_0^{\pi} d\theta_1 \cdots d\theta_N \prod_{1\le i<j\le N} (\cos\theta_j - \cos\theta_i)^2$$

$$\times \prod_{k=1}^{N} \sin^2\theta_k \prod_{n=1}^{N} (1-\cos\theta_n)^s, \tag{7}$$

which, after the transformation $x_j = \cos\theta_j$, becomes

$$\langle Z(U,0)^s \rangle_{USp(2N)} = N_{Sp}\, 2^{Ns+2N-N^2} \int_{-1}^{1} \cdots \int_{-1}^{1} dx_1 \cdots dx_N \prod_{1\le i<j\le N} (x_j - x_i)^2$$

$$\times \prod_{k=1}^{N} (1-x_k)^{1/2+s}(1+x_k)^{1/2}. \tag{8}$$

There is a form of Selberg's integral (detailed in [11]) which states that

$$\int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1\le j<l\le n} |(x_j - x_l)|^{2\gamma} \prod_{j=1}^{n} (1-x_j)^{\alpha-1}(1+x_j)^{\beta-1} dx_j$$

$$= 2^{\gamma n(n-1)+n(\alpha+\beta-1)} \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+\gamma(n+j-1))}, \tag{9}$$

if $\mathrm{Re}\alpha > 0$, $\mathrm{Re}\beta > 0$ and $\mathrm{Re}\gamma > -\min\left(\frac{1}{n}, \frac{\mathrm{Re}\alpha}{n-1}, \frac{\mathrm{Re}\beta}{n-1}\right)$.

In our case $\gamma = 1$, $\alpha = 3/2 + s$ and $\beta = 3/2$, so

$$\langle Z(U,0)^s \rangle_{USp(2N)} = N_{Sp}\, 2^{Ns+2N-N^2} 2^{N^2+N+Ns}$$

$$\times \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(3/2+s+j)\Gamma(3/2+j)}{\Gamma(2)\Gamma(3+s+N+j-1)}$$

$$= 2^{2Ns} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(1/2+s+j)}{\Gamma(1/2+j)\Gamma(1+s+N+j)}$$

$$\equiv M_{Sp}(N,s). \tag{10}$$

We now consider the coefficients c_j in the expansion

$$M_{Sp}(N,s) = e^{c_1 s + c_2 s^2/2 + c_3 s^3/3! + c_4 s^4/4! + \cdots}. \tag{11}$$

These coefficients are the cumulants of the distribution of $\log Z(U,0)$, because $M_{Sp}(N,s)$ is the generating function for the moments of this distribution:

$$M_{Sp}(N,s) = \sum_{j=0}^{\infty} \langle (\log Z(U,0))^j \rangle_{USp(2N)} \frac{s^j}{j!}. \tag{12}$$

Since

$$\log M_{Sp}(N,s) = 2sN\log 2 + \sum_{j=1}^{N}(\log\Gamma(1/2+s+j) + \log\Gamma(1+N+j) \\ - \log\Gamma(1/2+j) - \log\Gamma(1+s+N+j)), \tag{13}$$

we find that

$$c_1 = \frac{d}{ds}\log M_{Sp}(N,s)\big|_{s=0} = 2N\log 2 + \sum_{j=1}^{N}(\psi(1/2+j) - \psi(1+N+j)) \tag{14}$$

is the first cumulant, while the higher ones are given by

$$c_n = \frac{d^n}{ds^n}\log M_{Sp}(N,s)\big|_{s=0} = \sum_{j=1}^{N}\left(\psi^{(n-1)}(1/2+j) - \psi^{(n-1)}(1+N+j)\right), \tag{15}$$

where $\psi^{(j)}(z) = \frac{d^{j+1}}{dz^{j+1}}\log\Gamma(z)$ is a polygamma function.

We seek the behaviour of these cumulants for large N. For the first we use the asymptotic formula

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}}, \tag{16}$$

which holds when $z \to \infty$ with $|\arg z| < \pi$, where B_{2n} are the Bernoulli numbers. Also, we need the integral form of the digamma function,

$$\psi(z) + \gamma = \int_0^{\infty}\frac{e^{-t} - e^{-zt}}{1-e^{-t}}dt. \tag{17}$$

Applying (17), we obtain

$$c_1 = 2N\log 2 + \sum_{j=1}^{N}\left(\int_0^{\infty}\frac{e^{-t} - e^{-(j+1/2)t}}{1-e^{-t}}dt - \gamma - \int_0^{\infty}\frac{e^{-t} - e^{-(j+N+1)t}}{1-e^{-t}}dt + \gamma\right) \\ = 2N\log 2 + \sum_{j=1}^{N}\int_0^{\infty}\frac{e^{-(j+N+1)t} - e^{-(j+1/2)t}}{1-e^{-t}}dt. \tag{18}$$

We now interchange the summation and integration and perform the sum explicitly so that we can integrate by parts to arrive at

$$c_1 = 2N\log 2 - (N+1)\int_0^{\infty}\frac{e^{-(N+2)t}}{1-e^{-t}}dt + (2N+1)\int_0^{\infty}\frac{e^{-(2N+2)t}}{1-e^{-t}}dt \\ + \frac{1}{2}\int_0^{\infty}\frac{e^{-3t/2}}{1-e^{-t}}dt - (N+1/2)\int_0^{\infty}\frac{e^{-(N+3/2)t}}{1-e^{-t}}dt.$$

Converting back to polygamma function notation via (17), then applying (16) as N becomes large, we see that

$$\begin{aligned} c_1 &= 2N\log 2 + (N+1)\psi(N+2) - (2N+1)\psi(2N+2) \\ &\quad -\frac{1}{2}\psi(3/2) + (N+1/2)\psi(N+3/2) \\ &= \frac{1}{2}\log N + \frac{\gamma}{2} + O(N^{-1}). \end{aligned} \tag{19}$$

For the second cumulant we need to use the asymptotic formula for higher polygamma functions, valid as $z \to \infty$ with $|\arg z| < \pi$,

$$\psi^{(n)}(z) \sim (-1)^{n-1}\left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k}\frac{(2k+n-1)!}{(2k)!z^{2k+n}}\right]. \tag{20}$$

