GENERALISATIONS OF STATIONARITY, CLOSED AND UNBOUNDEDNESS, AND OF JENSEN'S $\Box$

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Abstract. The concepts of closed unbounded (club) and stationary sets are generalised to γ-club and γ-stationary sets, which are closely related to stationary reflection principles. We use these notions to define generalisations of Jensen's combinatorial principles $\Box$ as $\Box^{\gamma}$ and $\Box^{<\gamma}$ sequences. We define Π^1_γ-indescribability and and show first that in L if $\gamma < \kappa$ is an ordinal and κ is Σ^1_γ-indescribable but not Π^1_γ-indescribable, and $A \subseteq \kappa$ is γ-stationary, then there is $E_A \subseteq A$ and a $\Box^{<\gamma}$ sequence S on κ such that E_A is γ-stationary in κ and S avoids E_A. This generalises a result of Jensen for $\gamma = 1$. As a corollary we also extend the result of Jensen that in L a regular cardinal is stationary reflecting if and only if it is Π^1_1-indescribable by showing that such a κ as above is not γ-reflecting, yielding a different proof of a result appearing in [3]. Thus in L a cardinal is Π^1_γ-indescribable iff it reflects γ-stationary sets. We define $\Box^{\gamma}(\kappa)$, as stating that there is an *unthreadable* $\Box^{\gamma}$-sequence at κ; we show this implies that κ is not $\gamma+1$-reflecting. Certain assumptions on the γ-club filter allow us to prove that γ-stationarity is downwards absolute to L, and allows for splitting of γ-stationary sets. We define γ-ineffability, and look into the relation between γ-ineffability and various $\diamondsuit^{\gamma}$ principles.

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§**1. Introduction.** This paper is inspired by [2] where the authors introduced generalisations of stationarity. Essentially the idea is to iterate *stationary reflection*. A cardinal κ is called *stationary reflecting*, if for every stationary $S \subseteq \kappa$ there is an $\alpha < \kappa$ with $S \cap \alpha$ is stationary. The authors built simultaneously hierarchies of increasing strength of stationarity, and of stationary reflection at a cardinal κ. Such ideas have been studied over a long period (see *e.g.* [24] and [25]).

One conceptual difference here with the presentation in [2] is the introduction of γ-club sets, which are interesting combinatorial objects in their own right and also enable us to lift some arguments straight from the club and stationary set context. As combinatorial objects, γ-clubs allow us to generalise the combinatorial notion of a $\Box$ sequence, which is one of the main results of this paper.

Parts of the extant literature have close connections with γ-stationary and γ-club sets - for example in [8], Ben-Neria uses a "strong simultaneous reflection" property, this is stated as the aim of his Theorem 19, although the definition is only implicit in the proof. Let κ be an inaccessible cardinal and $\langle S_\alpha \mid \alpha < \kappa\rangle$ any sequence of sets stationary in κ. Strong simultaneous reflection requires that there is $\gamma < \kappa$ so that $\langle S_\alpha \cap \gamma \mid \alpha < \gamma\rangle$ are all stationary in γ. It is an exercise in the definitions to see that this principle is equivalent to the normality of our 1-club filter $\mathcal{C}^1(\kappa)$ (for which see Def. 2.18 below).

The notions of γ-club and γ-stationary sets $S \subseteq \kappa$ where we adopt the convention that κ will be a ordinal greater than γ, are natural, even without the study of stationary reflection. One way to see this is to think about how we might want to measure the *size* or *thickness* of subsets of a

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particular ordinal - not in terms of cardinality but rather in a way that takes into account more of the combinatorial aspects of ordinals. Starting with a stationary set $S \subseteq \alpha$, we can "thicken" this set by requiring that S is *stationary closed*, *i.e.* whenever S is stationary below any $\beta < \alpha$ then we have $\beta \in S$. We shall call such sets 1-*club*. Exactly this notion was defined, unbeknownst to us, much earlier by Sun in [29]. There Sun further defined κ to be a *stationary cardinal* if the 1-club filter was non-trivial and normal. It then follows from Sun, (*op.cit.*) and Meckler-Shelah ([25]) that the consistency strength of a stationary cardinal is strictly intermediate between that of a greatly Mahlo and a weakly compact cardinal.

Sun did not define n-clubs for $n > 1$, but a related notion called n-club is given by Hellsten in [15], and very recently explored further in [12]. (Hellsten's notion of, *e.g.*, 1-club is slightly different, in that it requires closure of sets at points that are strongly inaccessible, whilst Sun's and ours require this only for weakly inaccessibles. Hellsten's definitions of $n+1$-clubs build in closure of points α of a set X where $X \cap \alpha$ is Π^1_n -indescribable. Ours do not. Our philosophy here is that *simultaneous reflection* is the guiding concept, rather than *indescribability*. Further work in this area can be cited: see [16], [17], [11] and [9]. Such 1-club sets need not be club (a set can be unbounded without being stationary), so in terms of size they come between stationary and club sets. The family of 1-club sets share something of the structure of the club sets as they are defined by simply replacing "closure" with "stationary-closure", and the "unbounded" with "stationary" in the definition, but do the 1-club sets generate a normal filter? They do at Π^1_1-indescribables see Lemma 2.21 and at such indescribables the filter will equal the Π^1_1-indescribability filter (see Theorem 2.23) as Sun had also showed. However more generally this is where we see the connection with stationary reflection - if an ordinal α reflects any two stationary sets *simultaneously*, *i.e.* if for any stationary $A, B \subseteq \alpha$ there is some $\beta < \alpha$ such that $A \cap \beta$ and $B \cap \beta$ are both stationary in β, then the 1-clubs do indeed generate a filter - see Prop.2.6. At such ordinals, we can then define the 2-*stationary sets* as those which intersect every 1-club (*i.e.* are of positive measure with respect to the 1-club filter). We can continue by defining a 2-*club* set to be a 2-stationary set which is also 2-stationary closed, and then, where an ordinal simultaneously reflects 2-stationary sets we can show that these 2-clubs generate a filter. Hence, we can there define 3-stationary sets, and so on.

Our definitions of γ-club and γ-stationarity are similar to those of [2]; they are not quite the same, but the two types of definition, which were made independently of each other, coincide if $V = L$ (see the Remark before Prop.2.14). It can be seen from Prop.2.14 that our γ-stationarity is exactly the γ-s-stationarity Bagaria independently defined in a more recent paper [3]. He gives a rather different motivation for his notion of γ-s-stationarity, using *derived topologies* and the definition grew out of questions in proof theory and modal logic raised in [6]. We shall not discuss these connections to proof theory and modal logic here (for details see [3] and [6]), but this shows that the notions we shall define have very broad range of application. There are many other possible applications to explore with these notions.

Outside of L this is a different matter. The recent paper [1] examines the status of ξ-reflecting cardinals and they show for example that $n+1$-reflection does not imply that the cardinal is Π^1_n-indescribable (and similarly in the transfinite), thus generalising Meckler-Shelah mentioned above.

In this paper we generalise the combinatorial principles of $\Box$ (in Section 3) and $\diamondsuit$ (in Section 4) using γ-club and γ-stationary sets. We also look in some detail at how these properties manifest in Gödel's constructible universe L. The $\Box$ sequences constructed in Section 3 and 4 give important results about the γ-club filter in L, which are used to show that, under certain assumptions, γ-stationarity is downward absolute to L.

In Section 2 we define the notions of γ-closed and unbounded (γ-club) and γ-stationary set. We shall not give the definition in full here, but indicate its nature to show it extends that the usual notions.

Following [2] we denote this by $d_0(C)$ the 'first derivative' of C. In general we shall have that $d_\gamma(S) = \{\alpha < \kappa \mid S \cap \alpha \text{ is } \gamma\text{-stationary in } \alpha\}$.

Our definition of a 1-*reflecting cardinal* κ is that any two 1-stationary (which equals stationary here) subsets S, T of the cardinal, have a common point below which both are stationary: there is $\alpha < \kappa$ with both $S \cap \alpha$ and $T \cap \alpha$ stationary. Next κ is 2-*reflecting* if for any two 2-stationary subsets S, T there is $\alpha < \kappa$ with both $S \cap \alpha$ and $T \cap \alpha$ 2-stationary. This then generalises to "γ" replacing "2" in the last sentence. Thus for a γ-reflecting cardinal κ we shall obtain the simultaneous reflection of any two γ-stationary subsets of κ.

Just as the first derivative of a set unbounded in κ is a club in κ, so the γ'th derivative of a suitable set should be a γ-club:

PROPOSITION 2.9. *If κ is γ-reflecting and S is γ-stationary in κ then $d_\gamma(S)$ is γ-club in κ.*

The γ-club subsets of a γ-reflecting κ form a filter (Prop.2.6) but *prima facie* not necessarily a normal one. Conditions can be imposed to ensure its normality, and normality is provable in L.

After giving the basic properties of these sets in Section 2.1, we consider the main result of [2] which was that for $n < \omega$ we have that in L a regular cardinal κ is $n+1$-stationary in their sense, iff κ is Π^1_n-indescribable.

We give our statement of what is essentially this theorem of [2]:

THEOREM 2.14. ([2]) *Assume $V = L$. Then a regular cardinal κ is n-reflecting iff κ is Π^1_n-indescribable.*

We proceed to define the γ-club filter $\mathcal{C}^\gamma(\kappa)$ and then for finite γ we investigate the relation between the γ-club filter and Π^1_γ-indescribability.

LEMMA 2.20. *For any $n < \omega$ and Π^1_n-indescribable cardinal κ we have*

1. *$\mathcal{C}^n(\kappa)$ is contained in the Π^1_n-indescribability filter.*
2. *$\mathcal{C}^n(\kappa)$ is normal.*

A Solovay-style splitting theorem is provable:

COROLLARY 2.25. *Let κ be Π^1_n-indescribable, $n \geq 1$. Then any $n+1$-stationary subset of κ can be split into κ many disjoint $n+1$-stationary sets.*

This will be extended for transfinite γ once we have definitions of Π^1_γ-indescribability.

Section 3 contains one of the central results, where we prove the existence of certain $\Box$ sequences which give a characterisation of the γ-stationary reflecting cardinals in L in terms of indescribability:

THEOREM 3.32. $(V = L)$ *Let $\gamma < \kappa$ be an ordinal and κ be Π^1_η-indescribable for every $\eta < \gamma$ but not Π^1_γ-indescribable, and let $A \subseteq \kappa$ be γ-stationary. Then there is $E_A \subseteq A$ and a $\Box^{<\gamma}$ sequence $\mathcal{S}$ on κ such that E_A is γ-stationary in κ and $\mathcal{S}$ avoids E_A. Thus κ is not γ-reflecting.*

Section 3.1 gives the key tool used in the proof of Theorem 3.32 which here serves as the version of the primary club sequences derived from Σ_n-hulls which occur in the fine structural proof of $\Box$: the notion of a *trace*. In Section 3.2 we prove the existence of certain $\Box^\gamma$ sequences for finite γ (Theorem 3.2). Although this result falls out as a corollary from Theorem 3.32 in Section 3.4, the proof of Theorem 3.2 gives the key moves of the later proof, *i.e.* that of Theorem3.32, while in the more familiar context of Π^1_n-indescribability and without the added complication that limit cases necessitate.

Transfinite Π^1_γ-indescribability for infinite γ is defined in Section 3.3: this is essentially that of [27]. We point out (see the Remark before Lemma 3.25) that our definitions concerning Π^1_γ-indescribability all work to define a weak notion of indescribability using second order quantification over $(\kappa, \in, A_1, A_2, \dots)$ *etc.* However we seemingly have to quantify over strategies for games $G_\gamma(\beta, \varphi, A)$ for $\beta < \kappa$ if we wish to get that Π^1_γ-indescribability of κ is $\Pi^1_{\gamma+1}$ expressible, which requires a quantification over V_κ and this requires dealing with a strong indescribability notion.

We also need to change our type of $\square$ sequence to deal with limit cases. The reason becomes apparent at stage ω: we need to work with cardinals κ that are Π^1_n-indescribable for every n but are not Π^1_ω-indescribable. For these we need to show that they are not ω-reflecting. To this end, we define and use $\square^{<\omega}$ sequences. Here each element C_α of the sequence acquires a tag $n_\alpha < \omega$ indicating that C_α is an n_α-club. An enhanced coherency requirement states that for the relevant β, $C_\beta = C_\alpha \cap \beta$ and C_β should also be an $n_\alpha = n_\beta$-club.

In fact the definition $\square^{<\gamma}$ works for both limit and successor γ, so we use this for all transfinite γ, regarding now the finite case $\square^n$ as a proof-of-concept exercise for the general case.

Nevertheless for successor ordinals we have as a corollary for the more straightforward $\square^\gamma$-sequences:

COROLLARY 3.39. $(V = L)$ *Let* $\gamma < \kappa$ *and* κ *be a* Π^1_γ*-indescribable but not* $\Pi^1_{\gamma+1}$*-indescribable cardinal, and let* $A \subseteq \kappa$ *be* $\gamma + 1$ *stationary. Then there is* $E_A \subseteq A$ *and a* $\square^\gamma$*-sequence* S *on* κ *such that* E_A *is* $\gamma + 1$*-stationary in* κ *and* S *avoids* E_A.

In Section 3.4 we give the main result of Theorem 3.32 and then its application:

COROLLARY 3.25. $(V = L)$ *The following are equivalent for a regular cardinal* $\kappa > \omega$ *and ordinals* $0 < \gamma < \kappa$*:*
(i) κ *is* Π^1_γ*-indescribable;*
(ii) there is no $\Box^{<\gamma}$ *sequence on* κ *that avoids a* γ*-stationary* $E \subseteq \kappa$*;*
(iii) κ *is* γ*-reflecting.*

Using an alternate definition of Π^1_γ-indescribability, Bagaria independently in [3] proved a version of the result for (i) $\Longleftrightarrow$ (iii). It is not clear to us that these two notions of indescribability are equivalent. Bagaria does not define or use $\Box$-sequences. However here in clause (ii) then we have a new characterisation.

Jensen's theorem is Corollary 3.25 with $\gamma = 1$. It should be noted here that the proof of Jensen's theorem required a very close analysis of condensation and satisfaction in L - Jensen used fine-structure; Beller and Litman [7] gave a proof of this $\gamma = 1$ case using Silver machines. However we shall see that we do not need such a fine analysis for the cases when $\gamma > 1$ - essentially this will be replaced by a coarser analysis using traces and filtrations.

In Sections 3.4 and 3.5 we extend the results of Section 2.2 on Π^1_γ-indescribability and the γ-club filter to transfinite γ, including the splitting of stationary sets. This is fairly straight forward, proceeding very much as in the finite case once we have the relevant lemmas. In Section 3.5.3 we generalise the notion of non-threaded square. Recall that a *non-threaded square* is a sequence $\langle C_\alpha \mid \alpha < \kappa, Lim(\alpha)\rangle$ so that for such α (i) $C_\alpha \subseteq \alpha$ is club; (ii) $\forall \beta \in C^*, C_\beta = \alpha \cap C_\alpha$, which has no *thread*, that is a $S \subseteq \kappa$ which is club and $\forall \beta \in S^*, S \cap \beta = C_\beta$. Such a coherent sequence without a thread is called a $\Box(\kappa)$-sequence, and such implies that there are stationary sets $A, B \subseteq \kappa$ so that for any $Lim(\alpha)$, $C^*_\alpha \cap A = \emptyset$ or $C^*_\alpha \cap B = \emptyset$ (see [22, Prop.27]). Thus witnessing that simultaneous stationary reflection fails.

