

Derivatives of random matrix characteristic polynomials with applications to elliptic curves

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Abstract

The value distribution of derivatives of characteristic polynomials of matrices from $SO(N)$ is calculated at the point 1, the symmetry point on the unit circle of the eigenvalues of these matrices. We consider subsets of matrices from $SO(N)$ that are constrained to have at least n eigenvalues equal to 1, and investigate the first non-zero derivative of the characteristic polynomial at that point. The connection between the values of random matrix characteristic polynomials and values of L -functions in families has been well-established. The motivation for this work is the expectation that through this connection with L -functions derived from families of elliptic curves, and using the Birch and Swinnerton-Dyer conjecture to relate values of the L -functions to the rank of elliptic curves, random matrix theory will be useful in probing important questions concerning these ranks.

1 Introduction

1.1 Random matrix theory and number theory

The connection between random matrix theory and number theory began with the work of Montgomery [33] when he conjectured that the distribution of the non-trivial zeros of the Riemann zeta function follows the same statistics as the eigenvalues of a random matrix chosen from $U(N)$ generated uniformly with respect to Haar measure. This conjecture is supported by numerical evidence [35] and also by further work [21, 38, 2, 3] suggesting that the same conjecture is true for more general L -functions. For all these L -functions there is a Generalized Riemann Hypothesis that the non-trivial zeros lie on a vertical line in the complex plane. The conjectures mentioned above concern the statistics of the zeros high on this critical line.

The philosophy of Katz and Sarnak [24, 25] extended the connection with random matrix theory by proposing that rather than averaging over many zeros of a given L -function, if the zeros near to the critical point where the critical line crosses the real axis are averaged over a family of naturally connected L -functions, then they will be found to follow the statistics of the eigenvalues of one of the three classical compact groups of random matrices: $U(N)$, $O(N)$ or $USp(2N)$, where again the statistics are computed with respect

to the probability measure given by Haar measure. There is numerical evidence for this conjecture as well [37], and strong support is given to it by the rigorous work of Katz and Sarnak in the case of function field zeta functions.

For a review of applications of random matrix theory to questions in number theory see, for example, [5] or [28].

There has been a series of papers, starting with [27] and continuing with [7, 23, 22, 26, 9, 8], examining how random matrix theory can be used to predict the distribution of values of the Riemann zeta function and other L -functions, either averaged over an interval high on the critical line, or over a family at the critical point where the critical line crosses the real axis. An example of such a family is the collection of L -functions associated with real quadratic characters χ_d . These have a Dirichlet series given by $L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$, where d is a fundamental discriminant and $\chi_d(n) = \left(\frac{d}{n}\right)$ is the Kronecker symbol. (A fundamental discriminant is an integer other than 1 that is not divisible by the square of any prime other than two and that satisfies $d \equiv 1 \pmod{4}$ or $d \equiv 8, 12 \pmod{16}$.) For large values of the natural asymptotic parameter (in the above example this is the integer d that orders the L -functions within the family) the moments of L -functions in any given family are conjectured to split into a product of an arithmetic contribution, determined by the family being averaged over, and a component derived from a random matrix calculation - the corresponding moment of the characteristic polynomial of the matrices in one of the three matrix groups $U(N)$, $O(N)$ or $USp(2N)$. The asymptotic parameter on the random matrix side is the dimension of the matrix N and a natural equivalence can be made between the two.

As will be reviewed in the following section, this conjecture for the leading order behaviour of moments of L -functions has been used by Conrey, Keating, Rubinstein and Snaith [10] to predict the frequency, within a family of elliptic curves, of an L -function taking the value zero at the critical point. This family arises from the L -function associated to a given elliptic curve being twisted by the real quadratic Dirichlet characters mentioned in the previous paragraph. The prediction of the frequency of vanishing uses random matrix theory to predict the value distribution of the L -values at the critical point and then the discretization of these values [43, 39, 29] to calculate a probability of the L -value being zero. This work was extended in [12] to calculate the first two terms in the ratio of the number of vanishings of twists sorted according to arithmetic progressions. See also David, Fearnley and Kisilevsky [14] for a similar use of random matrix theory to predict frequency of vanishing at the critical point amongst families of elliptic curve L -functions twisted by cubic characters.

In Section 2.1 we calculate the value distribution of the first non-zero derivative of the characteristic polynomial at the point 1 (corresponding to the value at the critical point in the analogy with L -functions) when a given number of eigenvalues are conditioned to lie at the point 1. We find, at equation (2.20), that moment of the n th derivative grows like

$$\mathcal{M}(n, M, s) := \langle |\Lambda_U^{(n)}(1)|^s \rangle \sim (n!)^s (2\pi)^{s/2} 2^{-s^2/2 - s(n-1)} \frac{G(n+1/2)}{G(n+1/2+s)} M^{s^2/2 + s(n-1/2)}, \quad (1.1)$$

where $\Lambda_U(e^{i\theta})$ is the characteristic polynomial of U , $G(z)$ is the Barnes double gamma function (see 2.14) and the angle brackets denote an average over the subset of matrices from $SO(N)$ with at least n eigenvalues lying at the point 1 ($N = n + 2M$). We also show that the probability that $|\Lambda_U^{(n)}(1)| < X$ over those $U \in SO(N)$ with n eigenvalues at 1

(again, $N = n + 2M$) is, for small X , given by (see (2.12) and the sentence following)

$$\frac{2}{2n+1} X^{\frac{2n+1}{2}} f(n, M), \quad (1.2)$$

for the function f given by (2.13).

If a discretization of the values of derivatives of elliptic curve L -functions at the critical point were known, this would provide a way to predict the frequency of zeros of various orders occurring at this point. This has been investigated in the case of a single zero at the critical point [13]. In this situation, $n = 1$, the results of [13] give support to the validity of the model presented here.

The result presented in Section 2.1 on the moments of the first derivative of characteristic polynomials from $SO(2N + 1)$ (equation (1.1) with $n = 1$) has already been applied by Delaunay in [15] in order to predict the moments of the orders of Tate-Shafarevich groups and regulators of elliptic curves with odd rank belonging to a family of quadratic twists.

In Section 2.2, we briefly discuss the eigenvalue statistics of random matrices with a certain number of eigenvalues conditioned to lie at one.