There is also an integral formula for the higher polygamma functions which will prove useful:

$$\psi^{(n)}(z) = (-1)^{n-1}\int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}}dt. \tag{21}$$

This leads us to

$$\begin{aligned} c_2 &= \sum_{j=1}^{N}\Big(\psi^{(1)}(j+1/2) - \psi^{(1)}(1+N+j)\Big) \\ &= \sum_{j=1}^{N}\left(\int_0^\infty \frac{te^{-(j+1/2)t}}{1-e^{-t}}dt - \int_0^\infty \frac{te^{-(1+N+j)t}}{1-e^{-t}}dt\right). \end{aligned} \tag{22}$$

Again we interchange the order of the summation and integration, perform the sum and integrate by parts. The result, expressed in terms of polygamma functions, is

$$\begin{aligned} c_2 &= -\psi(3/2) - \frac{1}{2}\psi^{(1)}(3/2) + \psi(N+3/2) + (N+1/2)\psi^{(1)}(N+3/2) \\ &\quad + \psi(N+2) + (N+1)\psi^{(1)}(N+2) - \psi(2N+2) - (2N+1)\psi^{(1)}(2N+2) \\ &= \log N + 1 + \gamma + \log 2 - \frac{3}{2}\zeta(2) + O(N^{-1}). \end{aligned} \tag{23}$$

The higher cumulants follow in a similar manner

$$\begin{aligned} c_n &= \sum_{j=1}^{N}\left((-1)^n\int_0^\infty \frac{t^{n-1}e^{-(1/2+j)t}}{1-e^{-t}}dt - (-1)^n\int_0^\infty \frac{t^{n-1}e^{-(1+N+j)t}}{1-e^{-t}}dt\right) \\ &= (-1)^n\int_0^\infty \frac{t^{n-1}e^{-(3/2)t}}{1-e^{-t}}\;\frac{1-e^{-Nt}}{1-e^{-t}}dt \\ &\quad - (-1)^n\int_0^\infty \frac{t^{n-1}e^{-2-N}}{1-e^{-t}}\;\frac{1-e^{-Nt}}{1-e^{-t}}dt, \end{aligned} \tag{24}$$

and so

$$
\begin{aligned}
\lim_{N\to\infty} c_n &= (-1)^n \int_0^\infty \frac{t^{n-1}e^{-(3/2)t}}{(1-e^{-t})(1-e^{-t})}dt \\
&= (-1)^n \left[\frac{-t^{n-1}e^{-(3/2)t}}{1-e^{-t}}\Big|_0^\infty + (n-1)\int_0^\infty \frac{t^{n-2}e^{-(1/2)t}}{1-e^{-t}}dt \right. \\
&\qquad \left. -\frac{1}{2}\int_0^\infty \frac{t^{n-1}e^{-(1/2)t}}{1-e^{-t}}dt \right] \\
&= -(n-1)\psi^{(n-2)}(1/2) - \frac{1}{2}\psi^{(n-1)}(1/2) \\
&= (-1)^n (n-1)! \left[(2^{n-1}-1)\zeta(n-1) - \frac{1}{2}(2^n-1)\zeta(n) \right]. \qquad (25)
\end{aligned}
$$

These expressions for the cumulants, inserted into (11), allow us to write the leading order coefficient of the moment $M_{Sp}(N,s)$ as

$$
\begin{aligned}
f_{Sp}(s) &\equiv \lim_{N\to\infty} \frac{M_{Sp}(N,s)}{N^{s/2+s^2/2}} \\
&= \exp\left(\frac{\gamma}{2}s + \left(1+\gamma+\log 2 - \frac{3}{2}\zeta(2)\right)\frac{s^2}{2} \right. \qquad (26) \\
&\qquad \left. + \sum_{n=3}^{\infty} \left((-1)^n(2^{n-1}-1)\zeta(n-1) - \frac{1}{2}(-1)^n(2^n-1)\zeta(n) \right)\frac{s^n}{n} \right).
\end{aligned}
$$

This coefficient can be expressed as a combination of gamma functions and the Barnes G-function [1,16], which is defined by

$$
G(1+z) = (2\pi)^{z/2} e^{-[(1+\gamma)z^2+z]/2} \prod_{n=1}^{\infty} \left[(1+z/n)^n e^{-z+z^2/(2n)} \right], \qquad (27)
$$

and has zeros at the negative integers, $-n$, with multiplicity n ($n = 1, 2, 3 \ldots$). Other properties useful to us are that

$$
\begin{aligned}
G(1) &= 1, \qquad (28) \\
G(z+1) &= \Gamma(z)\ G(z),
\end{aligned}
$$

and furthermore, for $|z| < 1$,

$$
\log G(1+z) = (\log(2\pi)-1)\frac{z}{2} - (1+\gamma)\frac{z^2}{2} + \sum_{n=3}^{\infty}(-1)^{n-1}\zeta(n-1)\frac{z^n}{n}. \qquad (29)
$$

Combining this with

$$
\log\Gamma(1+z) = -\gamma z + \sum_{n=2}^{\infty}\zeta(n)\frac{(-z)^n}{n}, \qquad (30)
$$

which holds for $|z| < 1$, we see that, for $|s| < 1/2$,

$$\log G(1+s) - \frac{1}{2}\log G(1+2s) - \frac{1}{2}\log\Gamma(1+2s) + \frac{1}{2}\log\Gamma(1+s)$$
$$= \frac{\gamma}{2}s + (1+\gamma)\frac{s^2}{2} - \frac{3}{4}\zeta(2)s^2 + \sum_{n=3}^{\infty}\Big((-1)^n(2^{n-1}-1)\zeta(n-1)$$
$$-\frac{1}{2}(-1)^n(2^n-1)\zeta(n)\Big)\frac{s^n}{n}. \tag{31}$$

A comparison with (26) shows that

$$f_{Sp}(s) = 2^{s^2/2} \times \frac{G(1+s)\sqrt{\Gamma(1+s)}}{\sqrt{G(1+2s)\Gamma(1+2s)}}, \tag{32}$$

for $|s| < 1/2$, and hence by analytic continuation for all s.