Here $C^* = d_0(C)$ is the set of limit points of C. Roughly speaking then a $\Box^\gamma(\kappa)$-sequence is defined like a $\Box(\kappa)$ sequence, replacing d_0 by d_γ, club and stationary by γ-club and γ-stationary.

Generalising the non-simultaneous reflecting result above from $\Box(\kappa)$ we have the analogous statement: (Theorem 3.50) if the γ-club filter is normal and $\Box^\gamma(\kappa)$ holds then κ is not $\gamma+1$-reflecting, *i.e.* simultaneous reflection of $\gamma+1$-stationary sets fails.

COROLLARY 3.51. $(V = L)$. *Suppose* κ *is* Π^1_γ*-indescribable. Then the following are equivalent: (i)* κ *is not* $\Pi^1_{\gamma+1}$*-indescribable; (ii)* $\Box^\gamma(\kappa)$.

(The authors of [10] also considered a *prima facie* slightly weaker version of our threaded square for finite n which they call $\Box_n(\kappa)$. This involves the Π^1_{n-1}-indescribability ideal both in the definition of n-clubs and in its definition of where the C_α's must cohere. They show that our $\Box^n(\kappa)$ implies $\Box_n(\kappa)$, but leave the question of their equivalence open.)

In Section 4 we prove the main theorem Theorem 4.5 which shows that under certain assumptions, γ-stationarity is downwards absolute to L. The following corollary is a simpler, albeit weaker, statement of downward absoluteness.

COROLLARY 4.8. *Assume that for any ordinal* $\gamma < \kappa$ *with* κ *a* γ*-reflecting regular cardinal, that the* γ*-club filter on* κ *is normal. Then if* $S \subseteq \kappa$ *is* γ*-stationary with* $S \in L$ *we have* $(S$ *is* γ*-stationary in* $\kappa)^L$.

This extends the result of Magidor in [24] which essentially gives the case $\gamma = 2$. As well as being an interesting result in itself, Theorem 4.5 shows (together with the main result of Section 5) that if the existence of a γ-reflecting cardinal plus some mild assumptions is consistent, then so is the existence of a Π^1_γ-indescribable cardinal.

Finally, in Section 5 we use γ-stationarity to generalise the notion of ineffability and the combinatorial principle $\diamondsuit$. The notion of γ-ineffability is introduced in section 5.1 and we show that many results about ineffable cardinals generalise well to this context. These γ-ineffable cardinals are shown to satisfy the assumptions needed for Theorem 4.5, and hence (Theorem 5.12) any γ-ineffable cardinal is γ-ineffable in L. In section 5.2 we define generalisations of $\diamondsuit$ and $\diamondsuit^*$ and look into their connection with γ-ineffability. In particular we show that in L, the cardinals at which $\diamondsuit^{*\gamma}$ holds can be entirely characterised in terms of γ-ineffability (Corollary 5.23). Although many of the proofs in this chapter lift straight from the standard cases of club and stationary set for successor cases, the limit cases are generally a different matter.

§2. Generalising closed and unbounded sets.

2.1. Definitions and Basic Properties. We define here the central notions of γ-club set and γ-stationary set. The notion of γ-club is new, and helps us to generalise many of the basic properties of stationary sets. In order for γ-clubs to do what we want, we also define a notion of γ-reflecting ordinal which we restrict ourselves to when defining $\gamma+1$-stationary sets - just as we only define stationary sets at ordinals of uncountable cofinality.

DEFINITION 2.1. Let κ be an ordinal and S, C sets of ordinals. We define by simultaneous induction for $\gamma < \kappa$:

(1) S is *0-stationary* in κ if S is unbounded in κ.
(2) C is *γ-stationary-closed below* κ if for any $\alpha < \kappa$ such that C is γ-stationary in α, we have $\alpha \in C$.
(3) C is *γ-club* in κ if C is γ-stationary-closed below κ and γ-stationary in κ.
(4) κ is *γ-reflecting* if for any $A, B \subseteq \kappa$ with A and B γ-stationary in κ there is some $\alpha < \kappa$ such that A and B are γ-stationary in α.
(5) S is *γ-stationary* in κ for $\gamma > 0$ and for all $\eta < \gamma$ we have that κ is η-reflecting and for every C which is η-club in κ, $S \cap C \neq \emptyset$.

Whenever we speak of the notion of "γ-club in κ", or "γ-stationarity of κ" &c. it is to be understood that $\gamma < \kappa$, perhaps even without mentioning κ if the context is clear. (Thus γ will be less than κ for the rest of this section.) It is easy to see that our ordinary notions of club and stationary sets are the 0-clubs and 1-stationary sets. The 1-clubs in κ are then stationary-closed sets which are also stationary in κ. We shall see shortly that for $\gamma = \eta + 1$, (5) reduces to the usual definition of stationary sets in terms of clubs: S is $\eta + 1$ stationary if it intersects every η-club. For limit γ it is easy to see that S is γ-stationary if for every $\eta < \gamma$, S is η-stationary. The requirement of (4), a simple reflection property, is needed to ensure the η-clubs generate a filter - for 0-reflecting this is just having uncountable cofinality.

The following notation from [2] will be very useful in exploring these concepts:

DEFINITION 2.2. For a set $S \subseteq \kappa$ we set

$$d_\gamma(S) = \{\alpha < \kappa : S \cap \alpha \text{ is } \gamma\text{-stationary in } \alpha\}.$$

This is a version of *Cantor's derivative operator*, giving the limit points of a set in a certain topology (see [3]). Thus $d_0(S)$ is the set of limit points of S, $d_1(S)$ is the set of points below which S is stationary, *etc.*

REMARK. Using this notation we see that (2), γ-stationary-closure, is simply the condition that $d_\gamma(C) \subseteq C$. Further (4) can be restated as "κ is *γ-reflecting* if for any $A, B \subseteq \kappa$ with A and B γ-stationary in κ, $d_\gamma(A) \cap d_\gamma(B) \neq \emptyset$". Finally note that if $\gamma < \kappa$ and $d_\gamma(S)$ is γ-stationary in κ then $\min d_\gamma(S) > \gamma$ just by definition of γ-stationarity in κ.

For the rest of this section we look at some basic properties of such sets.

PROPOSITION 2.3. *Let $A \subseteq \kappa$ and $\gamma' < \gamma < \kappa$. Then:*
(i) If A is γ-stationary then it is γ'-stationary.

(ii) Thus $d_\gamma(A) \subseteq d_{\gamma'}(A)$.
(iii) If κ *is* η*-reflecting for all* $\eta < \gamma$*, then* A *is* γ'*-club implies* A *is* γ*-club.*

PROOF: Formally a proof by induction on κ. We suppose the proposition true for all $\kappa' < \kappa$. (i) is clear by (5) of the definition. (ii) then holds using clause (i) of the inductive hypothesis on $\alpha < \kappa$ and Definition 2.2. To show (iii): fix γ and $\gamma' < \gamma$. By (i) and (ii) γ'-stationary closure below κ implies γ-stationary closure below κ as if C is γ'-stationary closed then $d_\gamma(C) \subseteq d_{\gamma'}(C) \subseteq C$. Suppose C and C' are γ'-club. Then as κ is γ'-reflecting $d_{\gamma'}(C) \cap d_{\gamma'}(C') \neq \emptyset$ and $d_{\gamma'}(C) \cap d_{\gamma'}(C') \subseteq C \cap C'$. As C' was an arbitrary γ'-club, we have C is $\gamma' + 1$ stationary. By (i) therefore, C must be γ-stationary and hence γ-club. QED

REMARK. By (ii) above to see that a set is $\gamma+1$-stationary we need only check that it intersects every γ-club (and that κ is γ-reflecting).

The following three propositions we prove together by a simultaneous induction.

PROPOSITION 2.4. *If* S *is* γ*-stationary and* C *is* γ'*-club in* κ *for some* $\gamma' < \gamma$ *then* $S \cap C$ *is* γ*-stationary.*

PROPOSITION 2.5. *If* κ *is* γ*-reflecting and* A*,* B *are* γ*-stationary subsets of* κ *then* $d_\gamma(A) \cap d_\gamma(B)$ *is* γ*-stationary.*

PROPOSITION 2.6. *If* κ *is* γ*-reflecting and* C^1 *and* C^2 *are* γ*-club in* κ *then* $C^1 \cap C^2$ *is* γ*-club in* κ*. Thus the* γ*-clubs generate a filter on* κ*.*

PROOF: We prove this as a simultaneous induction on γ. First note that if $\gamma = 0$ then 2.4 is vacuous, and 2.5 and 2.6 are immediate from the definitions.

Suppose then Proposition 2.6 holds for every $0 \leq \gamma' < \gamma$. First we show Proposition 2.4 holds for γ. Let S be a γ-stationary subset of κ and $\gamma', \eta' < \gamma$. Let C be γ'-club in κ and let C' be η'-club in κ. Set $\eta = max\{\gamma', \eta'\}$. By Proposition 2.3 C and C' are both η-club. Now by Proposition 2.6 we have $C \cap C'$ is η-club. Thus $(S \cap C) \cap C' = S \cap (C \cap C') \neq \emptyset$, so as η' and C' were arbitrary $S \cap C$ is γ-stationary.

Now supposing we have Proposition 2.4 for γ we show Proposition 2.5 holds for γ. Let κ be γ-reflecting, $A, B \subseteq \kappa$ be γ-stationary and let $\gamma' < \gamma$ with C γ'-club. By Proposition 2.4 we have $C \cap A$ is γ-stationary, so as κ is γ-reflecting $d_\gamma(C \cap A) \cap d_\gamma(B) \neq \emptyset$. But $d_\gamma(C \cap A) \subseteq C$ as C γ-stationary in α implies that C is γ'-stationary in α, and the latter implies $\alpha \in C$, and then clearly also $d_\gamma(C \cap A) \subseteq d_\gamma(A)$. Thus $C \cap d_\gamma(A) \cap d_\gamma(B) \supseteq d_\gamma(C \cap A) \cap d_\gamma(B) \neq \emptyset$. Hence $d_\gamma(A) \cap d_\gamma(B)$ is γ-stationary.

Finally we show Proposition 2.5 implies Proposition 2.6. Take C^1 and C^2 to be γ-club in κ. By Proposition 2.5 we have $d_\gamma(C^1) \cap d_\gamma(C^2)$ is γ-stationary so as $d_\gamma(C^1) \cap d_\gamma(C^2) \subseteq C^1 \cap C^2$ we just need to show γ-stationary closure. But this is simple as if $C^1 \cap C^2$ is γ-stationary below $\alpha < \kappa$ then C^1 and C^2 are both γ-stationary below α so by the γ-stationary closure of C^1 and C^2 we must have $\alpha \in C^1 \cap C^2$. QED

PROPOSITION 2.7. *If* κ *is* γ*-reflecting and* C *is* γ*-club in* κ *then* $d_\gamma(C)$ *is also* γ*-club in* κ*.*

PROOF: By Proposition 2.5 we have $d_\gamma(C)$ is γ-stationary. To see we have closure, if $d_\gamma(C)$ is γ-stationary below α then, as $d_\gamma(C) \subseteq C$ we have C is γ-stationary below α, so by stationary closure of C we have $\alpha \in d_\gamma(C)$. QED

The following is a useful way of building γ-stationary sets.

LEMMA 2.8. *Let* $A \subseteq \kappa$ *be* γ*-stationary in* κ *and* $\langle A_\alpha : \alpha \in A\rangle$ *be a sequence of sets such that each* A_α *is* γ*-stationary in* α*. Then* $\bigcup\{A_\alpha : \alpha \in A\}$ *is* γ*-stationary in* κ*.*

PROOF: This is clearly true for $\gamma = 0$ so suppose $\gamma > 0$ and let $\gamma' < \gamma$ and $C \subseteq \kappa$ be γ'-club in κ. Then as A is γ-stationary in κ we must have κ is γ'-reflecting so by Proposition 2.7 $d_{\gamma'}(C)$ is γ'-club. Then we can find some α in $d_{\gamma'}(C) \cap A$. Now C is γ'-club in α so as A_α is γ-stationary in α we must have $C \cap A_\alpha \neq \emptyset$. Thus $C \cap \bigcup\{A_\alpha : \alpha \in A\} \neq \emptyset$ and we're done. QED

We can now prove a stronger result than 2.7:

PROPOSITION 2.9. *If κ is γ-reflecting and S is γ-stationary in κ then $d_\gamma(S)$ is γ-club in κ.*

PROOF: As κ is γ-reflecting $d_\gamma(S)$ is γ-stationary by Proposition 2.5. To show closure suppose $d_\gamma(S)$ is γ-stationary in α. Then $S \cap \alpha = \bigcup\{S \cap \beta : \beta \in d_\gamma(S) \cap \alpha\}$ with each $S \cap \beta$ being γ-stationary in β, so by Lemma 2.8 S is γ-stationary in α. Hence $\alpha \in d_\gamma(S)$ and we have γ-stationary closure. QED

The following shows that there are also many ordinals below which a set is not γ-stationary:

PROPOSITION 2.10. *If A is γ-stationary in κ then*

$$A \setminus d_\gamma(A) = \{\alpha \in A : A \cap \alpha \textit{ is not } \gamma\textit{-stationary}\}$$

is γ-stationary.

Note that $A \setminus d_\gamma(A)$ can never be $\gamma + 1$-stationary as if κ is γ-reflecting then Proposition 2.9 gives us that $d_\gamma(A)$ is γ-club.

PROOF: Let $\gamma' < \gamma$ and C be γ'-club in κ. Then $d_{\gamma'}(C)$ is γ'-club, using Prop.2.7 as κ is γ'-reflecting; hence it is γ'-stationary and thus its minimum is greater than γ'. So we can find α minimal in $A \cap d_{\gamma'}(C)$ and we shall have $\gamma' < \alpha$. Then C is γ'-club in α. If α is not γ'-reflecting then A cannot be γ-stationary in α so we're done. If α is γ'-reflecting then by Proposition 2.7 $d_{\gamma'}(C)$ is also γ'-club in α. But $A \cap \alpha \cap d_{\gamma'}(C) = \emptyset$ so $A \cap \alpha$ is not γ-stationary. QED

We can now show how closely this definition of γ-stationarity is related to that in [2], and to notions of reflection:

PROPOSITION 2.11. *Let κ be η-reflecting for all $\eta < \gamma$ and $S \subseteq \kappa$. Then S is γ-stationary iff for any $\eta < \gamma$ and any η-stationary $A \subseteq \kappa$ we have $d_\eta(A) \cap S \neq \emptyset$.*

PROOF: $(\Rightarrow)$ is clear as by Proposition 2.9 $d_\eta(A)$ is η-club. $(\Leftarrow)$: if C is η-club then C is η-stationary so we have $d_\eta(C) \cap S \neq \emptyset$, but $d_\eta(C) \subseteq C$. QED

The following is an easy consequence:

PROPOSITION 2.12. *If κ is γ-reflecting then κ is η-reflecting for all $\eta \leq \gamma$.*

REMARK. Comparing this characterisation to the definition in [2] (which we do not detail here) we see the only difference between our γ-stationary sets and those defined in [2] is that we always obtain *simultaneous* reflection as it is built in to our notion of γ-stationarity through reflection. Indeed it is easy to see that if $V = L$ then the two notions completely agree as we shall anyway get simultaneous reflection in this case. We do also see, however, that our definition of γ-stationary is equivalent to the definition of γ-s-stationary given in [3]:

DEFINITION 2.13. Define *γ-s-stationary in κ* inductively as follows, starting with 0-s-stationary being unbounded. Let S be γ-s-stationary in κ if for every $\eta < \gamma$ and A, B which are η-s-stationary in κ there is some $\alpha \in S$ such that $A \cap \alpha$ and $B \cap \alpha$ are both η-s-stationary in α.