We note that while we believe that the model presented in Section 2.1 should apply to the L -functions selected from families of quadratic twists of elliptic curves for the property that they have high order zeros at the critical point, there has been some very interesting theoretical work computing the one- and two-level densities for parametric families of elliptic curves that implies that the zeros of the associated L -functions follow a different model (see [32] and [46]). This does not seem to be a contradiction, as the zero statistics are examined over collections of L -functions selected in very different ways, but it certainly makes the question of the zero statistics of L -functions with high-order zeros at the critical point very intriguing. See Section 3 for further discussion on this issue.

1.2 Random matrix theory and elliptic curves

We review here the results of [10] which apply random matrix theory to predicting the frequency of vanishing at the critical point of the L -functions in the family of elliptic curves described below; or equivalently, assuming the Birch and Swinnerton-Dyer conjecture, the frequency of rank 2 or higher curves occurring in the family of elliptic curves. The motivation for the random matrix calculations presented in Section 2.1 is the expectation that they may be used similarly to examine curves of higher rank.

We consider an L -function (defined by the Dirichlet series and Euler product below when $\text{Re } s > 3/2$)

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \mathcal{L}_p(1/p^s) = \prod_{p|\Delta} (1 - a_p p^{-s})^{-1} \prod_{p \nmid \Delta} (1 - a_p p^{-s} + p^{1-2s})^{-1}, \quad (1.3)$$

that is associated to an elliptic curve E over \mathbb{Q}

$$E : y^2 = x^3 + Ax + B. \quad (1.4)$$

The coefficients a_p , for a generic prime p , are determined by $a_p = p + 1 - \#E(\mathbb{F}_p)$, where $\#E(\mathbb{F}_p)$ counts the number of pairs x, y , with $0 \leq x, y \leq p-1$, such that $y^2 \equiv x^3 + Ax + B \pmod{p}$, plus one for the point at infinity. $\Delta = -16(4A^3 + 27B^2)$ is the discriminant of the cubic $x^3 + Ax + B$. For an extremely clear introduction to elliptic curves in the context discussed here, see the review paper by Rubin and Silverberg [36]. The critical line for the

L -functions defined by (1.3) is at $\text{Re } s = 1$, and the critical point is $s = 1$. With the work of Wiles and others [45, 41, 4] it is known that there is an analytic continuation for these L -functions, so it makes sense to discuss their value at the critical point.

A family of elliptic curves is formed by

$$E_d : dy^2 = x^3 + Ax + B \tag{1.5}$$

for integers d that are fundamental discriminants. (A fundamental discriminant is an integer other than 1 that is not divisible by the square of any prime other than two and that satisfies $d \equiv 1 \pmod{4}$ or $d \equiv 8, 12 \pmod{16}$.) The corresponding family of L -functions, ordered by $|d|$, are

$$L_E(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n^s}, \tag{1.6}$$

where the characters $\chi_d(n) = \left(\frac{d}{n}\right)$ are Kronecker's extension of the Legendre symbol, defined for prime p as

$$\left(\frac{d}{p}\right) = \begin{cases} +1 & \text{if } p \nmid d \text{ and } x^2 \equiv d \pmod{p} \text{ is soluble} \\ 0 & \text{if } p \mid d \\ -1 & \text{if } p \nmid d \text{ and } x^2 \equiv d \pmod{p} \text{ is not soluble} \end{cases}. \tag{1.7}$$

The family of curves E_d is called the family of quadratic twists of E because the characters $\chi_d(n)$ are real, quadratic Dirichlet characters.

Evidence (see for example [37]) points to the zeros near the critical point (where the critical line crosses the real axis) of such a family of L -functions having statistics like the eigenvalues near 1 of matrices from $O(N)$ with Haar measure. To explain this correspondence more clearly, about half of the L -functions in this family are expected to have an even functional equation that relates $L_E(s, \chi_d)$ to $L_E(2-s, \chi_d)$, meaning that the zeros on the critical line ($\text{Re } s = 1$) appear in complex conjugate pairs and if there is a zero at the critical point then it must be of even order, while the other half have an odd functional equation, forcing a zero of order 1,3,5,... at the critical point and all other zeros appearing in complex conjugate pairs. Matrices in $O(N)$ also have eigenvalues that appear in complex conjugate pairs, with the possible exception of unpaired eigenvalues at 1 and -1. The low-lying zeros (that is, those near the critical point) of elliptic curve L -functions from the family above that have even functional equation are expected to show the statistics that eigenvalues of $SO(2N)$ display near the point one. These eigenvalues occur in complex conjugate pairs and an eigenvalue at 1 must have even multiplicity. In contrast, the low-lying zeros of the L -functions above that have an odd functional equation should display the same statistics as $SO(2N+1)$ eigenvalues near one, since in this case there is always an eigenvalue at 1 and it has to have odd multiplicity. (Note that we could just as well have chosen $O^-(2N)$ matrices (orthogonal with determinant -1) to model the L -functions with odd functional equation, since the statistics of these eigenvalues near to 1 are the same as those of $SO(2N+1)$, see for example [24].)

Both numerical and analytical evidence has been given already [10] that random matrix theory can be used to conjecture the frequency of L -functions with even functional equation vanishing at the critical point in a family such as that described above. This is particularly important because of the Birch and Swinnerton-Dyer conjecture which asserts that the order of the zero of an elliptic curve L -function at the critical point is the same as the rank of the elliptic curve itself. The rank is a non-negative integer r that characterizes

the number of rational points on the elliptic curve. The set of pairs $x, y \in \mathbb{Q}$ that satisfy equation (1.4), plus one point at infinity, form a finitely generated commutative group (with a standard law of addition defined on the curve) that is isomorphic to $\mathbb{Z}^r \oplus E(\mathbb{Q})_{tors}$. $E(\mathbb{Q})_{tors}$ is the subgroup of elements of finite order and r is the rank of the curve. The frequency of various ranks in a family such as the one defined at (1.5), or even whether the rank is bounded in such a family of elliptic curves, are important unanswered questions.

In [10] the first step is taken towards using random matrix theory to generate conjectural answers to these questions by addressing the frequency of L -functions from a family of quadratic twists of a given elliptic curve having any zero at all at the critical point. The argument used will be sketched below to illustrate the role played by random matrix theory. The purpose of this paper is to perform the random matrix calculations needed to predict the frequency of a zero of a *given order* at the critical point. Applying the random matrix results from Section 2.1 to the order of vanishing of L -functions involves the knowledge of complicated arithmetic quantities and much more work is needed in this area. Some preliminary numerical investigation has been done [13] for the case of third order vanishing.