For integer moments the formula is simpler. Using (28) we see that

$$G(n) = \prod_{j=1}^{n-1}\Gamma(j), \tag{33}$$

and so for integer n,

$$\begin{aligned}
f_{Sp}(n) &= 2^{n^2/2}\frac{G(1+n)\sqrt{\Gamma(1+n)}}{\sqrt{G(1+2n)\Gamma(1+2n)}}\\
&= 2^{n^2/2}\frac{\left(\prod_{j=1}^{n}\Gamma(j)\right)\sqrt{\Gamma(1+n)}}{\sqrt{\prod_{j=1}^{2n}\Gamma(j)\;\Gamma(1+2n)}}\\
&= 2^{n^2/2}\frac{\left(\prod_{j=1}^{n}\Gamma(j)\right)\sqrt{n!}}{\sqrt{\prod_{j=1}^{2n+1}(j-1)!}}\\
&= 2^{n^2/2}\frac{\left(\prod_{j=1}^{n}(j-1)!\right)\sqrt{n!}}{\sqrt{2^{2n-1}\,3^{2n-2}\,4^{2n-3}\cdots(2n-1)^2\,2n}}\\
&= 2^{n^2/2}\frac{\left(\prod_{j=1}^{n}(j-1)!\right)\sqrt{1}\sqrt{2}\cdots\sqrt{n-1}\sqrt{n}}{\sqrt{2}\sqrt{4}\cdots\sqrt{2n}\sqrt{2^{2n-2}3^{2n-2}4^{2n-4}\cdots(2n-2)^2(2n-1)^2}}\\
&= 2^{n^2/2}\frac{1^{n-1}2^{n-2}\cdots(n-2)^2(n-1)}{2^{n/2}2^{n-1}3^{n-1}4^{n-2}5^{n-2}\cdots(2n-2)(2n-1)}\\
&= 2^{n^2/2}\frac{1}{2^{n/2}2^{\sum_{j=1}^{n-1}j}\prod_{j=1}^{n}(2j-1)!!}\\
&= \left(\prod_{j=1}^{n}(2j-1)!!\right)^{-1}.
\end{aligned} \tag{34}$$

Following the ideas developed in [10], these integer coefficients have also been calculated independently by Brézin and Hikami [2].

Having the generating function, $M_{Sp}(N, s)$, it is a short step to find the value distributions of both $\log Z(U, 0)$ and $Z(U, 0)$ itself. The distribution of $\log Z(U, 0)/\log N$ is

$$\begin{aligned}
&\langle \delta(x - \log Z(U, 0)/\log N)\rangle_{USp(2N)} \\
&\quad = \left\langle \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iy(x-\log Z(U,0)/\log N)}dy \right\rangle_{USp(2N)} \\
&\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx} M_{Sp}(N, iy/\log N)dy \qquad (35)\\
&\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx} e^{c_1 iy/\log N + c_2(iy/\log N)^2/2 + c_3(iy/\log N)^3/3! + \cdots} dy \qquad (36)\\
&\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx} \exp\left[\left(\frac{1}{2}\log N + O(1)\right) iy/\log N \right.\\
&\qquad \left. -(\log N + O(1))\frac{y^2}{2(\log N)^2} - i(O(1))\frac{y^3}{3!(\log N)^3} + \cdots \right] dy.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{N\to\infty} \langle \delta(x - \log Z(U, 0)/\log N)\rangle_{USp(2N)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx+iy/2}dy \qquad (37)\\
&= \delta(x - 1/2),
\end{aligned}$$

and so the distribution of values of $\log Z(U, 0)/\log N$ tends to a delta function centred at $x = 1/2$.

If we instead retain the y^2 term in the exponent in (36), we have the central limit theorem

$$\lim_{N\to\infty} \left\langle \delta\left(x + \frac{c_1}{\sqrt{c_2}} - \frac{\log Z(U, 0)}{\sqrt{c_2}}\right)\right\rangle_{USp(2N)} = \sqrt{\frac{1}{2\pi}} \exp\left(-\frac{x^2}{2}\right), \qquad (38)$$

where c_1 and c_2 are related to N by (19) and (23), respectively. For finite N, the exact distribution is of course given by (35), where $M_{Sp}(N, s)$ is defined by (10).

It is not difficult to determine as well the distribution of the values of $Z(U, 0)^{\frac{1}{\log N}}$. Changing variables in (35), results in

$$\left\langle \delta(x - Z(U, 0)^{\frac{1}{\log N}})\right\rangle_{USp(2N)} = \frac{1}{2\pi x}\int_{-\infty}^{\infty} x^{-iy} M_{Sp}(N, iy/\log N)dy, \qquad (39)$$

and so

$$\begin{aligned}
\lim_{N\to\infty} \langle \delta(x - Z(U, 0)^{\frac{1}{\log N}})\rangle_{USp(2N)} &= \frac{1}{2\pi x}\int_{-\infty}^{\infty} e^{-iy\log x} e^{iy/2} dy \\
&= \frac{1}{x}\delta(\log x - 1/2). \qquad (40)
\end{aligned}$$

Alternatively, we can examine the value distribution of $Z(U, 0)$. Denoting it by $P_{Sp}(N, x)$, we see that

$$P_{Sp}(N, x) = \frac{1}{2\pi x}\int_{-\infty}^{\infty} x^{-iy} M_{Sp}(N, iy)dy. \qquad (41)$$

Although $P_{Sp}(N, x)$ does not have a limiting distribution as $N \to \infty$, we suggest the approximation

$$P_{Sp}(N, x) \approx \frac{1}{2\pi x} \int_{-\infty}^{\infty} \exp\left[-iy \log x + ic_1 y - \frac{c_2 y^2}{2}\right] dy$$
$$= \frac{1}{x\sqrt{2\pi c_2}} \exp\left(-\frac{(\log x - c_1)^2}{2c_2}\right), \qquad (42)$$

and plot it, for two values of N, in Fig. 1 along with the exact distribution (41). It should be noted that the approximation (42) is valid when x is fixed and $N \to \infty$, and more generally is expected to be a good approximation when $\log x >> -\log N$ and N is large, the lower bound being determined by the first pole of $M_{Sp}(N, s)$.