PROPOSITION 2.14. *A set S is γ-s-stationary in α iff S is γ-stationary in α.*

PROOF: By induction on γ. Suppose we have for $\eta < \gamma$ and any ordinal α that a set is η-stationary iff η-s-stationary.

Let S be γ-s-stationary in κ. First we see that κ must be η-reflecting for any $\eta < \gamma$: if A and B are η-stationary then they are η-s-stationary by the inductive hypothesis, and hence there is some $\alpha \in S$ such that $A \cap \alpha$ and $B \cap \alpha$ are both η-s-stationary in α. But then $A \cap \alpha$ and $B \cap \alpha$ are both η-stationary in α, so κ is η-reflecting. Now suppose C is η-club for some $\eta < \gamma$. Then C is η-s-stationary so there is some $\alpha \in S$ such that $C \cap \alpha$ is η-s-stationary in α, *i.e.* $C \cap \alpha$ is η-stationary in α. So by η-stationary closure, $\alpha \in C \cap S$. As C was arbitrary, we must have S is γ-stationary.

Now suppose S is γ-stationary and let $\eta < \gamma$ and A and B be η-s-stationary. By the inductive hypothesis, A and B are both η-stationary. By our definition of γ-stationarity we have that κ is η-reflecting. Thus by Proposition 2.9 $d_\eta(A)$ and $d_\eta(B)$ are both η-club. Then by Proposition 2.6, still using η-reflection, $d_\eta(A) \cap d_\eta(B)$ is η-club, and hence we can find $\alpha \in S \cap d_\eta(A) \cap d_\eta(B)$. But A and B are both η-stationary below such an α, and hence A and B are both η-s-stationary below such an α, so we have that S is γ-s-stationary. QED

It is easy to show (see Proposition 2.20) that any Π^1_n-indescribable is n-reflecting. So a simple induction shows that we get this result for our definitions too.

THEOREM 2.15. [2] *Assume $V = L$. Then a regular cardinal κ is n-reflecting iff κ is Π^1_n-indescribable.*

In Section 3 we shall give an alternative proof of this, and after defining Π^1_γ-indescribability in section 3.3 we shall extend it to replace n with any ordinal $\gamma < \kappa$. Nevertheless, our result is still of further interest as we are proving the existence of certain □-sequences, which are not constructed in either [2] or [3].

One question now is where the γ-stationary sets can occur if $V \neq L$. We have that if a cardinal is Π^1_1-indescribable then it is 2-stationary (*i.e.* 1-reflecting), and the above shows that it is consistent for these to be the only 2-stationary regular cardinals. However, known results show that it is also consistent, relative to certain large cardinal assumptions, that non-weakly compact cardinals be stationary reflecting, and indeed 1-reflecting (see [21], [29, Thms. 1.18, 1.25]).

It is well known that a regular cardinal cannot be stationary reflecting unless it is either the successor of a singular cardinal or weakly inaccessible. Observe that $E^{\lambda^+}_\lambda =_{df} \{\alpha < \lambda^+ \mid cf(\alpha) = \lambda\}$ does not reflect for $\lambda \in Reg$. Furthermore note that singular ordinals are not so interesting in this context as γ-stationarity for subsets of a singular reduces to γ-stationarity for subsets of its cofinality:

PROPOSITION 2.16. *Let α be a singular ordinal and $C \subseteq \alpha$ be club, with $\pi : \langle C, \in\rangle \cong \langle ot(C), \in\rangle$. Then for $\gamma \geq 1$ any $S \subseteq \alpha$ is γ-stationary in α iff $\pi``S \cap C$ is γ-stationary in $ot(C)$. Hence α is γ-reflecting if and only if $cf(\alpha)$ is.*

PROOF: This is proven by induction. Fix α and γ and suppose the claim is true for all $\beta < \alpha$ and for all $\eta < \gamma$. Fix C, S and $\pi : \langle C, \in\rangle \cong \langle ot(C), \in\rangle$. If D is η-club in α then by the inductive hypothesis and the closure of C, $\pi``D \cap C$ is $\eta$-club in $ot(C)$, and hence if $\pi``S \cap C$ is γ-stationary in $ot(C)$ then S is γ-stationary in α. Similarly if D is η-club in $ot(C)$ then $\pi^{-1}``D$ is $\eta$-club in $\alpha$, so if $S$ is $\gamma$-stationary in $\alpha$ then $\pi``S \cap C$ is γ-stationary in $ot(C)$. QED

A cardinal can be stationary reflecting without *simultaneously* reflecting stationary sets (*i.e.* without being 1-reflecting) as shown in [25], and in fact the consistency strength of the existence of a stationary reflecting cardinal is much less than the existence of a 1-reflecting cardinal. This follows from results of Mekler-Shelah in [25] and Magidor in [24]. In the latter it is shown that if a regular cardinal is 1-reflecting then it is weakly compact in L, and thus if the existence of a 1-reflecting cardinal is consistent then so is the existence of a weakly compact cardinal. we shall generalise this result on downward absoluteness in Section 4 (although we have to add some assumptions there).

Kunen has shown [21] that it is consistent relative to the existence of a weakly compact cardinal that there is a stationary reflecting cardinal that is not weakly compact - this is proven by giving a forcing which adds a Suslin tree to a cardinal κ that is weakly compact in the ground model. It is clear from the proof that in this model κ reflects any two stationary sets simultaneously and is thus 1-reflecting. It is also easy to see that the 1-club filter is normal there (see Definition 2.18).

Magidor in [24] starts from the much stronger assumption that there are infinitely many supercompact cardinals, and produces a forcing extension in which $\aleph_{\omega+1}$ reflects stationary sets. Again, it is clear from the proof that $\aleph_{\omega+1}$ is in fact 1-reflecting. Here, however, the 1-club filter is not even countably complete, as shown by the following easy argument. For each $n < \omega$ let $C_n = \{\alpha < \aleph_{\omega+1} : cf(\alpha) \geq \aleph_n\}$. Then each C_n is 1-club - it is clearly stationary, and cannot be stationary below any ordinal of cofinality $\leq \aleph_n$ so is also stationary closed. But $\bigcap_{n<\omega} C_n = \emptyset$. It is also easy to see that $\aleph_{\omega+1}$ cannot be 2-reflecting, as $\aleph_{\omega+1}$ is the *minimum* ordinal that can be 2-stationary so there is nowhere for 2-stationary sets to reflect to.

2.2. Results at Indescribables. Indescribability is a central concern of this paper. We recall its definition, and mention that an uncountable cardinal κ is *weakly compact* if and only if it is Π^1_1 indescribable.

DEFINITION 2.17. An uncountable cardinal κ is Π^1_n-*indescribable* if for any $R \subseteq V_\kappa$ and Π^1_n formula φ such that $\langle V_\kappa, \in, R\rangle \vDash \varphi$, there is some $\alpha < \kappa$ such that

$$\langle V_\alpha, \in, R \cap V_\alpha\rangle \vDash \varphi.$$

This turns out to be closely connected to n-stationarity.

2.2.1. *The Club and Indescribability Filters.* In this section we shall be focusing on the relationship between Π^1_n-indescribability and our generalised notions of club and stationarity, in particular the n-club filter. For this reason, we shall be mostly restricting to finite levels of the club and stationary set hierarchy here. We shall extend these results to the transfinite in section 3.5, after we have introduced a notion of Π^1_γ-indescribability. Analysing this relationship will allow us to generalise some deeper properties of stationary sets and the club filter to $n+1$-stationary sets and the n-club filter at a Π^1_n-indescribable cardinal. By Theorem 2.15 this generalisation will be full in L, but limited if we only have n-reflecting cardinals which are not Π^1_n-indescribable.

We first look at the relationship between the n-club filter and the Π^1_n-indescribability filter.

DEFINITION 2.18. For a γ-reflecting cardinal κ we denote by $\mathcal{C}^\gamma(\kappa)$ the γ-club filter on κ:

$$\mathcal{C}^\gamma(\kappa) := \{X \subseteq \kappa : X \text{ contains a } \gamma\text{-club}\}.$$

DEFINITION 2.19 (Levy). [23] If κ is Π^1_n-indescribable let $\mathcal{F}^n(\kappa)$ denote the Π^1_n-indescribable filter on κ:

$$\mathcal{F}^n(\kappa) := \{X \subseteq \kappa : \text{for some } \Pi^1_n \text{ sentence } \varphi \text{ with parameters such that } V_\kappa \vDash \varphi,\ V_\alpha \vDash \varphi \text{ implies } \alpha \in X\}.$$

REMARK. (1) By a $\mathcal{P}^1_n$ sentence here, we mean a usual Π^1_n of the second order language, but with both first and second order parameters allowed substituted in for the free variables.
(2) The statement "X is n-stationary in κ" is Π^1_n expressible over V_κ. This can be seen by induction - clearly "X is unbounded" is Π^1_0. "X is $n+1$-stationary" is equivalent (by Proposition 2.11) to "X is n-stationary $\wedge \forall S, T(S, T$ are n-stationary $\rightarrow \exists \alpha \in X\ S \cap \alpha, T \cap \alpha$ are n-stationary in $\alpha)$". Further assuming n-stationarity is Π^1_n expressible this is clearly Π^1_{n+1}.
(3) By a result of Levy if κ is Π^1_n indescribable then $\mathcal{F}^n(\kappa)$ is normal and κ-complete.

PROPOSITION 2.20. *If κ is a Π^1_n-indescribable cardinal then κ is n-reflecting and furthermore any set $X \in \mathcal{F}^n(\kappa)$ is $n+1$-stationary.*

PROOF: Suppose κ is Π^1_n-indescribable. We have seen that being n-stationary in κ is Π^1_n expressible over V_κ, so for any $S, T \subseteq \kappa$ if we have $V_\kappa \vDash$ "S, T are n-stationary" then for some $\alpha < \kappa$ we have $V_\alpha \vDash$ "$S \cap \alpha, T \cap \alpha$ are n-stationary" and hence κ is n-reflecting. For $X \in \mathcal{F}^n(\kappa)$ let φ be a Π^1_n formula such that $V_\kappa \vDash \varphi$ and $V_\alpha \vDash \varphi \to \alpha \in X$. For an n-club C, taking "$\psi = C$ is n-stationary $\wedge\, \varphi$", we have that ψ is Π^1_n and hence for some $\alpha < \kappa$, $V_\alpha \vDash \psi$. But then $V_\alpha \vDash \varphi$ so $\alpha \in X$ and $C \cap \alpha$ is n-stationary so by n-stationary closure, $\alpha \in C$. Thus X meets any n-club and so X is $n+1$-stationary. QED

First we show that at a Π^1_n-indescribable cardinal the n-club filter is normal. For $n = 1$ this is due to Sun [29].

LEMMA 2.21. *For any $n < \omega$ and Π^1_n-indescribable cardinal κ we have*

1. $\mathcal{C}^n(\kappa) \subseteq \mathcal{F}^n(\kappa)$,
2. $\mathcal{C}^n(\kappa)$ *is normal (and hence κ-complete)* .

PROOF: (1) Let C be n-club in κ. Then $d_n(C) \subseteq C$ and $d_n(C) = \{\alpha < \kappa : V_\alpha \vDash$ "C is n-stationary"$\}$. As being n-stationary is Π^1_n we see $d_n(C) \in \mathcal{F}^n$ so $C \in \mathcal{F}^n$.

(2) Let $\langle C_\alpha : \alpha < \kappa\rangle$ be a sequence of n-clubs. Defining $C := \triangle_{\alpha<\kappa} C_\alpha$ we have $C \in \mathcal{F}^n$ (using the normality of $\mathcal{F}^n$ and (1)). As each element of $\mathcal{F}^n$ is $n+1$-stationary we thus have C is n-stationary. To show closure suppose C is n-stationary in $\alpha < \kappa$. As $C \cap \alpha = \{\beta < \alpha : \beta \in \bigcap_{\gamma<\beta} C_\gamma\}$ we see that for any $\beta < \alpha$ we have $(\beta, \alpha) \cap C \subseteq C_\beta$ so $C_\beta \cap \alpha$ includes an end-segment of an n-stationary set and thus C_β is n-stationary in α. Then $\alpha \in C_\beta$ as C_β is n-club. Thus $\alpha \in \bigcap_{\beta<\alpha} C_\beta$, *i.e.* $\alpha \in C$. QED

REMARK. As the referee has kindly pointed out, there is a version of the last result where we weaken the notion of $\mathcal{F}^n$ to that of being the *weak indescribability filter* at a weakly indescribable cardinal κ. One then has the same conclusions - this corresponds to [29, Lemmata 1.14,1.15 & 1.16].

COROLLARY 2.22. *(Fodor's Lemma for n-stationary sets) If κ is Π^1_n-indescribable and $A \subset \kappa$ is n-stationary, then for any regressive function $f : A \to \kappa$ there is an n-stationary $B \subseteq A$ such that f is constant on B.*

In the next theorem the case of $n = 1$ is originally due to Sun [29].

THEOREM 2.23. *At any weakly compact cardinal κ, $\mathcal{C}^1(\kappa) = \mathcal{F}^1(\kappa)$ and assuming $V = L$ we have $\mathcal{C}^n(\kappa) = \mathcal{F}^n(\kappa)$ for any $n < \omega$ and Π^1_n-indescribable κ.*

PROOF: Firstly we show $\mathcal{C}^1(\kappa) = \mathcal{F}^1(\kappa)$; the later part will be an induction using that all n-reflecting cardinals are Π^1_n-indescribable, which holds in L.
Let $\forall X \varphi(X)$ be a Π^1_1 sentence (possibly with parameters) such that $V_\kappa \vDash \forall X \varphi(X)$. Note that $\varphi(X)$ is Σ^1_0 so $\neg\varphi(X)$ is Π^1_0. We define $A \subseteq \{\alpha < \kappa : V_\alpha \vDash \forall X\varphi(X)\}$ and show A is 1-club. Set $A := \{\alpha < \kappa : \alpha \text{ is inaccessible } \wedge V_\alpha \vDash \forall X\varphi(X)\}$. Then as inaccessibility is Π^1_1 expressible, $A \in \mathcal{F}^1(\kappa)$ and so A is stationary. To show A is stationary-closed take $\alpha < \kappa$ a limit of inaccessibles.
(i) If α is regular then α is inaccessible. Suppose $\alpha \notin A$. Then $V_\alpha \vDash \neg\varphi(Y)$ for some $Y \subseteq \alpha$ so by the inaccessibility of κ, $\{\beta < \alpha : V_\beta \vDash \neg\varphi(Y \cap \beta)\}$ is club in α. Thus if A is stationary in α then $\alpha \in A$.
(ii) If α is singular then A is not stationary in α: set $\lambda = cf(\alpha)$ and take C club in α with $ot(C) = \lambda$. Then no inaccessible above λ can be a limit point of C, so $d_0(C) \cap A$ is bounded in α. Thus A is stationary-closed and hence 1-club, so we have $\mathcal{C}^1(\kappa) = \mathcal{F}^1(\kappa)$.