We now review the results in [10]. We define the family of L -functions discussed above with even functional equation:

$$\mathcal{F}_{E^+} = \{L_E(s, \chi_d) \text{ having even functional equation}\}. \quad (1.8)$$

The zeros of these L -functions near the critical point, $s = 1$, are expected to show the same statistics as the eigenvalues of matrices from $SO(2N)$, and so the L -values at the critical point are modelled by the characteristic polynomial

$$\Lambda_U(e^{i\theta}) = \prod_{n=1}^N \left(1 - e^{i(\theta_n - \theta)}\right) \left(1 - e^{i(-\theta_n - \theta)}\right), \quad (1.9)$$

evaluated at the point $\theta = 0$. Here $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_N}$ are the eigenvalues of the matrix $U \in SO(2N)$. We discuss the relation between the size of the matrices N and the fundamental discriminants T below.

The moments of $\Lambda_U(1) = \prod_{n=1}^N |1 - e^{i\theta_n}|^2$ are easily calculated using Weyl's formula [44] for Haar measure on the conjugacy classes of $SO(2N)$ and a form of Selberg's integral to be [26], for $\text{Res} > -1/2$,

$$\begin{aligned} & \int_{SO(2N)} \Lambda_U(1)^s dU_{Haar} \\ &= 2^{2Ns} \prod_{j=1}^N \frac{\Gamma(N+j-1)\Gamma(s+j-1/2)}{\Gamma(j-1/2)\Gamma(s+j+N-1)} \\ &\equiv M_O(N, s). \end{aligned} \quad (1.10)$$

The L -function moments are then conjectured to have the form [7, 26]

$$\begin{aligned} M_E(T, s) &\equiv \frac{1}{T^*} \sum_{\substack{|d| \leq T \\ L_E(s, \chi_d) \in \mathcal{F}_{E^+}}} L_E(1, \chi_d)^s \\ &\sim a_s(E) M_O(N, s) \end{aligned} \quad (1.11)$$

for large T . Here $N = \log T$. The rationale behind this comes from equating the density of zeros near the critical point with the density of the matrix eigenvalues, as first suggested in [27]; all subsequent work has shown that this is the correct equivalence to obtain L -function moments and moments of characteristic polynomials of comparable sizes. In (1.11) the sum is over fundamental discriminants d , T^* is the number of terms in the sum and $a_s(E)$ is an Euler product that contains arithmetic information specific to the elliptic curve E and the family of L -functions being averaged over. In practice, it is often a subset of \mathcal{F}_{E^+} that is summed over. If, for example, we select those L -functions $L_E(s, \chi_d)$ in \mathcal{F}_{E^+} with $d > 0$ and further restricted by a condition on $d \pmod{Q}$, if Q is odd, and on $d \pmod{4Q}$, if Q is even, then the arithmetic factor would be

$$a_s(E) = \prod_{p \nmid Q} (1 - p^{-1})^{s(s-1)/2} \left(\frac{p}{p+1} \right) \left(\frac{1}{p} + \frac{1}{2} (\mathcal{L}_p(1/p)^s + \mathcal{L}_p(-1/p)^s) \right) \\ \times \prod_{p \mid Q} (1 - p^{-1})^{s(s-1)/2} \mathcal{L}_p(a_p/p)^s. \quad (1.12)$$

See [11] and [9], Section 4.4, for more examples.

Next we consider the distribution of the values of the characteristic polynomials of $SO(2N)$ matrices at the point 1. If $P_O(N, x)dx$ is the probability that the characteristic polynomial of a matrix chosen from $SO(2N)$ with Haar measure has a value between x and $x + dx$, then

$$P_O(N, x) = \frac{1}{2\pi ix} \int_{(c)} M_O(N, s) x^{-s} ds \\ \sim x^{-1/2} h(N) \quad (1.13)$$

for $x \rightarrow 0^+$, since for small x the behaviour is dominated by the pole of $M_O(N, s)$ at $s = -1/2$. Here (c) denotes the path of integration along the vertical line from $c - i\infty$ to $c + i\infty$ and $h(N)$ is determined by a residue calculation.

For large N , $h(N) \sim 2^{-7/8} G(1/2) \pi^{-1/4} N^{3/8}$ (G is the Barnes double gamma function, defined as [1, 42]:

$$G(1+z) = (2\pi)^{z/2} e^{-[(1+\gamma)z^2+z]/2} \prod_{n=1}^{\infty} \left[(1+z/n)^n e^{-z+z^2/(2n)} \right], \quad (1.14)$$

where γ is Euler's constant. See also (2.14) for more properties of this function.) Since the probability that an element of $SO(2N)$ has a characteristic polynomial whose value at 1 is X or smaller is $\int_0^X P_O(N, x)dx \sim 2X^{1/2}h(N)$, we find that the the behaviour of this probability for small X and large N is

$$\lim_{N \rightarrow \infty} N^{-3/8} \lim_{X \rightarrow 0^+} (X^{-1/2} \int_0^X P_O(N, x)dx) = 2^{1/8} G(1/2) \pi^{-1/4}. \quad (1.15)$$

We see from equation (1.11) that for large d , moments of L -functions are conjectured to be just $a_s(E)$ (the prime product) times the random matrix moment $M_O(N, s)$. (See [20] for further evidence that at leading order number theoretic moments take this form of an arithmetic factor multiplied by a random matrix factor.) If this is true, then we define

$P_E(T, x)dx$ as the probability, amongst members of \mathcal{F}_{E^+} , that $L_E(1, \chi_d)$, for $|d|$ around $T = e^N$, will take a value between x and $x + dx$, giving

$$P_E(T, x) = \frac{1}{2\pi i x} \int_{(c)} M_E(T, s) x^{-s} ds, \quad (1.16)$$

and an approximation for this probability for small x should be

$$P_E(T, x) \sim a_{-1/2}(E) x^{-1/2} h(N). \quad (1.17)$$

Here equating densities of zeros means $N \sim \log T$.

So, roughly, random matrix theory predicts that the probability that an L -function $L_E(s, \chi_d) \in \mathcal{F}_{E^+}$, with $|d|$ close to t , has a value at $s = 1$ that is X or smaller is

$$\sim a_{-1/2}(E) X^{1/2} 2^{1/8} G(1/2) \pi^{-1/4} \log^{3/8} t. \quad (1.18)$$

But these L -functions are constrained to take only certain discretized values at the critical point $s = 1$. The L -values have the form [43, 39, 29]:

$$L_E(1, \chi_d) = \kappa_E \frac{c_E(|d|)^2}{\sqrt{|d|}}, \quad (1.19)$$

where the $c_E(|d|)$ are integers, the Fourier coefficients of a half-integral weight form.