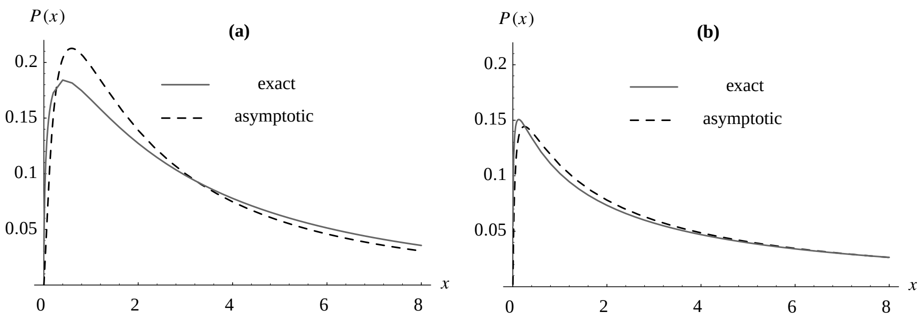

Fig. 1. Distribution of the values of $Z(U, 0)$ for matrices in $USp(2N)$, (**a**) $N = 6$, (**b**) $N = 42$. The solid curve is the exact distribution (41) and the dashed curve is the large N approximation (42)

It may be seen from Fig. 1 that $P_{Sp}(N, 0) = 0$. Although the approximation (42) also tends to zero as $x \to 0$, it does not predict the correct rate of approach. This may instead be obtained by examining the poles of the integrand of

$$P_{Sp}(N, x) = \frac{1}{2\pi x} \int_{-\infty}^{\infty} x^{-iy} 2^{2iNy} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(1/2+iy+j)}{\Gamma(1/2+j)\Gamma(1+iy+N+j)} dy. \qquad (43)$$

These poles occur at the points $y = i(2k+1)/2$ and are of order k, for $k = 1, 2, \ldots, N$, then of order N for all higher k. Due to the factor x^{-iy} it is evident that in the limit $x \to 0$, the lowest pole, that at $y = (3i)/2$, gives the dominant contribution. From the residue at that lowest pole we thus find that as $x \to 0$,

$$P_{Sp}(N, x) \sim x^{1/2} 2^{-3N} \frac{1}{\Gamma(N)} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(j)}{\Gamma(1/2+j)\Gamma(N+j-1/2)}. \qquad (44)$$

2.2. L-functions with symplectic symmetry. In the Introduction we gave a brief description of the mean values at $s = 1/2$ for families of L-functions and the relation of these to the symmetry type displayed by the low-lying zeros. Here we consider the case of symplectic symmetry in more depth.

If we again use Conrey and Farmer's notation, as in (1), then in the symplectic case they have $V(z) = z$ and find that $B(k) = \frac{1}{2}k(k+1)$ [3]. They also list several families which are conjectured to have low-lying zeros with symplectic symmetry, the simplest of which is the Dirichlet L-functions, $L(s, \chi_d)$, where χ_d is a quadratic Dirichlet character. The sum (1) is then over all such characters with conductor $|d| \le D$: as $D \to \infty$,

$$\frac{1}{D^*} \sum_{|d|\le D} L\left(\frac{1}{2}, \chi_d\right)^k \sim g_k \frac{a(k)}{\Gamma(1 + \frac{1}{2}k(k+1))} (\log D^{\frac{1}{2}})^{\frac{1}{2}k(k+1)}, \tag{45}$$

where D^* is the number of quadratic characters included in the sum.

For this case the first few values of g_k for integer k have been found using number-theoretic techniques to be [7,15,3] $g_1 = 1$, $g_2 = 2$, $g_3 = 2^4$ and, by conjecture, $g_4 = 3 \cdot 2^8$. It might be expected that g_k is related to the random matrix moment values calculated in Sect. 2.1, since it is believed to be purely symmetry-determined. Our purpose now is to provide evidence in favour of this.

Making the identification

$$N = \log(\mathcal{Q}^A), \tag{46}$$

and recalling that as $N \to \infty$ $M_{Sp}(N,k) \sim f_{Sp}(k) N^{\frac{1}{2}k(k+1)}$, we conjecture that for symplectic families of L-functions

$$\frac{g_k}{\Gamma(1 + \frac{1}{2}k(k+1))} = f_{Sp}(k). \tag{47}$$

Following the arguments of [10], the relation between N and $\mathcal{Q}$ should arise from equating the mean densities of zeros. For the L-functions we need the density near $s = 1/2$ because we are dealing with the L-functions just at this point. In the case of L-functions with quadratic Dirichlet characters, (45), the mean density at a fixed height up the critical line increases like $\frac{1}{2\pi} \log |d|$ as $|d| \to \infty$. Since the mean density of eigenvalues of a matrix in $USp(2N)$ is N/π, we equate $N = (1/2) \log D$, and obtain exactly the proposed relation, since $A = 1/2$ in this case.

It is then striking that the first few values of f_{Sp} at the integers, $f_{Sp}(1) = 1$, $f_{Sp}(2) = \frac{1}{3}$, $f_{Sp}(3) = \frac{1}{45}$ and $f_{Sp}(4) = \frac{1}{4725}$, agree precisely, via (47), with the values that Conrey and Farmer report for the symplectic L-functions.

Further evidence in favour of (47) is the success of a very similar conjecture relating moments of the Riemann zeta function to averages over $U(N)$ [10]. The only difference is that in the case of $\zeta(s)$ the average was along the critical line rather than over a family of functions. This is not a significant difference, however, and Conrey and Farmer in fact suggest that we think of the Riemann zeta function as a unitary family (with zeros showing the statistics of the eigenvalues of matrices from $U(N)$) in its own right, where we are averaging over special values of the family $\{\zeta(1/2 + it)\}$ as t ranges over the real numbers.

The validity of the conjecture (47) would imply many results on the value distribution of the central values of symplectic L-functions. The distribution for the logarithm of symplectic families of L-functions, for example, is expected to behave for asymptotically large $\mathcal{Q}$ in the same way as that of the characteristic polynomial Z, always remembering that N must be related to the L-function parameter via the density of zeros. This is because the conjecture (47) can also be written as

$$\lim_{Q\to\infty} \frac{1}{(\log Q^A)^{\frac{1}{2}k(k+1)} Q^*} \sum_{\substack{f\in\mathcal{F}\\ c(f)\le Q}} L_f(\tfrac{1}{2})^k = a(k)\times\left(\lim_{N\to\infty}\frac{M_{Sp}(N,k)}{N^{\frac{1}{2}k(k+1)}}\right), \quad (48)$$

so the value distribution of $\log L_f(\frac{1}{2})/\log\log Q^A$ defined by averages with $c(f)\le Q$, would be, for large Q and making the identification (46),

$$V_{Sp}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy} a(iy/\log N) M_{Sp}(N, iy/\log N)dy, \quad (49)$$

leading to

$$\lim_{N\to\infty} V_{Sp}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(0)e^{-ixy+iy/2}dy. \quad (50)$$

Since $a(0) = 1$, we see that this would imply that the distribution of $\log L_f(\frac{1}{2})/\log\log Q^A$ is asymptotic to $\delta(x-1/2)$, in just the same way as for $\log Z(U,0)/\log N$.