Now we assume $V = L$. Suppose κ is Π^1_{n+1} indescribable and for all $\alpha < \kappa$ if α is Π^1_n indescribable then $\mathcal{C}^n(\alpha) = \mathcal{F}^n(\alpha)$. Let $\forall X\varphi(X)$ be a Π^1_{n+1} sentence (possibly with parameters) such that $V_\kappa \vDash \forall X\varphi(X)$. Take $A := \{\alpha < \kappa : \alpha \text{ is inaccessible } \wedge V_\alpha \vDash \forall X\varphi(X)\}$. As before A is a subset of the points where $\forall X\varphi(X)$ is reflected, and A is $n+1$-stationary as $A \in \mathcal{F}^{n+1}(\kappa)$. Also as above, if $\alpha < \kappa$ is singular then A is not stationary (and hence not

$n+1$-stationary) in α. So suppose α is regular and A is $n+1$-stationary in α. Then A must be Π^1_n-indescribable as only n-reflecting ordinals admit $n+1$-stationary sets, and in L the n-reflecting regular cardinals are exactly the Π^1_n-indescribables (Theorem 2.15). Suppose for a contradiction $V_\alpha \models \neg\varphi(Y)$ for some $Y \subseteq \alpha$. Now $\neg\varphi(Y)$ is Π^1_n so setting $B = \{\beta < \alpha : V_\beta \models \neg\varphi(Y \cap \beta)\}$ we have $B \in \mathcal{F}^n(\alpha) = \mathcal{C}^n(\alpha)$ by the inductive hypothesis, so A is not $n+1$ stationary in α, contradiction. Thus A is $n+1$-club, so $\mathcal{F}^{n+1}(\kappa) = \mathcal{C}^{n+1}(\kappa)$. QED

2.2.2. *Splitting Stationary Sets.* We show that at a Π^1_n-indescribable κ each $n+1$-stationary set can be split into κ many disjoint $n+1$-stationary sets. This is a generalisation of Solovay's result [28] that any stationary subset of a regular κ can be split into κ many disjoint stationary sets, but the proof is very different. The difficult part of this is actually to show that each $n+1$-stationary set can be split into two disjoint $n+1$-stationary sets, *i.e.* to show that the ideal of sets which are not $n+1$-stationary is atomless. Splitting into κ sets then follows. After we have introduced Π^1_γ-indescribability in section 3.3 we shall extend the results below to $\gamma+1$-stationarity (see section 3.5.2).

LEMMA 2.24. $(n > 0)$ *If $\mathcal{C}^{n-1}(\kappa)$ is κ-complete then any n-stationary subset of κ is the union of two disjoint n-stationary sets.*

PROOF: Let S be n-stationary in κ and suppose S is not the union of two disjoint n-stationary sets. Define

$$F = \{X \subset \kappa : X \cap S \text{ is } n\text{-stationary in } \kappa\}$$

Claim: F is a κ complete ultrafilter

Upwards closure is clear. If A, $B \in F$ then we have $S\backslash A$ and $S\backslash B$ are both non-n-stationary sets, as S cannot be split. Thus $A \cup (\kappa\backslash S)$ and $B \cup (\kappa\backslash S)$ are both in the $n-1$-club filter, and hence their intersection contains an $n-1$-club C. But then $C \cap S$ is n-stationary and $C \cap S \subseteq A \cap B \cap S$, so $A \cap B \in F$. For maximality, if $X \in F$ then as S cannot be split we must have $\kappa\backslash X \notin F$. That $X \notin F \Rightarrow \kappa\backslash X \in F$ follows from the fact that the $n-1$-clubs form a filter. The κ-completeness of F follows from the κ-completeness of the $n-1$-club filter, in the same way as the intersection property. Also, F is clearly non-principal as it contains all end-segments.

Claim: F is normal

As we have shown that F is a κ complete non-principal ultrafilter on κ, we have that κ is measurable and hence κ is Π^1_{n-1}-indescribable. Thus $\mathcal{C}^{n-1}(\kappa)$ is normal by Proposition 2.21. Let $\langle X_\alpha : \alpha < \kappa\rangle$ be a sequence of sets in F. Then each $S\backslash X_\alpha$ is in the non-n-stationary ideal on κ, so $X_\alpha \cup (\kappa\backslash S) \in \mathcal{C}^{n-1}(\kappa)$. Now by the normality of $\mathcal{C}^{n-1}(\kappa)$, we have, setting

$$X := \triangle_{\alpha<\kappa} X_\alpha \cup (\kappa\backslash S)$$

that $X \in \mathcal{C}^{n-1}(\kappa)$, and so $X \cap S$ is n-stationary. But

$$\begin{aligned} X &= \{\alpha < \kappa : \forall\beta < \kappa\ \alpha \in X_\beta \cup (\kappa\backslash S)\} \\ &= \{\alpha < \kappa : \alpha \in \kappa\backslash S \vee \forall\beta < \kappa\ \alpha \in X_\beta\} \\ &= (\kappa\backslash S) \cup \triangle_{\alpha<\kappa} X_\alpha \end{aligned}$$

so $X \cap S = \triangle_{\alpha<\kappa}(X_\alpha \cap S)$, and thus $\triangle_{\alpha<\kappa}(X_\alpha \cap S)$ is n-stationary and hence in F.

As we have shown that F is a normal measure, and measurables are Π^2_1-indescribable: for any $R \subseteq V_\kappa$ and φ that is Π^2_1 we have that if $\langle V_\kappa, \in R\rangle \models \varphi$ then

$$\{\alpha < \kappa : \langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi\} \in F.$$

Thus setting $R = S$ and $\varphi =$ "S is n-stationary" we have

$$\{\alpha < \kappa : S \cap \alpha \text{ is } n\text{-stationary in } \alpha\} \in F.$$

By definition of F therefore,

$$A = \{\alpha \in S : S \cap \alpha \text{ is } n\text{-stationary}\}$$

is n-stationary. But by Proposition 2.10 we have

$$A' := \{\alpha \in S : S \cap \alpha \text{ is not } n\text{-stationary}\}$$

is n-stationary. This contradicts our assumption on S as A' and A are two disjoint, n-stationary subsets of S. QED

Adding the assumption of weak compactness we can now split S into κ many pieces.

THEOREM 2.25. $(n > 0)$ *If κ is weakly compact and $\mathcal{C}^{n-1}(\kappa)$ is κ-complete then any n-stationary subset of κ can be split into κ many disjoint n-stationary sets.*

PROOF: We defer the proof to that for arbitrary γ rather than n at Theorem 3.46. QED

COROLLARY 2.26. *$(n > 0)$ Let κ be Π^1_n-indescribable. Then any $n+1$-stationary subset of κ can be split into κ many disjoint $n+1$-stationary sets.*

PROOF: For $n \geq 1$, Π^1_n-indescribability implies weak compactness and Lemma 2.21 gives us the κ-completeness of $\mathcal{C}^n(\kappa)$, so we can apply Theorem 2.25. QED

§**3. Generalising.** Jensen proved [18] that in L, a regular cardinal being weakly compact is equivalent to being stationary-reflecting, by constructing a square sequence below a non-weakly compact κ which avoids a certain stationary set. We aim to generalise this, replacing the notion of stationary set with γ-stationary set, and defining a new notion of □-sequence using γ-clubs.

Firstly, for finite n, we shall use the natural generalisation of □ notions to the context of n-clubs by defining a $\square^n$-sequence that will witness the failure of a $n+1$-stationary set to reflect. This will subsequently be generalised for transfinite $\gamma > n$.

DEFINITION 3.1. A *$\square^n$-sequence on $\Gamma \subseteq \kappa$* is a sequence $\langle C_\alpha : \alpha \in \Gamma \cap d_n(\kappa)\rangle$ such that for each α:

1. C_α is an n-club subset of α and
2. for every $\beta \in d_n(C_\alpha)$ we have $\beta \in \Gamma$ and $C_\beta = C_\alpha \cap \beta$.

We say a $\square^n$-sequence $\langle C_\alpha : \alpha \in d_n(\kappa)\rangle$ *avoids* $E \subset \kappa$ if for all α we have $E \cap d_n(C_\alpha) = \emptyset$.

So, for instance, Jensen's characterisation in [18] of non-weakly compact cardinals κ is a $\square^0$-sequence below κ which avoids a certain stationary set. We can now state our first generalisation of Jensen's Theorem:

THEOREM 3.2. $(V = L)$ *Let $n \geq 0$ and κ be a Π^1_n- but not Π^1_{n+1}-indescribable cardinal, and let $A \subseteq \kappa$ be $n+1$-stationary. Then there is $E_A \subseteq A$ and a $\square^n$-sequence S on κ such that E_A is $n+1$-stationary in κ and S avoids E_A. Consequently κ is not $n+1$-reflecting.*

COROLLARY 3.3. *Under the assumptions of the last theorem, the sequence S is non-threadable, and hence $\square^n(\kappa)$ holds.*

For a proof of the Corollary assuming the Theorem, see that of Theorem 3.50. In order to prove this theorem for $n > 0$ we shall need some technical machinery, which will be introduced in the next subsection. In section 3.2 we give the proof of the above theorem.

However, we want to generalise this result further, replacing n in the above with an arbitrary ordinal $\gamma < \kappa$. To do this, we need a concept of Π^1_γ-indescribability, which is introduced in section 3.3.1. We shall also need a new type of □-sequence, because although Definition 3.30 makes sense for infinite ordinals γ, we shall need to deal with limit cases where that definition is not adequate, an intermediate form is needed. This is the $\square^{<\gamma}$-sequence introduced in 3.3.2, and then in section 3.4 we state and prove the most general version of our theorem.

3.1. Traces and Filtrations. We define some notation for familiar concepts.

DEFINITION 3.4. For a transitive set M together with a well-ordering of M which we fix for this purpose, if $X \subseteq M$, let $M\{X\}$ be the Skolem hull of X in M using the Skolem functions defined from the (suppressed) well-ordering.

Although stated generally, we shall just use structures M which are levels of the L-hierarchy, possibly with additional relations. The well-ordering is then the standard one of the levels of L.

DEFINITION 3.5 (Trace and Filtration). Let M be a transitive set equipped with a Skolem hull operator $M\{.\}$ and let $p \in M^{<\omega}$ (we say p is a *parameter* from M) and $\alpha \in M$. The *trace* of M, p on α is the set

$$\int(M, p, \alpha) := \{\beta < \alpha : \alpha \cap M\{p \cup \beta \cup \{\alpha\}\} = \beta\}$$

The *filtration of* M, p *in* α is the sequence

$$\langle M_\beta = M\{p \cup \beta \cup \{\alpha\}\} : \beta \in \int(M, p, \alpha)\rangle$$

REMARK. Note that the filtration is *continuous* and *monotone* increasing. If α is a regular cardinal we shall have as usual the filtration is unbounded, and the trace will be club in α.

Let Γ be a class of $\mathcal{L}_\in$ formulae such that each Γ formula φ has a distinguished variable v_0 .

DEFINITION 3.6. A model $\langle M, \in\rangle$ is Γ *correct over* α if $\alpha \in M$ and for any Γ formula $\varphi(v_0, \ldots v_n)$ with all free variables displayed and $A_1, \ldots A_n$ from $\mathcal{P}(\alpha) \cap M$,

$$M \models \varphi(\alpha, A_1, \ldots A_n) \text{ iff } \varphi(\alpha, A_1, \ldots A_n).$$

REMARK. Note that if we set $\neg\Gamma = \{\neg\varphi : \varphi \in \Gamma\}$ then Γ-correctness is the same as $\neg\Gamma$-correctness. We shall initially be using this for $\Gamma = \Pi^1_n$ and later for Π^1_γ.

We can thin out the trace and filtration by requiring that the hulls collapse to transitive models which are Γ correct over $\bar{\alpha}$, where $\bar{\alpha}$ is the collapse of α. More formally:

DEFINITION 3.7. The Γ-*trace of* M, p *on* α is denoted $\int^\Gamma(M, p, \alpha)$ and consists of all $\beta < \alpha$ such that $\beta \in \int(M, p, \alpha)$ and if $\pi : M\{p \cup \beta \cup \{\alpha\}\} \cong N$ is the transitive collapse then N is Γ-correct over $\beta = \pi(\alpha)$.

We now work under the assumption $V = L$ and prove some elementary properties of traces of levels of L.

LEMMA 3.8. *1. If $\alpha < \mu < \nu$ are limit ordinals with p a parameter from L_ν, q a parameter from L_μ and $L_\nu = L_\nu\{p \cup \alpha + 1\}$ then there is some $\beta_0 < \alpha$ such that*

$$\int(L_\mu, q, \alpha) \supseteq \int(L_\nu, p, \alpha)\backslash\beta_0$$

2. If in addition we assume L_ν and L_μ are Γ-correct over α, the same holds for the Γ-trace $\int^\Gamma(L_\nu, p, \alpha)$.

PROOF: (1) As $L_\nu = L_\nu\{p \cup \alpha + 1\}$ there is $\beta_0 < \alpha$ such that $L_\mu, q \in L_\nu\{p \cup \beta_0 \cup \{\alpha\}\}$. It is easy to see that this β_0 works as $L_\mu \models \varphi(\beta) \Rightarrow L_\nu \models \varphi(\beta)^{L_\mu}$.

(2) Suppose $\beta \in \int^\Gamma(L_\nu, p, \alpha)\backslash\beta_0$. Let $L_{\bar{\nu}} \cong L_\nu\{p \cup \beta \cup \{\alpha\}\}$ and $L_{\bar{\mu}} \cong L_\mu\{q \cup \beta \cup \{\alpha\}\}$, with π and π' the respective collapsing maps. As $\beta > \beta_0$, we have $L_\mu\{q \cup \beta \cup \{\alpha\}\} \subseteq L_\nu\{p \cup \beta \cup \{\alpha\}\}$ and $\alpha \cap L_\mu\{q \cup \beta \cup \{\alpha\}\} = \beta = \alpha \cap L_\nu\{p \cup \beta \cup \{\alpha\}\}$. Thus

$$\pi' {\restriction} \mathcal{P}(\alpha) = \pi {\restriction} \mathcal{P}(\alpha) \cap L_\mu\{p \cup \beta \cup \{\alpha\}\}.$$

Suppose $\varphi(\beta, A)$ for some Γ formula φ and $A \subseteq \beta$ with $A \in L_{\bar{\mu}}$. Then by Γ-correctness $L_{\bar{\nu}} \models \varphi(\beta, A)$ and so $L_\nu \models \varphi(\alpha, \pi^{-1}(A))$. But $\pi^{-1}(A) = {\pi'}^{-1}(A)$ by the remark above, so $\pi^{-1}(A) \in L_\mu$. Then as L_ν and L_μ are both Γ-correct over α we must have $L_\mu \models \varphi(\alpha, \pi^{-1}(A))$ and hence $L_{\bar{\mu}} \models \varphi(\beta, A)$. By essentially the same argument we have the converse: if $L_{\bar{\mu}} \models \varphi(\beta, A)$ then $\varphi(\beta, A)$. Thus $L_{\bar{\mu}}$ is Γ-correct over β so $\beta \in \int^\Gamma(L_\mu, p, \alpha)$. QED

LEMMA 3.9. *If p is a parameter from L_μ with $\mu > \alpha$ and $\pi : L_\mu\{p \cup \beta \cup \{\alpha\}\} \cong L_{\bar\mu}$ with $\pi(\alpha) = \beta$ and $\pi``p = q$ then*

$$\textstyle\int(L_{\bar\mu}, q, \beta) = \beta \cap \int(L_\mu, p, \alpha).$$

The same holds for the Γ trace etc.