So, if

$$L_E(1, \chi_d) < \frac{\kappa_E}{\sqrt{|d|}} \quad (1.20)$$

then

$$L_E(1, \chi_d) = 0 \quad (1.21)$$

and the first thought would be to take $X = \kappa_E/\sqrt{t}$ in (1.18) and then to integrate t from 0 to T . However, there is arithmetical information encoded in the $c_E(|d|)$ (one is not always the smallest allowed non-zero value) and so to avoid this problem, the conjecture stated in [10] is restricted to prime discriminants.

Conjecture 1.1 (*Conrey, Keating, Rubinstein, Snaith*):

Let E be an elliptic curve defined over \mathbb{Q} . Then there is a constant $c_E \geq 0$ such that

$$\sum_{\substack{p \leq T \\ L_E(1, \chi_p) = 0 \\ L_E(s, \chi_p) \in \mathcal{F}_{E^+}}} 1 \sim c_E T^{3/4} (\log T)^{-5/8}$$

(The conjecture was originally stated in [10] with $c_E > 0$, but in fact numerics have revealed that the constant can be zero [11] for certain families. An explanation of such a case with $c_E = 0$ was given by Delaunay [16].)

With the Birch and Swinnerton-Dyer conjecture, this suggests that out of all the elliptic curves associated with L -functions in \mathcal{F}_{E^+} with prime discriminant $p \leq T$ (there are of order $T/\log T$ of them), a number of order $T^{3/4}(\log T)^{-5/8}$ of them should have rank two or greater. The $T^{3/4}$ has been predicted previously by Sarnak using different arguments,

but random matrix theory adds more detailed information in the form of the power on the logarithm. For numerical evidence supporting the conjecture, see [10] and [11].

In the next section we will calculate the distribution of values equivalent to (1.13) that is required to probe questions of elliptic curves of a given rank occurring in families of quadratic twists (equation (1.5)).

2 Random matrix calculations

In this section we will calculate the probability of the n th derivative of a characteristic polynomial (2.1) of a random $SO(N)$ matrix taking a value less than X at $\theta = 0$ when all lower derivatives are constrained to be zero at this point. That is, we perform the random matrix average over the (measure zero) subset of $SO(N)$ of matrices with n eigenvalues at the point 1 on the unit circle, and $2M$ other eigenvalues occurring in complex conjugate pairs ($N = n + 2M$).

In Section 2.2 will also illustrate the n -level density of the $2M$ eigenvalues *not* constrained to lie at 1.

2.1 Value distribution of higher derivatives

For a matrix $U \in SO(N)$ with n eigenvalues equal to 1 ($N = n + 2M$), the characteristic polynomial is

$$\Lambda_U(e^{i\theta}) = (1 - e^{-i\theta})^n \prod_{j=1}^M (1 - e^{i(\theta_j - \theta)})(1 - e^{i(-\theta_j - \theta)}). \quad (2.1)$$

We consider the n th derivative (where α replaces $i\theta$)

$$\begin{aligned} \frac{d^n}{d\alpha^n} (1 - e^{-\alpha})^n \prod_{j=1}^M (1 - e^{i\theta_j - \alpha})(1 - e^{-i\theta_j - \alpha}) \Big|_{\alpha=0} \\ = \left[\frac{d^n}{d\alpha^n} (1 - e^{-\alpha})^n \right] \prod_{j=1}^M (1 - e^{i\theta_j - \alpha})(1 - e^{-i\theta_j - \alpha}) \Big|_{\alpha=0} \end{aligned}$$

and use this to define

$$\Lambda_U^{(n)}(1) = n! \prod_{j=1}^M (1 - e^{i\theta_j})(1 - e^{-i\theta_j}) = n! 2^M \prod_{j=1}^M (1 - \cos \theta_j). \quad (2.2)$$

The matrices with at least n eigenvalues equal to 1 form a set of measure zero within $SO(N)$ with Haar measure, for $n > 1$. For even $N = 2P$, Haar measure on the conjugacy classes of $SO(2P)$ is given by [44]

$$C' \prod_{1 \leq j < k \leq P} (\cos \theta_k - \cos \theta_j)^2 d\theta_1 \cdots d\theta_P, \quad (2.3)$$

with C' a normalization constant, and we define the probability of having $2q$ eigenvalues at

1 and the rest within infinitesimal neighbourhoods of $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_{P-q}}$ as

$$\begin{aligned} & \lim_{\alpha \rightarrow 0} \frac{\left(\int_0^\alpha \cdots \int_0^\alpha \prod_{1 \leq j < k \leq P} (\cos \theta_k - \cos \theta_j)^2 d\theta_{P-q+1} \cdots d\theta_P \right) d\theta_1 \cdots d\theta_{P-q}}{\int_0^\pi \cdots \int_0^\pi \int_0^\alpha \cdots \int_0^\alpha \prod_{1 \leq j < k \leq P} (\cos \theta_k - \cos \theta_j)^2 d\theta_{P-q+1} \cdots d\theta_P d\theta_1 \cdots d\theta_{P-q}} \\ &= C(P, q) \prod_{j=1}^{P-q} (1 - \cos \theta_j)^{2q} \prod_{1 \leq j < k \leq P-q} (\cos \theta_j - \cos \theta_k)^2 d\theta_1 \cdots d\theta_{P-q}. \end{aligned} \quad (2.4)$$

Here $C(P, q)$ is the normalization constant.

The same procedure for N odd, starting from the measure on $SO(2P + 1)$

$$C'' \prod_{j=1}^P (1 - \cos \theta_j) \prod_{1 \leq j < k \leq P} (\cos \theta_k - \cos \theta_j)^2 d\theta_1 \cdots d\theta_P, \quad (2.5)$$

with normalization constant C'' , leads to a general expression for the measure on the matrices in $SO(N)$ (where now N can be even or odd) with n eigenvalues equal to 1 ($N = n + 2M$):

$$C(M, n) \prod_{j=1}^M (1 - \cos \theta_j)^n \prod_{1 \leq j < k \leq M} (\cos \theta_j - \cos \theta_k)^2. \quad (2.6)$$