Following the same line of argument, we suggest that

$$\begin{aligned}
&\lim_{Q\to\infty}\left\langle \delta\left(x + \frac{\tilde{c_1}}{\sqrt{\tilde{c_2}}} - \frac{\log L_f\left(\frac{1}{2}\right)}{\sqrt{\tilde{c_2}}}\right)\right\rangle_{\mathcal{F}} \\
&\quad = \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} a\left(\frac{iy}{\sqrt{c_2}}\right) e^{-iyx-iyc_1/\sqrt{c_2}} e^{iyc_1/\sqrt{c_2}-y^2/2+c_3(iy)^3/(c_2^{3/2}3!)+\cdots}dy \\
&\quad = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right),
\end{aligned} \quad (51)$$

where $\langle\cdot\rangle_{\mathcal{F}}$ denotes an average over a family $\mathcal{F}$ of L-functions, as in (1), and $\tilde{c_1}$ and $\tilde{c_2}$ are given by (19) and (23), respectively, again with the identification (46).

If we now turn to the distribution of values of $L_f(\frac{1}{2})$ itself, $W_{Sp}(x)$, we can close the contour of

$$W_{Sp}(x) = \frac{1}{2\pi x}\int_{-\infty}^{\infty} x^{-is} a(is) M_{Sp}(N, is)ds \quad (52)$$

around the poles and obtain, as $x \to 0$, the dominant contribution from the pole at $s = (3i)/2$:

$$W_{Sp}(x) \sim x^{1/2} a(-3/2) 2^{-3N} \frac{1}{\Gamma(N)} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(j)}{\Gamma(1/2+j)\Gamma(N+j-1/2)}. \quad (53)$$

This is of particular note in the light of recent interest in the non-vanishing of the central values of L-functions, see for example [15,6,5] and references therein. Clearly (53) implies that as long as $a(-3/2)$ is finite for a particular family of symplectic L-functions, the probability that the central value of those L-functions lies in the range $(0, x)$ decreases like $x^{3/2}$ as $x \to 0$.

3. Orthogonal Symmetry

3.1. Random matrices in SO(2N). We now consider the characteristic polynomial of matrices from the group $SO(2N)$. Here the eigenphases also come in complex conjugate pairs, so $Z(U, \theta)$ takes the form (3), and the average at the point $\theta = 0$ is, once again using the joint probability density function for the eigenphases dictated by Haar measure [17],

$$\begin{aligned}\left\langle Z(U,0)^s \right\rangle_{SO(2N)} &= N_O \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N \prod_{1\le i<j\le N} \left(\frac{1}{2}(\cos\theta_j - \cos\theta_i)\right)^2 \\ &\quad \times 2^{Ns} \prod_{n=1}^{N} (1-\cos\theta_n)^s \\ &= N_O\, 2^{2N-N^2+Ns} \int_0^{\pi} \cdots \int_0^{\pi} d\theta_1 \cdots d\theta_N \prod_{1\le i<j\le N} (\cos\theta_j - \cos\theta_i)^2 \\ &\quad \times \prod_{n=1}^{N} (1-\cos\theta_n)^s \\ &= N_O\, 2^{2N-N^2+Ns} \int_{-1}^{1} \cdots \int_{-1}^{1} dx_1 \cdots dx_N \prod_{1\le i<j\le N} (x_j - x_i)^2 \\ &\quad \times \prod_{n=1}^{N} (1-x_n^2)^{-1/2} (1-x_n)^s, \end{aligned} \tag{54}$$

with a normalization constant

$$N_O = 2^{-N} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)}{\Gamma(1+j)(\Gamma(j-1/2))^2} = \frac{2^{2N^2-4N+1}}{\pi^N N!}. \tag{55}$$

We use the Selberg integral again, this time with $\gamma = 1$, $\alpha = s + 1/2$ and $\beta = 1/2$, obtaining

$$\begin{aligned}\left\langle Z(U,0)^s \right\rangle_{SO(2N)} &= N_O\, 2^{2N-N^2+Ns} \cdot 2^{N^2-N+sN+N/2+N/2-N} \\ &\quad \times \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(s+1/2+j)\Gamma(1/2+j)}{\Gamma(2)\Gamma(s+1+N+j-1)} \\ &= N_O\, 2^{N+2Ns} \prod_{j=1}^{N} \frac{\Gamma(1+j)\Gamma(s+j-1/2)\Gamma(j-1/2)}{\Gamma(s+N+j-1)} \\ &= 2^{2Ns} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(s+j-1/2)}{\Gamma(j-1/2)\Gamma(s+j+N-1)} \\ &\equiv M_O(N,s). \end{aligned} \tag{56}$$

As for $M_{Sp}(N,s)$, $M_O(N,s)$ is the generating function for the moments of the log of $Z(U,0)$, this time for the orthogonal ensemble, so if we write

$$M_O(N,s) = e^{q_1 s + q_2 s^2/2 + q_3 s^3/3! + q_4 s^4/4! + \cdots}, \tag{57}$$

then the parameters q_j are the cumulants of the value distribution of $\log Z(U,0)$. These cumulants can be obtained by taking derivatives of

$$\begin{aligned}\log M_O(N,s) = \; & 2Ns\log 2 + \sum_{j=1}^{N}(\log\Gamma(N+j-1) + \log\Gamma(s+j-1/2) \\ & -\log\Gamma(j-1/2) - \log\Gamma(s+j+N-1)),\end{aligned} \tag{58}$$

thus producing

$$\begin{aligned} q_1 &= \frac{d}{ds}\log M_O(N,s)\Big|_{s=0} \\ &= 2N\log 2 + \sum_{j=1}^{N}(\psi(j-1/2) - \psi(j+N-1)), \\ q_n &= \frac{d^n}{ds^n}\log M_O(N,s)\Big|_{s=0} \\ &= \sum_{j=1}^{N}\left(\psi^{(n-1)}(j-1/2) - \psi^{(n-1)}(j+N-1)\right), \end{aligned} \tag{59}$$

for $n = 2,3,4,\ldots$.