PROOF: Straightforward from the definitions. QED

LEMMA 3.10. *If $p \in \mathcal{P}(\alpha)^{<\omega}$ and $\nu > \alpha$ is the least limit ordinal such that (i) $p \in L_\nu$ and (ii) L_ν is Γ-correct over α, then $L_\nu = L_\nu\{p \cup \alpha + 1\}$.*
If in addition $\int^\Gamma(L_\nu, p, \alpha)$ is unbounded in α then L_ν is the union of the filtration, i.e.

$$L_\nu = \bigcup_{\beta \in \int^\Gamma(L_\nu, p, \alpha)} L_\nu\{p \cup \beta \cup \{\alpha\}\}$$

PROOF: Suppose $\pi : L_\nu\{p \cup \alpha + 1\} \cong L_{\bar\nu}$. Then $\pi``p = p$ as α is not collapsed. Also we must have $L_{\bar\nu}$ is Γ-correct: Let φ be a Γ formula with $A \in \mathcal{P}(\alpha) \cap L_{\bar\nu}$. As $\pi(\alpha) = \alpha$ and $\pi(A) = A$ we have:

$$L_{\bar\nu} \models \varphi(\alpha, A) \text{ iff } L_\nu \models \varphi(\alpha, A) \text{ iff } \varphi(\alpha, A)$$

So by the minimality of ν, we must have $\nu = \bar\nu$. The second part is an easy consequence. QED

The final, and most important lemma in this section gives a close relationship between the traces for specific classes Γ and generalised clubs. Just as, at a regular cardinal the trace will form a club, if we are at Π^1_n-indescribable cardinal then the Π^1_n-trace forms an n-club. Note that the claim here is not just that the Π^1_n-trace is in the n-club filter, but it is actually n-stationary-closed. This is important, as we shall use these Π^1_n-traces as our n-clubs when we define a $\square^n$-sequence in the next section. When in section 3.3 we define the class Π^1_γ, we shall extend this lemma to show that the Π^1_γ-trace is γ-club.

The class of Π^1_n formulae are defined here as the set of formulae of the form

$$\forall x_1 \subseteq v_0 \exists x_2 \subseteq v_0 \ldots Q x_n \psi(v_0, v_1, x_1, x_2, \ldots, x_n) \wedge v_1 \subseteq v_0$$

where ψ is Δ^0_ω and v_0 is the distinguished variable. Again, we simplify our notation by only allowing 1 parameter (the assignment of v_1). We also only quantify over subsets of α (rather than V_α) and as we are working in L this does not restrict us. Note that the Π^1_0-trace is just the trace. This is because the Π^1_0 formulae have no quantification over subsets of v_0, so as the models in the filtration are all transitive below the ordinal for which we require Π^1_0-correctness and Δ^0_ω satisfaction is absolute between transitive models, all the models in the filtration are Π^1_0- correct.

LEMMA 3.11. *If κ is Π^1_n-indescribable then for any limit $\nu > \kappa$ with L_ν Π^1_n-correct over κ we have $\int^{\Pi^1_n}(L_\nu, p, \kappa)$ is n-club in κ.*

PROOF: For each $\beta \in \int(L_\nu, p, \kappa)$ set $N_\beta = L_\nu\{p \cup \beta \cup \{\kappa\}\}$, and $\pi_\beta : N_\beta \cong L_{\nu_\beta}$. Note $\pi_\beta(\kappa) = \beta$. We first show that for n-stationary many β we have L_{ν_β} is Π^1_n-correct.

Let $A = \langle A_\alpha : \alpha < \kappa, lim(\alpha)\rangle$ enumerate the subsets of κ which occur in the filtration such that for any β in the trace, $\mathcal{P}(\kappa) \cap N_\beta$ is just some initial segment of A. As κ is regular we have on a club D that $\mathcal{P}(\kappa) \cap N_\beta = \{A_\alpha : \alpha < \beta\}$. Fix some ordering $\langle \varphi_n(v_0, v_1) : n \in \omega\rangle$ of Π^1_n formulae. Then for each limit $\alpha < \kappa$ we set

$$C_{\alpha+n} = \{\beta < \kappa : \varphi_n(\beta, A_\alpha \cap \beta) \text{ holds}\} \text{ if } \varphi_n(\kappa, A_\alpha) \text{ holds}$$

and $C_{\alpha+n} = \kappa$ otherwise. Now setting

$$C = D \cap \triangle_{\alpha<\kappa} C_\alpha$$

we have that C is n-stationary as it is the diagonal intersection of elements of the Π^1_n-indescribability filter, which is normal. We claim for each $\beta \in C$ that L_{ν_β} is Π^1_n-correct. So suppose $\beta \in C$. Then $\beta \in D$ so $\mathcal{P}(\kappa) \cup N_\beta = \{A_\alpha : \alpha < \beta\}$ and $\beta \in \triangle_{\alpha<\kappa} C_\alpha$ so for each $\alpha < \beta$ we have if $\varphi_n(\kappa, A_\alpha)$ then $\varphi_n(\beta, A_\alpha \cap \beta)$. Now we have $\pi^{-1} : L_{\nu_\beta} \to L_\nu$ is elementary, L_ν is Π^1_n-correct over κ and $\pi(\kappa) = \beta$, and for $\alpha < \beta$, $\pi(A_\alpha) = A_\alpha \cap \beta$. Thus if φ is Π^1_n and $Y \subseteq \beta$ with $Y \in L_{\nu_\beta}$ and $L_{\nu_\beta} \models \varphi(\beta, Y)$ then this is just $L_{\nu_\beta} \models \varphi_n(\beta, A_\gamma \cap \beta)$ for some $n < \omega$ and $\gamma < \beta$. Hence we have $L_\nu \models \varphi_n(\kappa, A_\gamma)$ and by Π^1_n-correctness of L_ν we have $\varphi_n(\kappa, A_\gamma)$ holds. Then as $\beta \in \triangle_{\alpha<\kappa} C_\alpha$ and $\gamma < \beta$ we have $\beta \in C_{\gamma+n}$, i.e $\varphi_n(\beta, A_\gamma \cap \beta)$ holds.

For the other direction, suppose $L_{\nu_\beta} \models \neg\varphi_n(\beta, A_\delta \cap \beta)$. This is $\exists x \subseteq \beta \psi(\beta, x, A_\delta \cap \beta)$ for some ψ which is Π^1_{n-1}. As a Π^1_{n-1} formula is also Π^1_n we have by the above that if $L_{\nu_\beta} \models \psi(\beta, x, A_\delta \cap \beta)$ then $\psi(\beta, x, A_\delta \cap \beta)$ holds, and thus so does $\exists x \subseteq \beta \psi(\beta, x, A_\delta \cap \beta)$, *i.e.* $\neg\varphi_n(\beta, A_\delta \cap \beta)$. Thus L_{ν_β} is Π^1_n-correct. So we have shown that $\int^{\Pi^1_n}(L_\nu, p, \kappa)$ is n-stationary. (This part of the argument goes through in V. We do need L to get n-stationary closure though in the next part.)

We now want to show that $\int^{\Pi^1_n}(L_\nu, p, \kappa)$ is n-stationary closed. So suppose $\beta < \kappa$ with $\int^{\Pi^1_n}(L_\nu, p, \kappa)$ n-stationary below β. We must have that β is regular: regularity of α is a Π^1_1 over V_α so as κ is regular so must be each $\alpha \in \int^{\Pi^1_n}(L_\nu, p, \kappa)$. Thus, as the regular cardinals do not form a stationary set below any singular, β must also be regular. Now as β has an n-stationary subset it must be $n-1$-stationary reflecting, and as we are in L and β is regular this means β is Π^1_{n-1}-indescribable. As $\int(L_\nu, p, \kappa)$ is unbounded below β we have $\beta \in \int(L_\nu, p, \kappa)$, so if $\beta \notin \int^{\Pi^1_n}(L_\nu, p, \kappa)$ we must have L_{ν_β} is not Π^1_n-correct. Thus for some Π^1_n sentence φ and $X \subseteq \kappa$ with $\varphi(\kappa, X)$ holding, we have $\neg\varphi(\beta, X \cap \beta)$. Fixing such φ and X we have by the Π^1_{n-1} indescribability of β that $\{\alpha < \beta : \neg\varphi(\alpha, X \cap \alpha)\}$ contains an $n-1$-club. But this cannot be, as $\int^{\Pi^1_n}(L_\nu, p, \kappa)$ is n-stationary below β. Thus we must have $\varphi(\beta, X)$ and so L_{ν_β} is Π^1_n-correct and $\beta \in \int^{\Pi^1_n}(L_\alpha, p, \kappa)$. So $\int^{\Pi^1_n}(L_\nu, p, \kappa)$ is n-stationary closed, and hence n-club. QED

3.2. The Finite Case. We can now generalise the construction of $\square$-sequences, constructing a $\square^n$-sequence below a cardinal which is Π^1_n-indescribable but not Π^1_{n+1}-indescribable. This theorem will in fact be a corollary of the more general Theorem 3.32, but we give the proof of this first, as here we can give the essence of the construction without getting caught up in too many new concepts and tricky details. In this subsection we assume $V = L$ throughout.

We say $S' = \langle C'_\alpha : \alpha \in d_n(\kappa)\rangle$ is a *refinement* of $S = \langle C_\alpha : \alpha \in d_n(\kappa)\rangle$ iff for each α we have $C'_\alpha \subseteq C_\alpha$.

THEOREM. 3.2. ($V = L$) *Let $n \geq 0$ and κ be a Π^1_n-indescribable but not Π^1_{n+1}-indescribable cardinal, and let $A \subseteq \kappa$ be $n+1$ stationary. Then there is $E_A \subseteq A$ and a $\square^n$-sequence S on κ such that E_A is $n+1$-stationary in κ and S avoids E_A.*
Thus κ is not $n+1$-reflecting.

PROOF: For $n > 0$ we produce the $\square^n$-sequence in two steps. For the first step, we define $S' = \langle C'_\alpha : \alpha \in Reg \cap d_n(\kappa)\rangle$ and show that it is a $\square^n$-sequence below κ. This does not yet depend on the particular A. Then we set

$$E_A = \{\alpha \in A \cap Reg : C'_\alpha \cap A \text{ is not } n\text{-stationary in } \alpha \text{ or } d_n(C'_\alpha) = \emptyset\}$$

and in the second step we construct a refinement of S' which avoids E_A. If we are just looking for *any* $n+1$-stationary set with a square sequence avoiding it (so we can take $A = \kappa$) then this second step is superfluous. This is because if $A = \kappa$ then $E_A = \{\alpha \in \kappa \cap Reg : \alpha \notin d_n(\kappa)\}$ as C'_α is always n-stationary by definition, and hence the coherence of S' already guarantees that S' avoids E_A. We now fix κ and and an $n+1$-stationary set $A \subseteq \kappa$.

Constructing S':

As κ is Π^1_n- but not Π^1_{n+1}-indescribable we can fix $\varphi(v_0, v_1, v_2)$ a Π^1_n formula (for ease of the construction, we use a formula with three free variables) and $Z \subseteq \kappa$ such that:

$$\forall X{\subseteq}\kappa\, \neg\varphi(\kappa, X, Z)$$

but for all $\alpha < \kappa$

$$\exists X{\subseteq}\alpha\, \varphi(\alpha, X, Z \cap \alpha)$$

Let $\alpha \in d_n(\kappa)$. We set X_α to be the $<_L$-least subset of α such that $\varphi(\alpha, X_\alpha, Z \cap \alpha)$ holds. We take $\nu_\alpha > \alpha$ to be the least limit ordinal such that L_{ν_α} is Π^1_n-correct over α and $X_\alpha, Z\cap\alpha \in L_{\nu_\alpha}$. We set $p_\alpha = \{X_\alpha, Z \cap \alpha\}$ and then we set:

$$C'_\alpha = \begin{cases} \int^{\Pi^1_n}(L_{\nu_\alpha}, p_\alpha, \alpha) & \text{if this is } n\text{-club in } \alpha \\ \text{an arbitrary non-reflecting } n\text{-stationary set otherwise} \end{cases}$$

Note that if α is n-reflecting we shall have that it is Π^1_n-indescribable and so by Lemma 3.11 we shall be in the first case. Thus C'_α is always well defined. We set

$$S' = \langle C'_\alpha : \alpha \in d_n(\kappa)\rangle$$

$$E = E_A = \{\alpha \in A \cap \mathit{Reg} : C'_\alpha \cap A \text{ is not } n\text{-stationary in } \alpha \text{ or } d_n(C'_\alpha) = \emptyset\}$$

as above.

CLAIM. *S' is a $\square^n$-sequence.*

PROOF: It is immediate from the definition that each C'_α is n-club (in the second case trivially so), so we just need to show that we have the coherence property. So let $\alpha \in d_n(\kappa)$ and suppose C'_α is defined as in the first case (otherwise $d_n(C'_\alpha) = \emptyset$ so coherence is trivial). Let $\beta \in d_n(C'_\alpha)$. Let $N_\beta = L_{\nu_\alpha}\{p_\alpha \cup \beta \cup \{\alpha\}\}$ and $\pi : N_\beta \cong L_{\bar\nu_\beta}$ be the collapsing map. We need to show $\pi``p_\alpha = p_\beta$ and $\nu_\beta = \bar\nu_\beta$.

Clearly $\pi(Z \cap \alpha) = Z \cap \beta$. Let $X = \pi(X_\alpha) = X_\alpha \cap \beta$. Then we have

$$L_{\nu_\alpha} \models \forall U \subseteq \alpha\, [U <_L X_\alpha \to \neg\varphi(\alpha, U, Z \cap \alpha)]$$

Then by elementarity we have

$$L_{\bar\nu_\beta} \models \forall U \subseteq \beta\, [U <_L X \to \neg\varphi(\beta, U, Z \cap \beta)]$$

and so by absoluteness

$$\forall U \subseteq \beta\, [U <_L X \to L_{\bar\nu_\beta} \models \neg\varphi(\beta, U, Z \cap \beta)].$$

Thus by Π^1_n-correctness of $L_{\bar\nu_\beta}$ we do not have have $X_\beta <_L X$. Also

$$L_{\bar\nu_\beta} \models \varphi(\beta, X, Z \cap \alpha)$$

and so by Π^1_n-correctness $X_\beta = X$.

It remains to show that $\nu_\beta = \bar\nu_\beta$. We already have that $p_\beta \subseteq L_{\bar\nu_\beta}$ and that $L_{\bar\nu_\beta}$ is Π^1_n-correct over β, so we only need to show the minimality requirement. Now for each limit ordinal $\gamma > \alpha$ with $\gamma < \nu_\alpha$ we have:

$$\Theta(\gamma): \quad (X_\alpha \notin L_\gamma) \vee (Z \cap \alpha \notin L_\gamma) \vee \exists U \notin L_\gamma \exists n \in \omega (U \text{ is minimal with } \psi(n, U))$$

where $\psi(\dots)$ is the universal Π^1_{n-1} sentence. A little thought will show that this is one way to formalise the requirement that ν_α is minimal.