Thus the quantity to calculate (the s th moment of the n th derivative of the characteristic polynomial averaged over the subset of matrices with n eigenvalues conditioned to lie at 1) is

$$\begin{aligned} & \mathcal{M}(n, M, s) := \\ & C(M, n) \int_0^\pi \cdots \int_0^\pi |\Lambda_U^{(n)}(1)|^s \prod_{j=1}^M (1 - \cos \theta_j)^n \prod_{1 \leq j < k \leq M} (\cos \theta_j - \cos \theta_k)^2 d\theta_1 \cdots d\theta_M \\ &= C(M, n) \int_0^\pi \cdots \int_0^\pi (n!)^s 2^{Ms} \prod_{j=1}^M (1 - \cos \theta_j)^{n+s} \prod_{1 \leq j < k \leq M} (\cos \theta_j - \cos \theta_k)^2 d\theta_1 \cdots d\theta_M \\ &= C(M, n) (n!)^s 2^{Ms} \int_{-1}^1 \cdots \int_{-1}^1 \prod_{j=1}^M \frac{(1 - x_j)^{n-1/2+s}}{(1 + x_j)^{1/2}} \prod_{1 \leq j < k \leq M} (x_j - x_k)^2 dx_1 \cdots dx_M. \end{aligned} \quad (2.7)$$

This can be evaluated using a form of Selberg's integral (for details see [30]):

$$\begin{aligned} & \int_{-1}^1 \cdots \int_{-1}^1 \prod_{1 \leq j < l \leq K} |(x_j - x_l)|^{2\gamma} \prod_{j=1}^K (1 - x_j)^{\alpha-1} (1 + x_j)^{\beta-1} dx_j \\ &= 2^{\gamma K(K-1) + K(\alpha + \beta - 1)} \prod_{j=0}^{K-1} \frac{\Gamma(1 + \gamma + j\gamma) \Gamma(\alpha + j\gamma) \Gamma(\beta + j\gamma)}{\Gamma(1 + \gamma) \Gamma(\alpha + \beta + \gamma(K + j - 1))}, \end{aligned} \quad (2.8)$$

if $\text{Re} \alpha > 0$, $\text{Re} \beta > 0$ and $\text{Re} \gamma > -\min\left(\frac{1}{K}, \frac{\text{Re} \alpha}{K-1}, \frac{\text{Re} \beta}{K-1}\right)$.

We have $\gamma = 1$, $\alpha = n + 1/2 + s$ and $\beta = 1/2$, so the integral in (2.7) is

$$\begin{aligned} & \mathcal{M}(n, M, s) = C(M, n) (n!)^s 2^{Ms + M(M-1) + Mn} \\ & \quad \times \prod_{j=1}^M \frac{\Gamma(j+1) \Gamma(n - 1/2 + s + j) \Gamma(j - 1/2)}{\Gamma(n + M + s + j - 1)}. \end{aligned} \quad (2.9)$$

But the normalization constant can also be evaluated by Selberg's integral to be $C(M, n) = \left(2^{M(M-1)+Mn} \prod_{j=1}^M \frac{\Gamma(j+1)\Gamma(n-1/2+j)\Gamma(j-1/2)}{\Gamma(n+M+j-1)}\right)^{-1}$, giving us

$$\mathcal{M}(n, M, s) = (n!)^s 2^{2Ms} \prod_{j=1}^M \frac{\Gamma(n-1/2+s+j)\Gamma(n+M+j-1)}{\Gamma(n-1/2+j)\Gamma(n+s+M+j-1)}. \quad (2.10)$$

Consider n fixed and finite and M large. If the probability that $|\Lambda_U^{(n)}(1)|$ takes a value between x and $x+dx$ is given by $P(n, M, x)dx$ then from a standard result in probability (see, for example, chapter XV of [18]: $\phi(s) := \mathcal{M}(n, M, is)$ is essentially the characteristic function of the logarithm of the random matrix characteristic polynomial)

$$\begin{aligned} P(n, M, x) &= \frac{1}{2\pi ix} \int_{(c)} x^{-s} \mathcal{M}(n, M, s) ds \\ &= \frac{1}{2\pi ix} \int_{(c)} x^{-s} (n!)^s 2^{2Ms} \prod_{j=1}^M \frac{\Gamma(n-1/2+s+j)\Gamma(M+n+j-1)}{\Gamma(n-1/2+j)\Gamma(n+s+M+j-1)} ds. \end{aligned} \quad (2.11)$$

Here (c) denotes the path of integration along the vertical line from $c-i\infty$ to $c+i\infty$.

We are interested in the behaviour at small x , and this is dominated by the nearest pole to zero of the integrand: the pole at $s = -(n+1/2)$ of $\Gamma(n-1/2+s+j)$. Thus for small x

$$P(n, M, x) \sim x^{n-1/2} f(n, M), \quad (2.12)$$

where $f(n, M)$ is determined from (2.11) by a residue calculation.

The probability that $|\Lambda_U^{(n)}(1)| < X$ over those $U \in SO(N)$ with n eigenvalues at 1 is therefore $\sim \frac{2}{2n+1} X^{\frac{2n+1}{2}} f(n, M)$ for small X and the dependency on n and M is encoded in $f(n, M)$:

$$f(n, M) = (n!)^{\frac{-2n-1}{2}} 2^{-2M(\frac{2n+1}{2})} \frac{1}{\Gamma(M)} \prod_{j=1}^M \frac{\Gamma(j)\Gamma(M+n+j-1)}{\Gamma(n-1/2+j)\Gamma(M+j-3/2)}. \quad (2.13)$$

We can discover the behaviour of $f(n, M)$ for large M by introducing the Barnes G-function [1, 42]:

$$G(1+z) = (2\pi)^{z/2} e^{-[(1+\gamma)z^2+z]/2} \prod_{n=1}^{\infty} \left[(1+z/n)^n e^{-z+z^2/(2n)} \right], \quad (2.14)$$

which has zeros at the negative integers, $-n$, with multiplicity n ($n = 1, 2, 3, \dots$). Other properties useful to us are that

$$\begin{aligned} G(1) &= 1, \\ G(z+1) &= \Gamma(z) G(z), \end{aligned} \quad (2.15)$$

and furthermore, for large $|z|$

$$\log G(z+1) \sim z^2 \left(\frac{1}{2} \log z - \frac{3}{4} \right) + \frac{1}{2} z \log(2\pi) - \frac{1}{12} \log z + \zeta'(-1) + O\left(\frac{1}{z}\right). \quad (2.16)$$