The asymptotic behaviour of these cumulants for large N may be recovered using the techniques of Sect. 2.1. Starting with q_1 and using (16) and (17),

$$\begin{aligned} q_1 &= 2N\log 2 + \sum_{j=1}^{N}\left(\int_0^\infty \frac{e^{-t} - e^{-(j-1/2)t}}{1-e^{-t}}dt - \gamma - \int_0^\infty \frac{e^{-t} - e^{-(j+N-1)t}}{1-e^{-t}}dt + \gamma\right) \\ &= 2N\log 2 + \sum_{j=1}^{N}\int_0^\infty \frac{e^{-(j+N-1)t} - e^{-(j-1/2)t}}{1-e^{-t}}dt. \end{aligned} \tag{60}$$

At this point we interchange the sum and integral, evaluate the sum and integrate by parts, resulting in

$$\begin{aligned} q_1 &= 2N\log 2 - (N-1)\int_0^\infty \frac{e^{-Nt}}{1-e^{-t}}dt + (2N-1)\int_0^\infty \frac{e^{-2Nt}}{1-e^{-t}}dt \\ &\quad - \frac{1}{2}\int_0^\infty \frac{e^{-t/2}}{1-e^{-t}}dt - (N-1/2)\int_0^\infty \frac{e^{-(N+1/2)t}}{1-e^{-t}}dt \\ &= 2N\log 2 + (N-1)\psi(N) - (2N-1)\psi(2N) \\ &\quad + \frac{1}{2}\psi(1/2) + (N-1/2)\psi(1/2+N) \\ &= -\frac{1}{2}\log N - \frac{\gamma}{2} + O\left(\frac{1}{N}\right). \end{aligned} \tag{61}$$

The second cumulant is determined similarly (with the help of (20) and (21)) to be

$$\begin{aligned}
q_2 &= \sum_{j=1}^{N}\left(\int_0^\infty \frac{te^{-(j-1/2)t}}{1-e^{-t}}dt - \int_0^\infty \frac{te^{-(j+N-1)t}}{1-e^{-t}}dt\right)\\
&= \int_0^\infty \frac{te^{-t/2}(1-e^{-Nt})}{(1-e^{-t})^2}dt - \int_0^\infty \frac{te^{-Nt}(1-e^{-Nt})}{(1-e^{-t})^2}dt\\
&= \int_0^\infty \frac{e^{-t/2}}{1-e^{-t}}dt + \frac{1}{2}\int_0^\infty \frac{te^{-t/2}}{1-e^{-t}}dt\\
&\quad - \int_0^\infty \frac{e^{-(N+1/2)t}}{1-e^{-t}}dt + (N-1/2)\int_0^\infty \frac{te^{-(N+1/2)t}}{1-e^{-t}}dt\\
&\quad - \int_0^\infty \frac{e^{-Nt}}{1-e^{-t}}dt + (N-1)\int_0^\infty \frac{te^{-Nt}}{1-e^{-t}}dt\\
&\quad + \int_0^\infty \frac{e^{-2Nt}}{1-e^{-t}}dt - (2N-1)\int_0^\infty \frac{te^{-2Nt}}{1-e^{-t}}dt\\
&= -\psi(1/2) + \frac{1}{2}\psi^{(1)}(1/2) + \psi(N+1/2) + (N-1/2)\psi^{(1)}(N+1/2)\\
&\quad + \psi(N) + (N-1)\psi^{(1)}(N) - \psi(2N) - (2N-1)\psi^{(1)}(2N)\\
&= \log N + 1 + \gamma + \log 2 + \frac{3}{2}\zeta(2) + O\left(\frac{1}{N}\right).
\end{aligned} \tag{62}$$

Finally, all the higher cumulants converge asymptotically to a constant,

$$\begin{aligned}
\lim_{N\to\infty} q_n &= \lim_{N\to\infty}\left((-1)^n\int_0^\infty \frac{t^{n-1}e^{-t/2}}{1-e^{-t}}\frac{1-e^{-Nt}}{1-e^{-t}}dt - (-1)^n\right.\\
&\qquad \left.\times\int_0^\infty \frac{t^{n-1}e^{-Nt}}{1-e^{-t}}\frac{1-e^{-Nt}}{1-e^{-t}}dt\right)\\
&= (-1)^n\int_0^\infty \frac{t^{n-1}e^{-t/2}}{(1-e^{-t})^2}dt.
\end{aligned} \tag{63}$$

Evaluating the integral in (63) by integrating by parts and then rewriting it as a pair of polygamma functions,

$$\begin{aligned}
\lim_{N\to\infty} q_n &= -(n-1)\psi^{(n-2)}(-1/2) + \frac{1}{2}\psi^{(n-1)}(-1/2)\\
&= (-1)^n(n-1)!\left[(2^{n-1}-1)\zeta(n-1) + \frac{1}{2}(2^n-1)\zeta(n)\right].
\end{aligned} \tag{64}$$

It is thus clear from (57) and the asymptotic form of the cumulants that the leading order coefficient of the moments of $Z(U,0)$ is

$$f_O(s) \equiv \lim_{N\to\infty} \frac{M_O(N,s)}{N^{s^2/2-s/2}}$$
$$= \exp\Bigg[-\frac{\gamma}{2}s + \left(1+\gamma+\log 2+\frac{3}{2}\zeta(2)\right)\frac{s^2}{2}$$
$$+\sum_{n=3}^{\infty}\left((-1)^n(2^{n-1}-1)\zeta(n-1)+\frac{1}{2}(-1)^n(2^n-1)\zeta(n)\right)\frac{s^n}{n}\Bigg]. \quad (65)$$