By Π^1_n-correctness we have for each $\gamma < \nu_\alpha$ that $L_{\nu_\alpha} \models \Theta(\gamma)$ and thus

$$L_{\nu_\alpha} \models \forall\gamma\Theta(\gamma) \Rightarrow L_{\bar\nu_\beta} \models \forall\gamma\Theta(\gamma) \Rightarrow \forall\gamma < \bar\nu_\beta\;\; L_{\bar\nu_\beta} \models \Theta(\gamma)$$

So by Π^1_n-correctness we have $\Theta(\gamma)$ for each limit $\gamma < \bar\nu_\beta$, so we do indeed have $\nu_\beta = \bar\nu_\beta$. QED

We can now define a part of our system S. We set

$$\Gamma^1 = \{\alpha \in d_n(\kappa) : C'_\alpha \cap A \text{ is } n\text{-stationary in } \alpha\}.$$

If $\alpha \in \Gamma^1$ we set

$$C_\alpha = \begin{cases} d_n(C'_\alpha \cap A) & \text{if } d_n(C'_\alpha \cap A) \text{ is } n\text{-stationary} \\ \text{an arbitrary non-reflecting } n\text{-stationary set otherwise} \end{cases}$$

C_α is well-defined because as $\alpha \in \Gamma^1$ we have $C'_\alpha \cap A$ is n-stationary so if α is n-stationary reflecting then α is Π^1_n-indescribable and hence $d_n(C'_\alpha \cap A)$ must be n-club. Now it is clear that this defines a $\square^n$-sequence on Γ^1: each C_α is n-club by definition and coherence follows from the coherence of S', as if $\beta \in d_n(C'_\alpha \cap A)$ then $C'_\beta = C_\alpha \cap \beta$ and so $\beta \in \Gamma^1$. Also, each such C_α avoids E_A as either $d_n(C_\alpha) = \emptyset$ or $d_n(C_\alpha) \subseteq d_n(d_n(C'_\alpha \cap A))$.

Now we need to define C_α for $\alpha \in \Gamma^2$, where

$$\Gamma^2 = \{\alpha \in d_n(\kappa) : C'_\alpha \cap A \text{ is not } n\text{-stationary in } \alpha\} = d_n(\kappa)\backslash\Gamma^1.$$

For such α we shall find $C_\alpha \subseteq C'_\alpha$ such that (i) C_α avoids A and hence E and (ii) for $\beta \in d_n(C_\alpha)$ we have $C'_\beta \cap A$ is not n-stationary (*i.e.* $\beta \in \Gamma^2$) and $C_\beta = C_\alpha \cap \beta$. Once we have (i) and (ii) it is easy to see that $S = \langle C_\alpha : \alpha \in d_n(\kappa)\rangle$ will satisfy Theorem 3.2, and it will only remain to show that E_A is $n+1$-stationary.

For $\alpha \in \Gamma^2$ set $p'_\alpha = \{C'_\alpha, A \cap \alpha\} \cup p_\alpha$, and take $\eta_\alpha \geq \nu_\alpha$ to be the minimal limit ordinal such that $C'_\alpha \in L_{\eta_\alpha}$ and L_{η_α} is Π^1_n-correct over α. Note that if D_α is the $<_L$ least $n-1$-club avoiding $A \cap C'_\alpha$ then $D_\alpha \in L_{\eta_\alpha}$ by Π^1_n-correctness. We now set

$$C_\alpha = \begin{cases} D_\alpha \cap \int^{\Pi^1_n}(L_{\eta_\alpha}, p'_\alpha, \alpha) & \text{if this is } n\text{-club in } \alpha \\ \text{an arbitrary non-reflecting } n\text{-stationary subset of } C'_\alpha \text{ otherwise.} \end{cases}$$

Then C_α is well defined because if α reflects n-stationary sets then α is Π^1_n-indescribable. Then the Π^1_n-trace of L_{ν_α} is n-club, and hence so also is its intersection with D_α.

If C_α is defined as in the first case, then $C_\alpha \subseteq d_n(C'_\alpha)$ because for each $\beta \in \int^{\Pi^1_n}(L_{\eta_\alpha}, p'_\alpha, \alpha)$ we have $C' \cap \beta$ is n-stationary. Thus $C_\alpha \subseteq C'_\alpha$. Also because $C_\alpha \subseteq D_\alpha$ or $d_n(C_\alpha) = \emptyset$ we have (i): C_α avoids A. If $\beta \in d_n(C_\alpha)$ then C_α is defined as in the first case so $\beta \in d_n(C'_\alpha)$ and hence $C'_\beta = C'_\alpha \cap \beta$. Then we have $A \cap C'_\beta$ is not n-stationary in β by elementarity and Π^1_n-correctness, so $\beta \in \Gamma^2$. Also by elementarity and Π^1_n-correctness, $D_\beta = D_\alpha \cap \beta$ and so by Lemma 3.9

$$\int^{\Pi^1_n}(L_{\eta_\beta}, p'_\beta, \beta) = \int^{\Pi^1_n}(L_{\eta_\alpha}, p'_\alpha, \alpha) \cap \alpha.$$

Thus

$$C_\beta = D_\beta \cap \int(L_{\eta_\beta}, p'_\beta, \beta) = \beta \cap D_\alpha \cap \int(L_{\eta_\alpha}, p'_\alpha, \alpha) = C_\alpha \cap \beta.$$

So we have (ii). This completes our construction of the $\square^n$-sequence on the regulars avoiding E_A.

This defines the square sequence for regular $\alpha \in d_n(\kappa)$; for singular $\alpha \in d_n(\kappa)$ we argue as follows: as E_A has been defined, it is a set of regular cardinals, and so cannot even be stationary below a singular ordinal. Thus we can simply use Jensen's global square sequence below singulars - let $\langle D_\alpha : \alpha \in Sing\rangle$ be a global square sequence. Then for any singular α, we have $d_0(D_\alpha) \subseteq Sing$ and thus D_α avoids E_A, and of course we have coherence.

It remains to show that E_A is $n+1$-stationary.

DEFINITION 3.12. We define $H \subseteq A$ by letting $\alpha \in H$ iff $\alpha \in A$ and there is $\mu_\alpha > \alpha$ and q a parameter from L_{μ_α} such that:

1. L_{μ_α} is Π^1_n-correct over α;
2. $\mu_\alpha < \nu_\alpha$;
3. $A \cap \int^{\Pi^1_n}(L_{\mu_\alpha}, q, \alpha) = \emptyset$.

LEMMA 3.13. $H \subseteq E_A$.

PROOF: Suppose $\alpha \notin E_A$, so we have $C'_\alpha \cap A$ is n-stationary and $d_n(C'_\alpha) \neq \emptyset$. Thus C'_α was defined as in the first case: $C'_\alpha = \int(L_{\nu_\alpha}, p_\alpha, \alpha)$. Let $\mu < \nu_\alpha$ and $q \in L_\mu$. Then using lemma 3.8 for some $\beta < \alpha$ we have $q \in L_{\nu_\alpha}\{\beta \cup p_\alpha \cup \{\alpha\}\}$ and so $\int(L_\mu, q, \alpha) \backslash \beta \supseteq \int(L_{\nu_\alpha}, p_\alpha, \alpha) = C'_\alpha$. Thus $\alpha \notin H$. QED

LEMMA 3.14. *H is $n+1$-stationary.*

PROOF: Let $C \subseteq \kappa$ be n-club. Take $\mu > \kappa$ minimal such that $C, Z \in L_\mu$ and L_μ is Π^1_n-correct. Set $D = \int(L_\mu, \{C, Z\}, \kappa)$ and note that $D \subseteq d_n(C) \subseteq C$, and D is n-club by Lemma 3.11. Take $\delta = min(D \cap A)$. Set μ_δ to be the ordinal such that $L_\mu\{\{C, Z, \kappa\} \cup \delta\} \cong L_{\mu_\delta}$. We show that $\delta \in H$, with μ_δ and $\{C \cap \delta, Z \cap \delta\}$ witnessing this.

Claim 1: $\mu_\delta < \nu_\delta$.
We have that for every $X \in \mathcal{P}(\delta) \cap L_{\mu_\delta}$

$$L_{\mu_\delta} \models \neg\varphi(\delta, X, Z \cap \delta).$$

As L_{μ_δ} is Π^1_n-correct this means:

$$\forall X \in \mathcal{P}(\delta) \cap L_{\mu_\delta} \ \neg\varphi(\delta, X, Z \cap \delta).$$

But we know by the choice of ν_δ that

$$\exists X \in \mathcal{P}(\delta) \cap L_{\nu_\delta} \ \varphi(\delta, X, Z \cap \delta).$$

Hence we must have $\nu_\delta > \mu_\delta$.

Claim 2: $A \cap \int(L_{\mu_\delta}, \{C \cap \delta, Z \cap \delta\}, \delta) = \emptyset$.
We have (Lemma 3.9) $A \cap \int(L_{\mu_\delta}, \{C \cap \delta, Z \cap \delta\}, \delta) = A \cap \delta \cap \int(L_\mu, \{C, Z\}, \kappa) = A \cap \delta \cap D = \emptyset$.

To justify the final sentence of the Theorem, if we had that κ was $n+1$-reflecting, then $d_{n+1}(E_A)$ would be $n+1$-club; choosing some $\delta \in d_n(\kappa) \cap d_{n+1}(E_A)$ we should have the n-club C_δ defined, and with C_δ intersecting the $n+1$-stationary in δ set $E_A \cap \delta$. But $C_\delta \cap E_A = \emptyset$ as S avoids E_A- a contradiction. QED (Theorem 3.2)

3.3. Q_γ games, Π^1_γ-indescribability and $\square^{<\gamma}$. In this section we introduce the definitions we shall need to generalise Theorem 3.2 of the previous section: Π^1_γ-indescribability and $\square^{<\gamma}$-sequences.

3.3.1. *Π^1_γ-indescribability.* We now turn to a generalisation of the notions Π^1_n and Σ^1_n introduced in [27]; there the authors introduced Π^1_γ and Σ^1_γ classes of properties for transfinite γ to extend those of Π^1_n and Σ^1_n for finite n.[1] To our knowledge, this was the first time that such second order classes were defined game theoretically. This approach was thus also behind the concomitant notions of Π^1_η-indescribabilty. We shall use these ideas to strengthen Theorem 3.2, obtaining $\square^{<\gamma}$-sequences below cardinals which are Π^1_η-indescribable for all $\eta < \gamma$ but not Π^1_γ-indescribable. There are some comments to be made about the possible choices. The notions of γ-clubs, γ-stationarity etc., lend themselves naturally to a concept of indescribability that is more akin to the *weak* indescribability of [23], [4]. There the basic structure is some $(\kappa, \in, R)$ (Baumgartner includes a pairing function $p : \kappa^2 \to \kappa$) and the second order quantifiers have domain $\mathcal{P}(\kappa)$. Then a Π^1_n-indescribable becomes a *weakly* inaccessible cardinal with reflection properties. These were first analysed by Levy in [23]. Indescribability over $(V_\kappa, \in, R)$ for an $R \subseteq V_\kappa$ and quantifiers over $\mathcal{P}(V_\kappa)$ lead to strongly inaccessible cardinals. (And Kunen showed [20] that it was consistent that for all $n < \omega$ the least Π^1_1-indescribable cardinal - thus weakly compact - was greater than all the least weakly Π^1_n-indescribables.) The sequence of definitions and lemmata to come can be taken to conform to a 'weak' notion and be read as being second

[1] Later Bagaria in [3] introduced a definition of Π^1_γ property different to the one we present and showed that in L a cardinal is γ-stationary reflecting iff it is Π^1_γ-indescribable in his sense. We believe the proof of the following section constructing a $\square^{<\gamma}$-sequence would also work with his definition.

order statements over some $(\kappa, \in, p, R)$. However the definitions involve a game-theoretic notion. Obtaining universal Π^1_γ-formulae, and, say, the classification that being Π^1_γ-indescribable is $\Pi^1_{\gamma+1}$ expressible then involves quantifying in our setting over strategies for those games. This then is something more naturally done by using quantification over V_κ. So for this reason if we wish to analyse Π^1_γ-indescribability this far we should take the statements involving reflection of game theoretic statements as being evaluated over a $(V_\kappa, \in, R)$.

The following definitions are variants of Definitions 3.15, 3.16 and 3.21 of [27].

DEFINITION 3.15. (*The Q_α game on κ*, [27]) We define a two player game $G_\alpha(\kappa, \varphi, A)$ of finite length, with parameter $A \subseteq \kappa$ and a Δ^0_ω formula φ with three free set variables (and no other parameters), as follows.

Case 1: α is an even ordinal (including the case α is a limit).

In round n ($n \geq 1$) Player Σ plays a pair, (α_n, X_n) and player Π follows, playing Y_n, with the following constraints:

1. Each α_n is an odd ordinal and setting $\alpha_0 = \alpha$ we have $\alpha_n < \alpha_{n-1}$;
2. $X_n, Y_n \subseteq \kappa$;
3. Player Π must play Y_n such that $\varphi(\vec{X}_n, \vec{Y}_n, A)$ holds, where $\vec{X}_n = \langle X_1, \ldots, X_n\rangle$ and $\vec{Y}_n = \langle Y_1, \ldots, Y_n\rangle$.

The first player to be unable to move loses.

Case 2: α is an odd ordinal.

The game here is similar: the players switch roles but Σ still starts. Again for $n \geq 1$ player Σ plays a Y_n and player Π follows, playing (α_n, X_n), with the following constraints:

1. Each α_n is an odd ordinal and setting $\alpha_0 = \alpha$ we have $\alpha_n < \alpha_{n-1}$;
2. $X_n, Y_n \subseteq \kappa$;
3. Player Σ must play Y_n such that $\varphi(\vec{X}_{n-1}, \vec{Y}_n, A)$ holds (setting $\vec{X}_0 = \emptyset$), where $\vec{X}_{n-1} = \langle X_1, \ldots, X_{n-1}\rangle$ and $\vec{Y}_n = \langle Y_1, \ldots, Y_n\rangle$.

REMARK. (1) The decreasing sequence of ordinals ensures the games are always finite in length, so by the familiar Gale-Stewart argument they are determined - one player always has a winning strategy. So without ambiguity, we shall say "Σ wins G_α" to mean Σ has a winning strategy for G_α.

(2) We write " that $\varphi(\vec{X}_n, \vec{Y}_n, A)$ holds..." to abbreviate $(V_\kappa, \vec{X}_n, \vec{Y}_n, A) \models \varphi(\dot{X}, \dot{Y}, \dot{A})$ (although we might just as easily take this to mean that $(\kappa, p, \vec{X}_n, \vec{Y}_n, A) \models \varphi(\dot{X}, \dot{Y}, \dot{A})$ if φ were appropriate). Recall that we have definable pairing functions over any V_κ (for cardinals κ) and thus we can view $\vec{X}_n$ as a single subset of κ which can be decoded as $\langle X_1, \ldots, X_n\rangle$ &c. We can then assume φ has just the 3 second order variables shown.

(3) Again note any first order object (in other words a set or ordinal of V_κ) can be coded as a subset of κ and could be absorbed into such a sequence code $\vec{X}_n$. We thus can safely ignore fussiness about first order parameters or longer sequences of second order variables than three. We leave it to the reader to make adaptations to explicit bring out these features if they so desire. In short, in the sequel we shall write as if for a pair $A, Z \subseteq \kappa$, $\langle A, Z\rangle$ were also a subset of κ *via* the pairing function (and similarly for longer sequences without further comment).

There are a number of facts which are easy consequents of the above definitions which we give as lemmata, some of which we state without arguments but for which the reader can easily provide arguments.