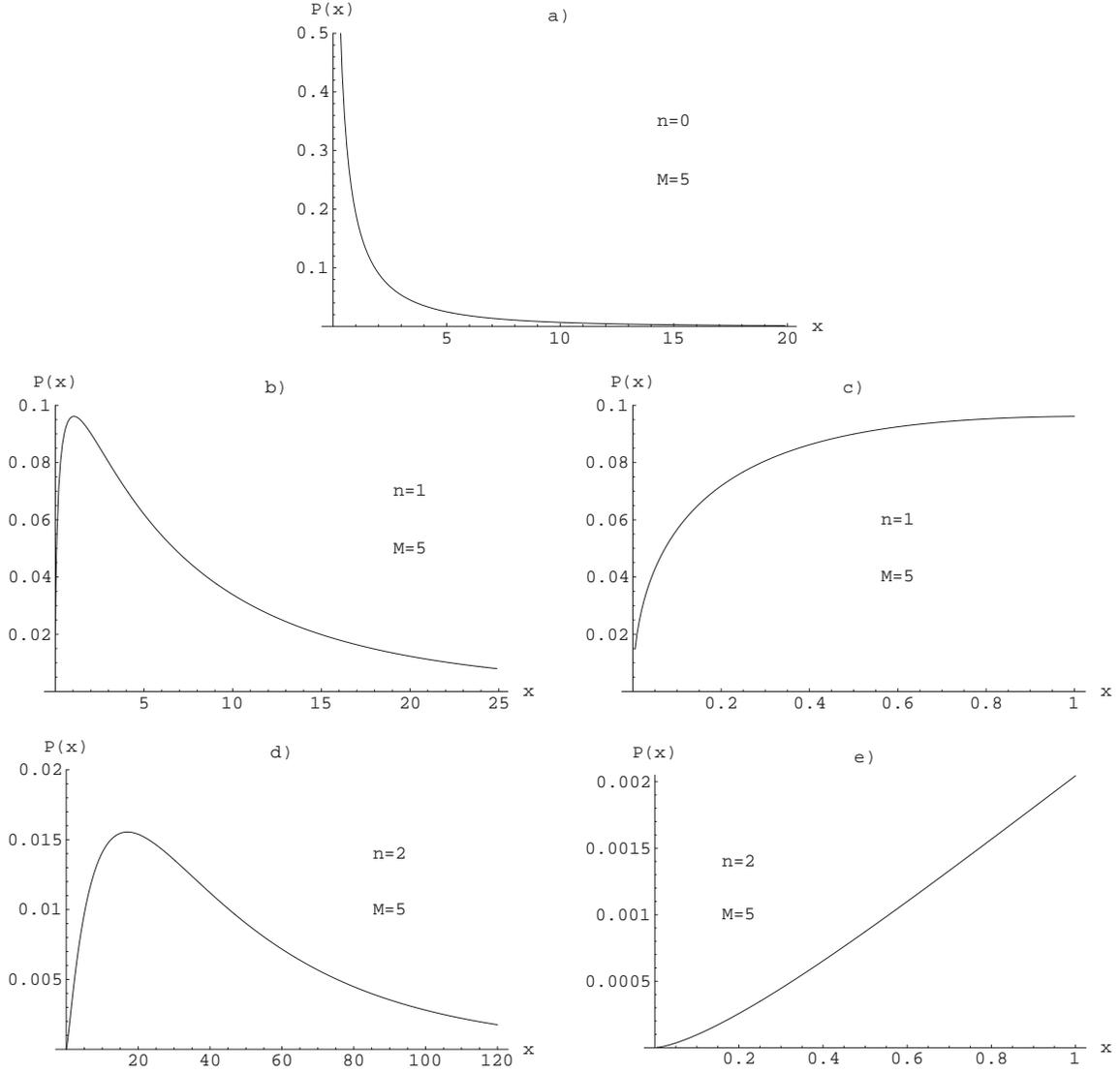


Figure 1: Distribution of values ($P(n, M, x)$ from (2.11)) at the point 1 of the n th derivative of the characteristic polynomial of the subset of matrices from $SO(n + 2M)$ that are conditioned to have at least n eigenvalues at 1. In figure a) $n = 0$, in b) and c) $n = 1$ and in d) and e) $n = 2$. In all cases $M = 5$. Figures c) and e) show in more detail the behaviour at the origin of figures b) and d) (see equation (2.12)).

Thus

$$\prod_{j=1}^{\ell} \Gamma(j+r) = \frac{G(\ell+r+1)}{G(1+r)}. \quad (2.17)$$

So we can write

$$f(n, M) = (n!)^{\frac{-2n-1}{2}} 2^{-2M(\frac{2n+1}{2})} \frac{G(M)G(2M+n)G(n+1/2)G(M-1/2)}{G(M+n)G(n+M+1/2)G(2M-1/2)} \quad (2.18)$$

and expanding the G -functions for large M gives

$$f(n, M) \sim (n!)^{\frac{-2n-1}{2}} G(n+1/2) M^{-\frac{n^2}{2} + \frac{n}{2} + \frac{3}{8}} 2^{\frac{n^2}{2} - \frac{3n}{2} - \frac{7}{8}} \pi^{-\frac{n}{2} - \frac{1}{4}}. \quad (2.19)$$

Note that we can also use (2.16) to revisit $\mathcal{M}(n, M, s)$ (defined at 2.7) and examine that asymptotically for large M . This gives us the large M behaviour of the moments at the point 1 of the n th derivative of characteristic polynomials for which all lower derivatives vanish. We have

$$\begin{aligned} \mathcal{M}(n, M, s) &= (n!)^s 2^{2Ms} \frac{G(n+1/2+s+M)}{G(n+1/2+s)} \frac{G(n+1/2)}{G(n+1/2+M)} \frac{G(n+2M)}{G(n+M)} \frac{G(n+s+M)}{G(n+s+2M)} \\ &\sim (n!)^s (2\pi)^{s/2} 2^{-s^2/2-s(n-1)} \frac{G(n+1/2)}{G(n+1/2+s)} M^{s^2/2+s(n-1/2)}. \end{aligned} \quad (2.20)$$

2.2 Eigenvalue statistics

To get an idea of what the eigenvalue statistics of the remaining eigenvalues look like when we condition at least n eigenvalues to lie at 1 in the manner described in (2.4), we briefly review the m -level densities of the unconditioned eigenvalues. These calculations are not new; these same matrix ensembles were studied by Nagao and Wadati [34] and also Dueñez [17] and are discussed in the book by Forrester [19]. Concurrently with the present work, this matrix model has also been discussed in connection with the statistics of the zeros of elliptic curve L -functions by Miller [31] (with appendix by Duenez). However, we include a brief discussion of the statistics here to illustrate the effect that forced eigenvalues at 1 have on the remaining eigenvalues.