Examining the product form of $M_O(N,s)$ we see that the coefficient is expected to have poles of order k at $s=-(2k-1)/2$, for $k=1,2,3\dots$. Using (29) and (30), we see that, for $|s|<1/2$,

$$\log G(1+s) - \frac{1}{2}\log G(1+2s) + \frac{1}{2}\log\Gamma(1+2s) - \frac{1}{2}\log\Gamma(1+s)$$
$$= -\frac{\gamma}{2} + \left(1+\gamma+\frac{3}{2}\zeta(2)\right)\frac{s^2}{2}$$
$$+\sum_{n=3}^{\infty}\left((-1)^n(2^{n-1}-1)\zeta(n-1)+\frac{1}{2}(-1)^n(2^n-1)\zeta(n)\right)\frac{s^n}{n}, \quad (66)$$

and comparing with (65) we thus find that

$$f_O(s) = 2^{s^2/2}\times\frac{G(1+s)\sqrt{\Gamma(1+2s)}}{\sqrt{G(1+2s)\Gamma(1+s)}}, \quad (67)$$

for $|s|<1/2$, and hence by analytic continuation in the rest of the complex plane. This clearly has the correct combination of poles.

This leading order coefficient reduces for integer moments, again using (28), to

$$f_O(n) = 2^{n^2/2}\frac{\left(\prod_{j=1}^{n}\Gamma(j)\right)\sqrt{\Gamma(1+2n)}}{\sqrt{\left(\prod_{j=1}^{2n}\Gamma(j)\right)\Gamma(1+n)}}$$
$$= 2^{n^2/2}\frac{\left(\prod_{j=1}^{n}(j-1)!\right)\sqrt{(2n)!}}{\sqrt{2^{2n-2}3^{2n-3}\cdots(2n-2)^2(2n-1)n!}}$$
$$= 2^{n^2/2}\frac{\left(\prod_{j=1}^{n}(j-1)!\right)\sqrt{2^n n!}\sqrt{(2n-1)!!}}{2^{n-1}4^{n-2}\cdots(2n-2)\sqrt{3^{2n-3}5^{2n-5}\cdots(2n-1)n!}}$$
$$= 2^{n^2/2}\frac{\left(\prod_{j=1}^{n}(j-1)!\right)\sqrt{2^n}}{2^{\sum_{j=1}^{n-1}j}1^{n-1}2^{n-2}\cdots(n-1)\sqrt{3^{2n-4}5^{2n-6}\cdots(2n-3)^2}}$$
$$= 2^{n^2/2}\frac{1^{n-1}2^{n-2}\cdots(n-2)^2(n-1)2^{n/2}}{2^{n(n-1)/2}3^{n-2}5^{n-3}\cdots(2n-3)1^{n-1}2^{n-2}\cdots(n-1)}$$
$$= 2^n\left(\prod_{j=1}^{n-1}(2j-1)!!\right)^{-1}. \quad (68)$$

This result was also obtained independently in [2].

Once more, we can examine the value distribution of $Z(U, 0)$ and its logarithm. The value distribution of $\log Z(U, 0)/\log N$ is

$$\begin{aligned}
&\langle \delta(x - \log Z(U,0)/\log N)\rangle_{SO(2N)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx} M_O(N, iy/\log N)dy \qquad (69)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-iyx + \left(-\frac{1}{2}\log N + O(1)\right) iy/\log N \right.\\
&\left. - (\log N + O(1))\frac{y^2}{2(\log N)^2} - i(O(1))\frac{y^3}{3!(\log N)^3} + \cdots\right] dy,
\end{aligned}$$

yielding the limiting distribution

$$\begin{aligned}
\lim_{N\to\infty}\langle \delta(x - \log Z(U,0)/\log N)\rangle_{SO(2N)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyx - iy/2}dy \\
&= \delta(x + 1/2). \qquad (70)
\end{aligned}$$

This is a delta distribution as in the symplectic case, but this time centred at $x = -1/2$.

Keeping the y^2 term in the exponent in the integral above leads to the central limit theorem:

$$\lim_{N\to\infty}\left\langle \delta\left(x + \frac{q_1}{\sqrt{q_2}} - \frac{\log Z(U,0)}{\sqrt{q_2}}\right)\right\rangle_{SO(2N)} = \sqrt{\frac{1}{2\pi}}\exp\left(-\frac{x^2}{2}\right). \qquad (71)$$

The value distribution of $Z(U, 0)^{\frac{1}{\log N}}$ is similarly straightforward to compute. We see that

$$\langle \delta(x - Z(U,0)^{\frac{1}{\log N}})\rangle_{SO(2N)} = \frac{1}{2\pi x}\int_{-\infty}^{\infty} x^{-iy} M_O(N, iy/\log N)dy, \qquad (72)$$

and so

$$\begin{aligned}
\lim_{N\to\infty}\langle \delta(x - Z(U,0)^{\frac{1}{\log N}})\rangle_{SO(2N)} &= \frac{1}{2\pi x}\int_{-\infty}^{\infty} e^{-iy\log x} e^{-iy/2}dy \\
&= \frac{1}{x}\delta(\log x + 1/2). \qquad (73)
\end{aligned}$$

We also examine the distribution simply of $Z(U, 0)$, $P_O(N, x)$. As

$$P_O(N, x) = \frac{1}{2\pi x}\int_{-\infty}^{\infty} x^{-iy} M_O(N, iy)dy, \qquad (74)$$

we can make the approximation

$$\begin{aligned}
P_O(N, x) &\approx \frac{1}{2\pi x}\int_{-\infty}^{\infty} \exp\left[-iy\log x + iq_1 y - \frac{q_2 y^2}{2}\right] dy \\
&= \frac{1}{x\sqrt{2\pi q_2}}\exp\left(\frac{-(\log x - q_1)^2}{2q_2}\right), \qquad (75)
\end{aligned}$$

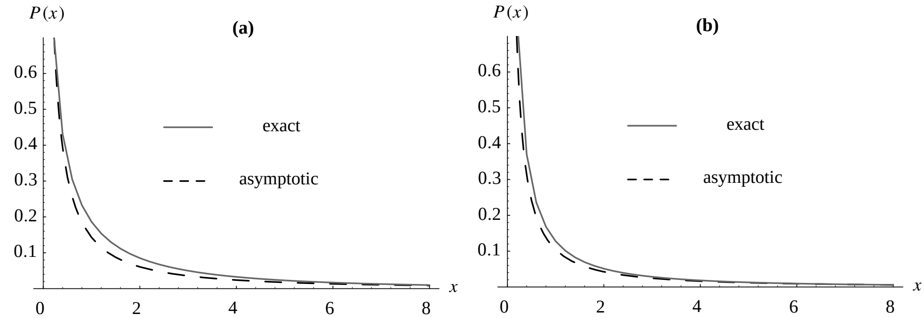

Fig. 2. Distribution of the values of $Z(U, 0)$ for matrices in the group $SO(2N)$, with **(a)** $N = 6$ and **(b)** $N = 42$. The solid curve is the exact distribution (74) and the dashed curve is the large N approximation in (75)

valid as $N \to \infty$ when x is fixed (and, like (42), expected to be a good approximation when $\log x >> -\log N$ and N is large). The result (75) is plotted in Fig. 2 for $N = 6$ and $N = 42$ along with the numerically calculated exact distribution, from (74).