LEMMA 3.16.
(1) If $\gamma < \kappa$ is odd (respectively even) and Σ (respectively Π) wins $G_\gamma(\kappa, \varphi, A)$ then

$$\forall\alpha < \gamma(\alpha \textit{ odd (resp. even) } \to \Sigma \textit{ (resp. } \Pi) \textit{ wins } G_\alpha(\kappa, \varphi, A).$$

(2) If $\gamma < \kappa$ is even (respectively odd) and Σ (respectively Π) wins $G_\gamma(\kappa, \varphi, A)$ then

$$\forall\alpha \geq \gamma(\alpha \textit{ even (resp. odd) } \to \Sigma \textit{ (resp. } \Pi) \textit{ wins } G_\alpha(\kappa, \varphi, A).$$

PROOF: This is just by observing ordinals, for example in (2), if Σ wins $G_\gamma(\kappa, \varphi, A)$ then she can win any $G_\alpha(\kappa, \varphi, A)$ for $\alpha \geq \gamma$ by using as first move in the latter game the move her strategy gives her in the former. The ordinal associated with that move is then less than $\gamma < \alpha$. The other implications are justified in the same way. QED

And similarly:

LEMMA 3.17. *Suppose $Lim(\gamma) \wedge \gamma < \kappa$. Then*
(1) Σ wins $G_\gamma(\kappa, \varphi, A)$ if and only if

$$\exists\alpha < \gamma(\alpha \textit{ even } \wedge \Sigma \textit{ wins } G_\alpha(\kappa, \varphi, A).$$

(2) Π wins $G_\gamma(\kappa, \varphi, A)$ if and only if

$$\forall\alpha < \gamma(\alpha \textit{ even } \wedge \Pi \textit{ wins } G_\alpha(\kappa, \varphi, A).$$

Let φ_o and φ_e be the following formulae (recalling that $X_0 = \emptyset$).

$$\varphi_e(\langle X_0 X_1 \cdots X_{n-1}\rangle, \langle Y_1 \cdots Y_n\rangle, \langle A, Z\rangle) \leftrightarrow \varphi(\langle Z X_1 \cdots X_{n-1}\rangle, \langle Y_1 \cdots Y_n\rangle, A)$$
$$\varphi_o(\langle X_1 \cdots X_n\rangle, \langle Y_1 \cdots Y_n\rangle, \langle A, Z\rangle) \leftrightarrow \varphi(\langle X_0 \cdots X_{n-1}\rangle, \langle Z Y_1 \cdots Y_n\rangle, A).$$

It can be then argued that with this notation:

LEMMA 3.18. *(1) If $\alpha + 1$ is even, then Σ wins $G_{\alpha+1}(\kappa, \varphi, A)$ iff there exists $Z \subseteq \kappa$ such that Π wins $G_\alpha(\kappa, \varphi_e, \langle A, Z\rangle)$.*
(2) If $\alpha + 1$ is odd, then Π wins $G_{\alpha+1}(\kappa, \varphi, A)$ iff for all $X \subseteq \kappa$ we have that Σ wins $G_\alpha(\kappa, \varphi_o, \langle A, X\rangle)$.

Using the above we have first the following reformulation of Lemma 3.17.

LEMMA 3.19. *Suppose $Lim(\gamma) \wedge \gamma < \kappa$. Then Σ wins $G_\gamma(\kappa, \varphi, A)$ if and only if*

$$\exists\alpha < \gamma\exists Z \subseteq \kappa(\alpha \textit{ odd } \wedge \Pi \textit{ wins } G_\alpha(\kappa, \varphi_e, \langle A, Z\rangle).$$

This leads to the following definitions:

DEFINITION 3.20. A property expressed by a formula of the second order language $\Phi(A)$ in the parameter $A \subseteq \kappa$ is said to be Π^1_γ (respectively Σ^1_γ) *in A, over* V_κ, for any $\gamma < \kappa$, if it is equivalent to the set-theoretical statement "Π (resp. Σ) wins the game $G_\gamma(\kappa, \varphi, A)$" for some Δ^0_ω formula φ with 3 free second order variables (for subsets of κ).

Interpreted over V_κ a Δ^0_ω formula φ has the second order variables interpreted as subsets of κ. But note that with a judicious use of pairing and unpairing functions a subset of κ may code a sequence of such subsets, but also of ordinals less than or equal to κ as well as finitely many code sets for set-parameters $x \in V_\kappa$. In this way first order truth for V_κ, can be decoded from truth for such Δ^0_ω formulae with such parameters allowed.

We could define a notion of Π^1_γ set here as the class of subsets of V_κ of the form $\{x : \Pi$ wins the game $G_\gamma(\kappa, \varphi, \langle A, x\rangle)\}$.

The following shows that this is a good candidate for a generalisation of Π^1_n for our purposes. (Again a version, *mutatis mutandis*, of this appears independently in [3].)

PROPOSITION 3.21. *The property of being γ-stationary in κ is Π^1_γ (uniformly in γ) over V_κ.*

PROOF: We consider the case γ even, the other case is only superficially different. We in fact show that generalised stationarity is *uniformly* expressible in the sense that there is a Δ^0_ω formula Θ with three free variables such that for any κ and any (even) $\gamma < \kappa$ and $A \subseteq \kappa$, Π wins $G_\gamma(\kappa, \Theta, A)$ iff A is γ-stationary in κ. We shall have that in each round i, Σ plays $\langle \alpha_i, X_i \rangle$ with $X_i = \langle \beta_i, S^1_i, S^2_i \rangle$ and Π plays $Y_i = \langle \delta_i, T^1_i, T^2_i \rangle$. In round 1, Σ tries to show that A is not γ-stationary by choosing $\beta_1 < \gamma$ and β_1-stationary $S^1_1, S^2_1 \subseteq \kappa$ such that $d_{\beta_1}(S^1_1) \cap d_{\beta_1}(S^2_1) \cap A = \emptyset$. Then Π tries to show that Σ fails, by choosing $\delta_1 < \beta_1$ and T^1_1, T^2_1 to witness that either S^1_1 or S^2_1 is not β_1-stationary. Then Σ tries to show that either T^1_1 or T^2_1 is not δ_1-stationary, and so on. The game will end when either player chooses 0. To parse the displayed formula below which will be our Θ, note that it is a disjunction of three square-bracketed expressions. The first disjunct, on the second line, has Σ asserting that the sets S^1_1, S^2_1 witness the non-β_1-stationarity of A. The third line registers Σ's attempt to show that one of Π's sets T^1_i, T^2_i is not β_{i+1}-stationary. In the second disjunct we see these sets T^1_i, T^2_i deployed by Π to attempt to show that one of Σ's sets S^1_i, S^2_i is not δ_i-stationary.

A slight adjustment has to be made as we don't want to have γ as a parameter in Θ. This is effected by the third disjunct.

$$\begin{aligned}
&\exists i \leq lh(\vec{X})\, \neg\Big[\big(X_i = \langle \beta_i, S^1_i, S^2_i \rangle \text{ with } S^1_i, S^2_i \text{ unbounded}\big)\\
&\qquad\qquad \wedge \big(d_{\beta_1}(S^1_1) \cap d_{\beta_1}(S^2_1) \cap A = \emptyset\big) \wedge \big(\beta_{i+1} < \delta_i\big)\\
&\wedge \big(d_{\beta_{i+1}}(S^1_{i+1}) \cap d_{\beta_{i+1}}(S^2_{i+1}) \cap T^1_i = \emptyset \vee d_{\beta_{i+1}}(S^1_{i+1}) \cap d_{\beta_{i+1}}(S^2_{i+1}) \cap T^2_i = \emptyset\big)\Big]\\
&\vee \forall i \leq lh(\vec{X})\Big[\big(Y_i = \langle \delta_i, T^1_i, T^2_i \rangle \text{ with } T^1_i, T^2_i \text{ unbounded}\big) \wedge (\delta_i < \beta_i)\\
&\qquad\qquad \wedge \big(d_{\delta_i}(T^1_i) \cap d_{\delta_i}(T^2_i) \cap S^1_i = \emptyset \vee d_{\delta_i}(T^1_i) \cap d_{\delta_i}(T^2_i) \cap S^2_i = \emptyset\big)\Big]\\
&\vee \forall i < lh(\vec{X})\Big[\beta_i \neq 0 \wedge \delta_i \neq 0 \wedge \beta_{i+1} \neq 0\Big]
\end{aligned}$$

This final disjunct ensures that Σ cannot "cheat" by choosing $\beta_1 \geq \gamma$ - *i.e.* for a γ-stationary A, Π has a winning strategy in $G_\gamma(\kappa, \Theta, A)$ even if $\beta_1 \geq \gamma$. Recall Σ must choose odd ordinals $\alpha_{i+1} < \alpha_i$ with $\alpha_1 < \gamma$. In the first round, Π chooses $Y_1 = \langle \alpha_1, A, A \rangle$ if $\beta_1 \geq \gamma$, noting $\alpha_1 < \gamma$ so $\alpha_1 < \beta_1$. The game then proceeds as if starting from round 2, but this time Σ cannot "cheat". The final disjunct means that it doesn't matter that Π's first move doesn't satisfy the second disjunct, unless Σ gets down to choosing $\beta_i = 0$ and S^1_i, S^2_i with S^1_i, S^2_i unbounded and $d_0(S^1_i) \cap d_0(S^2_i) \cap T^1_{i-1} = \emptyset$ or $d_0(S^1_i) \cap d_0(S^2_i) \cap T^2_{i-1} = \emptyset$ - but then Σ could have won fair and square, without starting with $\beta_1 \geq \gamma$. QED

DEFINITION 3.22. A cardinal κ is Π^1_γ-*indescribable* (respectively Σ^1_γ-*indescribable*) if for every Δ^0_ω formula φ with 3 free variables and every parameter $A \subseteq \kappa$ such that Π (resp. Σ) wins the game $G_\gamma(\kappa, \varphi, A)$ we have that there is some $\alpha < \kappa$ such that Π (resp. Σ) also wins the game $G_\gamma(\alpha, \varphi, A \cap \alpha)$.

LEMMA 3.23. *(i) A Π^1_γ property is also both a $\Pi^1_{\gamma+1}$ property, and a $\Sigma^1_{\gamma+1}$ property.*
(ii) If $\alpha < \beta < \kappa$, and κ is Π^1_β-indescribable, then it is Π^1_α-indescribable.

PROOF: We outline the ideas, but leave the details are left to the reader. (i) To show that a Π^1_γ property is also a $\Pi^1_{\gamma+1}$ property, the point is to construe a run of play of a $G_\gamma(\kappa, \varphi, A)$ game which Π wins, as one for a $G_{\gamma+1}(\kappa, \tilde{\varphi}, A)$ game, which again Π wins (and conversely). Suppose that γ is odd. The sequence of moves in $G_\gamma(\kappa, \varphi, A)$: $Y_1, (\alpha_1, X_1), Y_2, \ldots$ with the requirement on Σ that she must play Y_n such that $\varphi(\vec{X}_{n-1}, \vec{Y}_n, A)$ holds (setting $\vec{X}_0 = \emptyset$), become in $G_{\gamma+1}(\kappa, \tilde{\varphi}, A)$ a sequence of moves $(\beta_1, Y_1), X'_1, (\beta_2, Y_2), \ldots$ with $\beta_1 < \gamma + 1$ *etc.* but now X'_i is a subset of κ, and is required to be of the form $\langle \alpha_i, X_i \rangle$. The requirement $\tilde{\varphi}(\vec{Y}_n, X'_n, A)$ on Π is that for all $i \leq n$ (a) the X'_i are chosen of this form; (b) if $\varphi(\vec{X}_{i-1}, \vec{Y}_i, A)$

then Π must have chosen the ordinal components $\alpha_i < \alpha_{i-1} < \gamma$ of X_i' correctly. But Π can win this game using the same α_i and $\vec{X}$, as from $G_\gamma(\kappa,\varphi,A)$.

The argument when γ is even is similar. To show that a Π^1_γ property is also a $\Sigma^1_{\gamma+1}$ property, again suppose, *e.g.*, γ is odd. To the sequence of moves $Y_1,(\alpha_1,X_1),Y_2,\ldots$ in a $G_\gamma(\kappa,\varphi,A)$ game with the requirement on Σ that she must play Y_n such that $\varphi(\vec{X}_{n-1},\vec{Y}_n,A)$ holds (setting $\vec{X}_0=\emptyset$) we can set up a $G_{\gamma+1}(\kappa,\varphi',A)$ game with moves $(\gamma,\emptyset),Y_1,(\alpha_1,X_1),Y_2,\ldots$ and have

$$\varphi'(\langle X_1\cdots X_n\rangle,\langle Y_2\cdots Y_n\rangle,\langle Y_1,A\rangle)\equiv\varphi(\langle X_0\cdots X_n\rangle,\langle Y_1\cdots Y_n\rangle,A).$$

Then, for (ii), we first remark that $\Pi^1_{\gamma+1}$-indescribability, implies Π^1_γ-indescribability. Let $G_\gamma(\kappa,\varphi,A)$ be a γ-game that Π wins. We saw above how to recast this as a $G_{\gamma+1}(\kappa,\tilde{\varphi},A)$-game which Π also wins. By $\Pi^1_{\gamma+1}$-indescribability then, we have that Π also wins some $G_{\gamma+1}(\zeta,\tilde{\varphi},A)$. Translating back they also win the game $G_\gamma(\zeta,\varphi,A)$. This justifies Π^1_γ-indescribability. That Π^1_λ-indescribability, implies Π^1_γ-indescribability for $\gamma<\lambda$ with λ a limit, again follows by similar variations that we leave for the reader. QED

LEMMA 3.24. *(i) A cardinal κ is Π^1_γ-indescribable if and only if it is $\Sigma^1_{\gamma+1}$-indescribable; (ii) a cardinal κ is Π^1_γ-indescribable for all $\gamma<\lambda$ if and only if it is Σ^1_λ-indescribable.*

PROOF: In the substantiative direction of (i) suppose, for example, γ is odd and κ is Π^1_γ-indescribable. Let Σ have a winning strategy for $G_{\gamma+1}(\kappa,\varphi,A)$. Let $\langle\alpha_1,X_1\rangle$ be their first move according to this strategy. Then taking

$$\tilde{\varphi}(\langle X_2\cdots X_n\rangle,\langle Y_0\cdots Y_{n-1}\rangle,\langle A,X_1\rangle)\equiv\varphi(\langle X_1\cdots X_n\rangle,\langle Y_1\cdots Y_n\rangle,A)$$

we have that Π wins this $G_\gamma(\kappa,\tilde{\varphi},\langle A,X_1\rangle)$ using Σ's strategy in the $G_{\gamma+1}$ game, with the sequence of moves $Y_1,(\alpha_2,X_2),Y_2,\ldots$. Now use Π^1_γ-indescribability, and there is some $\zeta<\kappa$ with a winning strategy for Π in $G_\gamma(\zeta,\tilde{\varphi},\langle A\cap\zeta,X_1\cap\zeta\rangle)$. But then Σ has a winning strategy in $G_{\gamma+1}(\zeta,\varphi,A\cap\zeta)$ beginning with first move $(\gamma,X_1\cap\zeta)$.