The terminology for these statistics varies (for example, Mehta [30] refers to them as m -point correlation functions) so we will write (with C the normalization constant)

$$\begin{aligned} &C \int_0^\pi \cdots \int_0^\pi \sum_{1 \leq j_1 < \cdots < j_m \leq N} f(\theta_{j_1}, \dots, \theta_{j_m}) (1 - \cos \theta_j)^n \\ &\quad \times \prod_{1 \leq j < k \leq M} (\cos \theta_j - \cos \theta_k)^2 d\theta_1 \cdots d\theta_M \\ &= \frac{1}{m!} \int_0^\pi \cdots \int_0^\pi f(\theta_1, \dots, \theta_m) \\ &\quad \times \det(K_M^{(n-1/2, -1/2)}(\theta_j, \theta_k))_{j,k=1, \dots, m} \prod_{j=1}^m (1 - \cos \theta_j)^n d\theta_1 \cdots d\theta_m \\ &= \frac{1}{m!} \int_0^\pi \cdots \int_0^\pi f(\theta_1, \dots, \theta_m) \\ &\quad \times \det(2^n \sin^n \frac{\theta_j}{2} \sin^n \frac{\theta_k}{2} K_M^{(n-1/2, -1/2)}(\theta_j, \theta_k))_{j,k=1, \dots, m} d\theta_1 \cdots d\theta_m \end{aligned} \quad (2.21)$$

and refer to $\det(2^n \sin^n \frac{\theta_j}{2} \sin^n \frac{\theta_k}{2} K_M^{(n-1/2, -1/2)}(\theta_j, \theta_k))_{j,k=1, \dots, m}$ as the m -level density (see [6] for further discussion of these densities). By standard techniques (see for example chapter 5 of [30] or chapter 4 of [40])

$$\begin{aligned}
& K_M^{(n-1/2, -1/2)}(\theta_j, \theta_k)_{j,k=1, \dots, m} \\
&= \sum_{j=0}^M (h_j^{(n-1/2, -1/2)})^{-1} P_j^{(n-1/2, -1/2)}(\cos \theta_j) P_j^{(n-1/2, -1/2)}(\cos \theta_k) \\
&= \frac{2^{-n+1}}{2M+n+1} \frac{\Gamma(M+2)\Gamma(M+n+1)}{\Gamma(M+n+1/2)\Gamma(M+1/2)} \\
&\times \left(\frac{P_{M+1}^{(n-1/2, -1/2)}(\cos \theta_j) P_M^{(n-1/2, -1/2)}(\cos \theta_k) - P_M^{(n-1/2, -1/2)}(\cos \theta_j) P_{M+1}^{(n-1/2, -1/2)}(\cos \theta_k)}{\cos \theta_j - \cos \theta_k} \right)
\end{aligned} \tag{2.22}$$

is defined in terms of the Jacobi polynomials $P_j^{(\alpha, \beta)}(\cos \theta)$ (described in detail in [40]) with the orthogonality condition

$$\int_0^\pi P_j^{(\alpha, \beta)}(\cos \theta) P_k^{(\alpha, \beta)}(\cos \theta) (1 - \cos \theta)^n d\theta = \delta_{jk} h_m^{(\alpha, \beta)}, \tag{2.23}$$

where

$$h_j^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1}}{2j+\alpha+\beta+1} \frac{\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{\Gamma(j+1)\Gamma(j+\alpha+\beta+1)}. \tag{2.24}$$

Scaling the eigenvalues and using asymptotic formulae for the Jacobi polynomials (see [40], page 197) we find that in the large-matrix limit (where $J_\alpha(x)$ are Bessel functions of the first kind)

$$\begin{aligned}
L^{(n-1/2, -1/2)}(\theta, \phi) &:= \lim_{M \rightarrow \infty} \frac{\pi}{M} K_M^{(n-1/2, -1/2)}\left(\frac{\pi\theta}{M}, \frac{\pi\phi}{M}\right) 2^n \sin^n\left(\frac{\pi\theta}{2M}\right) \sin^n\left(\frac{\pi\phi}{2M}\right) \\
&= (\pi\theta)^{1/2} (\pi\phi)^{1/2} \frac{\theta J_{n-3/2}(\theta\pi) J_{n-1/2}(\phi\pi) - \phi J_{n-3/2}(\phi\pi) J_{n-1/2}(\theta\pi)}{\phi^2 - \theta^2}.
\end{aligned} \tag{2.25}$$

The scaled 1-level density is therefore

$$L^{(n-1/2, -1/2)}(\theta, \theta) = \frac{\pi^2}{2} \theta (J_{n-3/2}^2(\theta\pi) + J_{n-1/2}^2(\theta\pi) - \frac{2n-1}{\theta\pi} J_{n-1/2}(\theta\pi) J_{n-3/2}(\theta\pi)), \tag{2.26}$$

and can be seen to behave like a constant times θ^{2n} for small θ , where we remember that n is the number of degenerate eigenvalues. This increasing repulsion of the first eigenvalue not located at 1 can be seen in Figure 2.

3 Discussion

We have shown that the probability that $|\Lambda_U^{(n)}(1)| < X$ over those $U \in SO(N)$ with n eigenvalues at 1 ($N = n + 2M$) is, for small X , given by

$$\frac{2}{2n+1} X^{\frac{2n+1}{2}} f(n, M) \tag{3.1}$$

and the mean value of the n th derivative grows like

$$\mathcal{M}(n, M, s) := \langle |\Lambda_U^{(n)}(1)|^s \rangle \sim (n!)^s (2\pi)^s 2^{-s^2/2-s(n-1)} \frac{G(n+1/2)}{G(n+1/2+s)} M^{s^2/2+s(n-1/2)}, \tag{3.2}$$