Unlike the symplectic case (and unlike the approximation (75)), $P_O(N, x)$ diverges as $x \to 0$. This can be seen by considering the poles of the integrand, which occur at $i/2, 3i/2, 5i/2, \ldots$. Once again it is the lowest pole, the simple one at $i/2$, that dominates the integral as $x \to 0$. In this case we find that

$$P_O(N, x) \sim x^{-1/2} 2^{-N} \frac{1}{\Gamma(N)} \prod_{j=1}^{N} \frac{\Gamma(N + j - 1)\Gamma(j)}{\Gamma(j - 1/2)\Gamma(j + N - 3/2)}, \tag{76}$$

in that limit.

3.2. L-functions with orthogonal symmetry. We now turn our attention to families of L-functions with a symmetry governed by an ensemble of orthogonal matrices. L-functions of this type fall into two categories, even and odd, which are related to the ensembles $SO(2N)$ and $SO(2N + 1)$ respectively. Of the L-functions comprising an orthogonal family, approximately one half will have even symmetry, and the other half odd symmetry, these latter vanishing at $s = 1/2$.

Examples of such families are given in [3]. Referring to (1), in the orthogonal case $V(z) = z$ and $B(k) = \frac{1}{2}k(k-1)$. As in the symplectic case, the first few of the coefficients g_k with integer coefficients have been calculated. The known values are $g_1 = 1$, $g_2 = 2$, $g_3 = 2^3$ and it is conjectured that $g_4 = 2^7$ [3].

With N taking the place of $\log(Q^A)$, we conjecture this time that

$$\lim_{Q\to\infty} \frac{1}{(\log Q^A)^{\frac{1}{2}k(k-1)} Q^*} \sum_{\substack{f \in \mathcal{F} \\ c(f) \le Q}} L_f\left(\tfrac{1}{2}\right)^k = a(k) \times f_O(k)/2. \tag{77}$$

The right-hand side is divided by two because the random matrix average was just over $SO(2N)$, whereas the sum over central values of the L-functions contains an equal number of functions contributing zero to the average; namely the L-functions with odd symmetry about $s = 1/2$. Once again, we expect the relation (46) to follow from equating the density of zeros of the L-functions and the density of eigenphases of the matrices.

Having posed the conjecture (77), we check it against the known values of g_k. It is clear that the first four coefficients $f_O(1) = 2$, $f_O(2) = 4$, $f_O(3) = \frac{8}{3}$ and $f_O(4) = \frac{16}{45}$ satisfy conjecture (77); that is $g_k/\Gamma(1 + \frac{1}{2}k(k-1)) = f_O(k)/2$, for $k = 1, 2, 3, 4$.

As for the symplectic case, we can examine what (77) implies about the value distributions of L-functions and their logarithms. Since the L-functions with odd symmetry are zero at $s = 1/2$, we now restrict ourselves to averages over the orthogonal L-functions with even symmetry. These are expected to satisfy (77) without the factor 1/2 on the right-hand side.

The value distribution of $\log L_f\left(\frac{1}{2}\right) / \log\log \mathcal{Q}^A$ for L-functions with even symmetry (defined by averaging as in (77)) is expected to be given, for large $\mathcal{Q}$, and with the identification (46), by

$$V_O(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixs} a(is/\log N) M_O(N, is/\log N) ds, \tag{78}$$

and following the argument laid out for the symplectic case, this converges to $\delta(x+1/2)$ as $N \to \infty$.

We can once again state a conjectural central limit theorem, this time for averages over a family $\mathcal{F}$ of L-functions with $c(f) \le \mathcal{Q}$ governed by the symmetry $SO(2N)$:

$$\begin{aligned}
&\lim_{\mathcal{Q}\to\infty} \left\langle \delta\left(x + \frac{\tilde{q}_1}{\sqrt{\tilde{q}_2}} - \frac{\log L_f\left(\frac{1}{2}\right)}{\sqrt{\tilde{q}_2}}\right)\right\rangle_{\mathcal{F}} \\
&= \lim_{N\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} a\left(\frac{iy}{\sqrt{q_2}}\right) e^{-iyx - iyq_1/\sqrt{q_2}} e^{iyq_1/\sqrt{q_2} - y^2/2 + q_3(iy)^3/(q_2^{3/2}3!)+\cdots} dy \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right),
\end{aligned} \tag{79}$$

where $\tilde{q}_1$ and $\tilde{q}_2$ are related to (61) and (62), respectively, via (46).

For the value distribution of $L_f(\frac{1}{2})$ itself, which the conjecture suggests for large $\mathcal{Q}$ is

$$W_O(x) = \frac{1}{2\pi x}\int_{-\infty}^{\infty} x^{-is} a(is) M_O(N, is) ds, \tag{80}$$

we expect that near $x = 0$,

$$W_O(x) \sim x^{-1/2} a(-1/2) 2^{-N} \frac{1}{\Gamma(N)} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(j)}{\Gamma(j-1/2)\Gamma(j+N-3/2)}; \tag{81}$$

the contribution to the integral (80) from the simple pole at $s = i/2$. For L-functions with even symmetry from an orthogonal family for which $a(-1/2) \ne 0$, this analysis therefore suggests that the likelihood that the central value vanishes is integrably singular, and that the probability of a value in the range $(0, x)$ vanishes as $x^{1/2}$ when $x \to 0$.

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