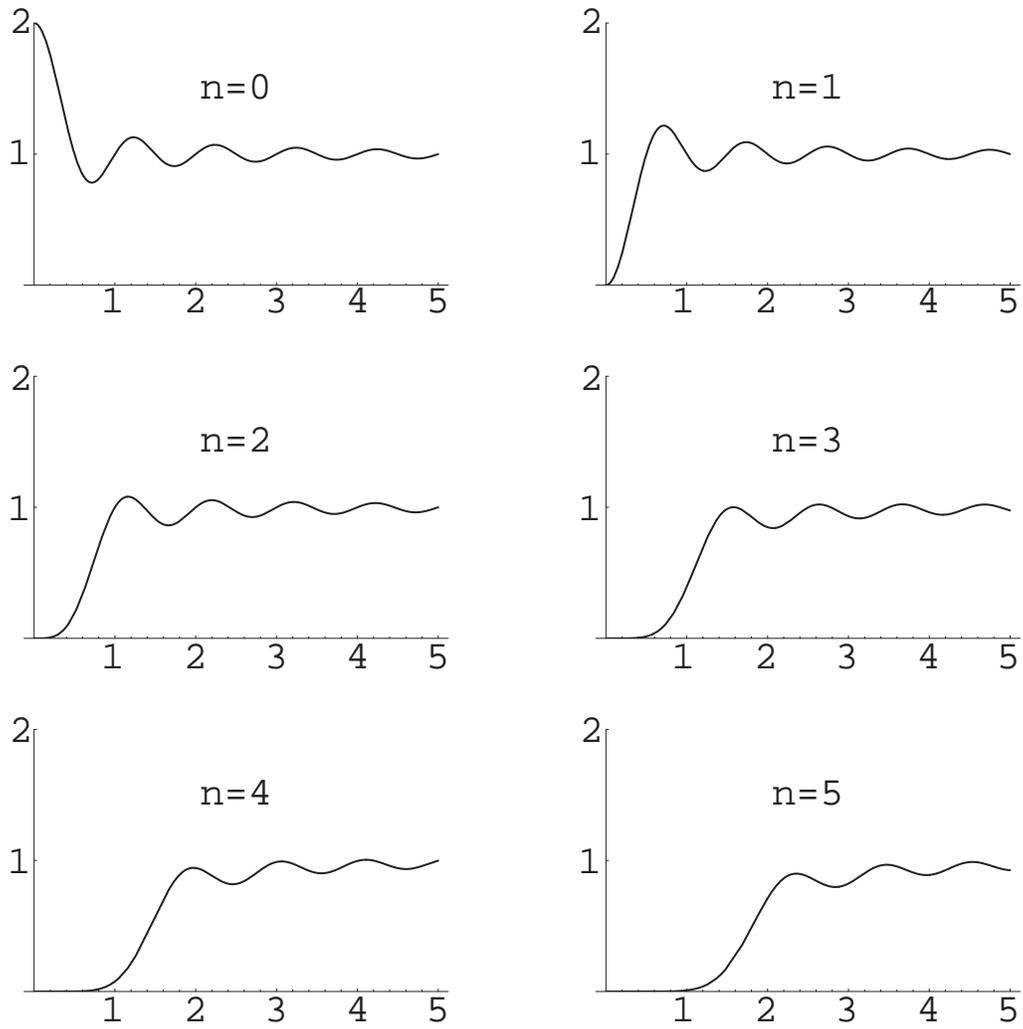


Figure 2: Scaled 1-level densities, from equation (2.26) for various values of n , the degeneracy of the eigenvalue at one.

where the angle brackets denote an average over the set of matrices from $SO(N)$ with n eigenvalues lying at the point 1 ($N = n + 2M$).

The result (3.2), see also (2.20), was applied by Delaunay in [15] to predict moments of the orders of Tate-Shafarevich groups and the regulators of elliptic curves belonging to a family of quadratic twists, see (1.5). The Birch and Swinnerton-Dyer conjecture provides a formula for the first non-zero derivative of an L -function in terms of various quantities related to the associated elliptic curve. One of these quantities is the order of the Tate-Shafarevich group, and another, in the case of the first derivative of L -functions with odd functional equation, is the regulator. For families of L -functions with even functional equation the result (2.20) is used with $n = 0$ by Delaunay to predict the asymptotic form of moments of the order of the Tate-Shafarevich group for the associated family of elliptic curves, and for families with odd functional equation the case $n = 1$ was used to conjecture the form of moments of the regulator.

To predict, for example, the number of L -functions in \mathcal{F}_{E^-} (the family defined like \mathcal{F}_{E^+} in (1.8) but with odd functional equation) that have a zero of order at least three at the critical point we need the result (3.1) for $n = 1$ plus some information analogous to (1.20) giving the smallest non-zero value that can be taken by the derivative of an L -function from this family. A formula like (1.20) is not known at the moment for derivatives of L -functions, but numerics suggest [13] that there is a gap between zero and the smallest non-zero value of L' , which allows random matrix theory to be applied to the question of the number of L -functions with a zero of order at least three at the critical point. Further work in this area is ongoing, but it is also worth noting that the numerics in [13] support the random matrix model presented here, as the numerical cumulative distribution of values of L' for L -functions in \mathcal{F}_{E^-} shows the expected $x^{3/2}$ behaviour near $x = 0$, agreeing with the exponent $3/2$ in (3.1) when $n = 1$.

This leads us to believe that the measure in (2.6) should correctly model the zeros near the point 1 of L -functions with at least an n th order zero at 1 selected from the family \mathcal{F}_{E^-} or \mathcal{F}_{E^+} (depending on whether n is odd or even, respectively), but we note again that this is not the only model for L -functions with n zeros at 1. The measure presented here is what Miller [31] calls the Interaction Model, and he contrasts this with the Independent Model where the n eigenvalues at 1 have no effect on the statistics of the remaining eigenvalues. This gives, for example for matrices from $SO(2M + r)$ with r eigenvalues at 1, the set of matrices

$$\left\{ \begin{pmatrix} I_{r \times r} & \\ & g \end{pmatrix} : g \in SO(2M) \right\},$$

with a $r \times r$ identity block in the upper left corner and with the joint probability density on the remaining eigenvalues being Haar measure on $SO(2M)$, that is (up to a normalization constant)

$$\prod_{1 \leq j < k \leq M} (\cos \theta_k - \cos \theta_j)^2 \prod_{1 \leq j \leq M} d\theta_j.$$

Miller finds numerical evidence [31], and for theoretical results see [32] and [46], that the Independent Model agrees with the zero statistics of L -functions of a parametric family of elliptic curves constructed so that each member of the family has rank r . Through the Birch and Swinnerton-Dyer conjecture, the associated L -functions are expected to have a zero of order r at the point 1, and Miller finds evidence for this high-order zero, as well as for the fact that its presence does not affect the statistics of the other zeros in the way

that the model presented here would (for example as illustrated in Figure 2), but rather the zeros follow Miller’s Independent Model. This is not a contradiction to the proposal that the model presented here (the same as Miller’s Interaction Model) is correct for the family of quadratic twists of elliptic curve L -functions. It will be the subject of further work to produce convincing numerics for the zero statistics of the elliptic curve L -function families discussed in this paper, although this is a difficult task due to the small number of curves in these families with rank greater than or equal to two.

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