(ii) ($\Leftarrow$) By (i) and Lemma 3.23 it suffices to show that for arbitrarily large odd $\gamma<\lambda$ that κ is $\Sigma^1_{\gamma+1}$-indescribable. Let $\gamma<\lambda$ be then an odd ordinal, and let $G_{\gamma+1}(\kappa,\varphi,A)$ be a game that Σ wins. Then this would also be a a $G_\lambda(\kappa,\varphi,A)$ game that Σ wins *if* Σ's first ordinal move is with an $\alpha_1<\gamma+1$. To ensure this we tweak the game $G_{\gamma+1}(\kappa,\varphi,A)$ to a game $G_{\gamma+1}(\kappa,\varphi',\langle A,\gamma\rangle)$ by requiring that Σ's first move is of the form $(\alpha_1,\langle\alpha_1,X_1\rangle)$ (with, as usual $\langle\alpha_1,X_1\rangle$ coded as some $X_1'\subseteq\kappa$), and unless $\alpha_1<\gamma$ then $\varphi'(\langle\alpha_1,X_1\rangle,Y_1,\langle A,\gamma\rangle)$ imposes no constraints whatsoever on the subsequent $\vec{Y}_i$ (which would lead to Π winning), but otherwise the constraints of $\varphi(\vec{X}_i,\vec{Y}_i,A)$ should hold. So $G_\lambda(\kappa,\varphi',\langle A,\gamma\rangle)$ is a game that Σ wins, and by Σ^1_λ-indescribability there is some $\zeta<\kappa$ so that Σ wins $G_\lambda(\zeta,\varphi',\langle A,\gamma\rangle)$ (w.l.o.g. we may assume $\gamma<\zeta$). However then by using this strategy, Σ can win $G_{\gamma+1}(\zeta,\varphi,A)$. ($\Rightarrow$) is easier, again using (i), as then for arbitrarily large even $\gamma<\lambda$ then κ is Σ^1_γ-indescribable, and we leave it as an exercise. QED

Note that a consequence of Proposition 3.21 is that any Π^1_γ-indescribable cardinal is γ-reflecting.

REMARK. Up to this point our definitions and arguments concerning Π^1_γ-indescribability could have been simply with the intention of defining a weak notion using second order quantification over $(\kappa,\in,A_1,A_2,\ldots)$ *etc.* But in the next lemma we need to quantify over strategies for games $G_\gamma(\beta,\varphi,A)$ for $\beta<\gamma$ to get that Π^1_γ-indescribability is $\Pi^1_{\gamma+1}$ expressible, which requires a quantification over V_κ and thus requires dealing with a strong indescribability notion.

LEMMA 3.25. *(i) For each γ, we have a Π^1_γ-property over V_α which is uniform in γ and for limit $\alpha>\gamma$ and A, and which is* universal *for $\Pi^1_\gamma(A)$ properties over V_α*[2]*. That is there is some $\Psi(\vec{X}_m,\vec{Y}_m,\langle A,n\rangle)\in\Delta^0_\omega$, with*

[2]Bagaria [3] also introduces a Π^1_γ formula, universal for all Π^1_γ properties in his sense.

Π *wins* $G_\gamma(\alpha, \Psi, \langle A, n\rangle)$ *iff* Π *wins* $G_\gamma(\alpha, \varphi_n, A)$.

(ii) "α is Π^1_γ-indescribable" is expressible, uniformly in γ, α, as a $\Pi^1_{\gamma+1}$-property over V_α.

PROOF: (i). We essentially appeal to the fact that there is a canonically defined predicate $Sat(R_{\vec{X}_m,\vec{Y}_m})$ which holds if and only if $R_{\vec{X}_m,\vec{Y}_m}$ is a satisfaction class for first order formulae with set parameters over $(V_\alpha, \in, \vec{X}_m, \vec{Y}_m, A)$. Thus $Sat(R_{\vec{X}_m,\vec{Y}_m})$ holds just in case $\forall n\forall\vec{x}(R_{\vec{X}_m,\vec{Y}_m}(n,\vec{x}) \leftrightarrow \varphi_n(\vec{X}_m, \vec{Y}_m, A, \vec{x}))$, where we have fixed a recursive enumeration $\langle\varphi_n : n \in \omega\rangle$ of Δ^0_ω formulae with 3 free second order variables (with first order variables for $\vec{x}$ allowed, but which we think of as coded as (bounded) subsets of κ). Then there is a first order formula $\bar{\Psi}$, so in Δ^0_ω, so that for any limit ordinal α we have $(V_\alpha, \in, \vec{X}_m, \vec{Y}_m, A) \models Sat(R_{\vec{X}_m,\vec{Y}_m})$ iff $(V_\alpha, \in, \vec{X}_m, \vec{Y}_m, A, R) \models \forall n\forall\vec{x}\bar{\Psi}(\vec{X}_m, \vec{Y_m}, A, n, \vec{x})$. (See the discussion in [13], pp.96-97 for a construction of $\bar{\Psi}$, or at [23] Thm. 8.)

Let Ψ be the Δ^0_ω formula such that:

$$\Psi(\vec{X}_m, \vec{Y_m}, \langle A, n, \vec{x}\rangle) \leftrightarrow \varphi_n(\vec{X}_m, \vec{Y}_m, A, \vec{x}).$$

Then Π wins $G_\gamma(\alpha, \Psi, \langle A, n, \vec{x}\rangle)$ iff Π wins $G_\gamma(\alpha, \varphi_n, (A, \vec{x}))$. Thus the Π^1_γ property "Π wins $G_\gamma(\alpha, \Psi, X, \langle A, n, \vec{x}\rangle)$" is universal.

(ii) In the usual way: "α is Π^1_γ-indescribable" is expressible by a $\Pi^1_{\gamma+1}$-property over V_α as:

$\forall Z[$"II wins $G_\gamma(\alpha, \Psi, Z)$" $\to \exists\beta($"II wins $G_\gamma(\beta, \Psi, Z\cap\beta))$"$]$

the consequent here being first order over (V_κ, Z), the matrix within square brackets is Σ^1_γ. QED

COROLLARY 3.26. *A cardinal κ being Π^1_η-indescribable for all $\eta < \lambda$ for limit λ, is Π^1_λ expressible.*

PROOF: Consider the following $G = G_\lambda(\kappa, \tilde{\varphi}, A)$ game which Π will win, and demonstrates that κ is Π^1_η-indescribable for all $\eta < \lambda$. From the last lemma let $G_{\gamma+1}(\kappa, \bar{\varphi}, \varnothing)$ be the game (uniform in γ) which Π wins iff κ is Π^1_γ-indescribable. In G, Σ must first play (as specified by $\bar{\varphi}$) some $(\alpha_1, \langle\alpha_1, X_1\rangle)$, and further it will be required that α_1 be odd, but greater than $\eta + 2$ where η is the largest limit less than α_1.

Then Π's first move here in G is some (any) Y_1. We consider this as just as a preamble to a $G_{\alpha_1-1}(\kappa, \bar{\varphi}, \varnothing)$ game where Σ's first move is posited as (α_2, X_2). Π's first move in this $G_{\alpha_1-1}(\kappa, \bar{\varphi}, \varnothing)$ is a response X_2 to Y_2. (Thus essentially X_1 and Y_1 are discarded.) Now Π wins this game iff κ is $\Pi^1_{\alpha_1-2}$-indescribable. Consequently Π wins G if and only if it is the case that for any odd $\alpha + 1 < \lambda$ of this form, we have that κ is $\Pi^1_{\alpha+1}$-indescribable. By Lemma 3.24 this means it is in fact Π^1_α-indescribable for all $\alpha < \lambda$. QED

Note that, by Lemma 3.24, being $\Sigma^1_{\gamma+1}$-indescribable is $\Pi^1_{\gamma+1}$-expressible. The Π^1_γ-indescribability filter is defined in the obvious way. (Independently from this, [3] defines in Section 4.1 a similar filter analogous to the notion of indescribability given there.)

DEFINITION 3.27. The Π^1_γ-*indescribability filter* on κ, $\mathcal{F}^\gamma_\kappa$, for $\gamma < \kappa$, is the filter generated by the sets $C_{\varphi,A}$, where $A \subseteq \kappa$, $\varphi(X_0, X_1, X_2) \in \Delta^0_\omega$ has 3 free variables and:

$$C_{\varphi,A} =_{df} \{\alpha < \kappa : \Pi \textit{ wins both } G_\gamma(\kappa, \varphi, A) \textit{ and } G_\gamma(\alpha, \varphi, A \cap \alpha)\}.$$

REMARK. As being γ-stationary is Π^1_γ expressible, and the collection of Π^1_γ properties is closed under conjunction, each $F \in \mathcal{F}^\kappa_\gamma$ is $\gamma + 1$-stationary.

The proof of the following Lemma is similar to that of Levy for the case when $\gamma = n < \omega$. We give the proof here for the sake of completeness. (A similar version of this appears independently, in [3], see Prop 4.4.)

LEMMA 3.28. *If κ is Π^1_γ-indescribable, for some $\gamma < \kappa$, then the Π^1_γ-indescribable filter is normal and κ-complete.*

PROOF: Fix an ordinal γ and $\kappa > \gamma$ with κ Π^1_γ-indescribable, and let $X \subseteq \kappa$ be positive with respect to $\mathcal{F}^\gamma_\kappa$ and $f : X \to \kappa$ be regressive. Suppose for a contradiction that for each $\beta < \kappa$ there is an element of $\mathcal{F}^\gamma_\kappa$ that avoids f^{-1}"$\{\beta\}$. For each $\beta < \kappa$ choose n_β and $A_\beta \subseteq \kappa$ witnessing this, *i.e.* such that Π wins $G_\gamma(\kappa, \varphi_{n_\beta}, A_\beta)$ but if $f(\alpha) = \beta$ then Σ wins $G_\gamma(\alpha, \varphi_{n_\beta}, A_\beta)$. Thus for each $\alpha \in X$ we have

$$(*) \qquad \Sigma \text{ wins } G_\gamma(\alpha, \varphi_{n_{f(\alpha)}}, A_{f(\alpha)})$$

Claim: "$\forall \beta < \kappa\ \Pi$ wins $G_\gamma(\beta, \varphi_{n_\beta}, A_\beta)$" is Π^1_γ.

Set $A = \{\langle \beta, 0, \delta\rangle : \delta \in A_\beta\} \cup \{\langle \beta, 1, n_\beta\rangle\}$, so that $A \subseteq \kappa$ codes A_β and n_β for each $\beta \in \kappa$. The following Δ^0_ω sentence Φ with parameter A produces an appropriate game: X_0 is / is not a pair $X_0 = \langle \beta, X_0'\rangle$ with $\beta \in \kappa$, $X_0' \subseteq \kappa$ and/or $\Psi(\beta, X_0' {}^\frown \vec{X}_m, \vec{Y}_m, \langle n_\beta, A_\beta\rangle)$, with "is ... and" if γ is odd and "is not ... or" if γ is even, and ψ is universal as in Lemma 3.25.

Now we have Π wins $G_\gamma(\kappa, \Phi, A)$ so by the Π^1_γ-indescribability of κ, for some $\alpha \in X$ we have Π wins $G_\gamma(\alpha, \Phi, A \cap \alpha)$. Thus, by definition of Φ and the fact that f is regressive, Π wins $G_\gamma(\alpha, \varphi_{n_{f(\alpha)}}, A_{f(\alpha)})$. But this contradicts $(*)$. QED

Now we have introduced these notions, we apply them to prove the analogue of Lemma 3.11 for Π^1_γ. The proof is essentially the same as before, except that in the proof of Lemma 3.11 we used the fact proven in [2] that in L, any regular cardinal which reflects n-stationary sets, and hence any which admits $n+1$-stationary sets, is Π^1_n-indescribable. As we have not yet shown this for γ-stationary sets and Π^1_γ-indescribability, this lemma must be proven inductively along with Theorem 3.32.

LEMMA 3.29. $(V = L)$ *Let γ be an ordinal and assume that for all $\gamma' < \gamma$ we have that any regular cardinal which is γ'-reflecting is $\Pi^1_{\gamma'}$-indescribable. If $\kappa > \gamma$ is Π^1_γ-indescribable then for any limit $\nu > \kappa$ such that L_ν is Π^1_γ-correct over κ, we have $\int^{\Pi^1_\gamma}(L_\nu, p, \kappa)$ is γ-club in κ.*

PROOF: We proceed as in 3.11. For each $\beta \in \int(L_\nu, p, \kappa)$ set $N_\beta = L_\nu\{p \cup \beta \cup \{\kappa\}\}$, and $\pi_\beta : N_\beta \cong L_{\nu_\beta}$. Note $\pi_\beta(\kappa) = \beta$. We first show that for γ-stationary many β we have L_{ν_β} is Π^1_γ-correct.

Let $A = \langle A_\alpha : \alpha < \kappa \wedge lim(\alpha)\rangle$ enumerate the subsets of κ which occur in the filtration such that for any β in the trace, $\mathcal{P}(\kappa) \cap N_\beta$ is just some initial segment of A. As κ is regular we have on a club D that $\mathcal{P}(\kappa) \cap N_\beta = \{A_\alpha : \alpha < \beta\}$. Fix some ordering $\langle \varphi_n(v_0, v_1, v_2) : n \in \omega\rangle$ of Δ^0_ω formulae with all free variables displayed. Then for each limit $\alpha < \kappa$ we set

$$C_{\alpha+n} = \begin{cases} \{\beta < \kappa : \Pi \text{ wins } G_\gamma(\beta, \varphi_n, A_\alpha \cap \beta)\} & \text{if } \Pi \text{ wins } G_\gamma(\beta, \varphi_n, A_\alpha) \\ \kappa \text{ otherwise.} & \end{cases}$$

Now setting

$$C = D \cap \triangle_{\alpha<\kappa} C_\alpha$$

we have that C is γ-stationary as it is the diagonal intersection of elements of the Π^1_γ-indescribability filter, which is normal by Lemma 3.28. We claim for each $\beta \in C$ that L_{ν_β} is Π^1_γ-correct. So suppose $\beta \in C$. Then $\beta \in D$ so $\mathcal{P}(\kappa) \cup N_\beta = \{A_\alpha : \alpha < \beta\}$ and $\beta \in \triangle_{\alpha<\kappa} C_\alpha$ so for each $\alpha < \beta$ we have if Π wins $G_\gamma(\kappa, \varphi_n, A_\alpha)$ then Π wins $G_\gamma(\beta, \varphi_n, A_\alpha \cap \beta)$. Now we have $\pi^{-1} : L_{\nu_\beta} \to L_\nu$ is elementary, L_ν is Π^1_γ-correct over κ and $\pi(\kappa) = \beta$, $\pi(A_\alpha) = A_\alpha \cap \beta$ for $\alpha < \beta$. Thus if ψ is Π^1_γ then $L_{\nu_\beta} \vDash \psi(\beta, X)$ is just $L_{\nu_\beta} \vDash \Pi$ wins $G_\gamma(\beta, \varphi_n, A_\delta \cap \beta)$ for some $n < \omega$ and $\delta < \beta$. Hence we have $L_\nu \vDash \Pi$ wins $G_\gamma(\kappa, \varphi_n, A_\delta)$ and by Π^1_γ-correctness Π does indeed win $G_\gamma(\kappa, \varphi_n, A_\delta)$. Then as $\beta \in \triangle_{\alpha<\kappa} C_\alpha$ and $\delta < \beta$ we have $\beta \in C_{\delta+n}$, *i.e.* Π wins $G_\gamma(\beta, \varphi_n, A_\delta \cap \beta)$.

As for $\gamma' < \gamma$, $\Pi^1_{\gamma'}$ properties are also Π^1_γ, we have the other direction by induction. Thus L_{ν_β} is Π^1_γ-correct. So we have shown that $\int^{\Pi^1_n}(L_\nu, p, \kappa)$ is γ-stationary. (Again, this part of the argument goes through in V, but we need L to get γ-stationary closure.)

We show that $\int^{\Pi^1_n}(L_\nu, p, \kappa)$ is γ-stationary closed. Suppose $\beta < \kappa$ with $\int^{\Pi^1_\gamma}(L_\nu, p, \kappa)$ γ-stationary below β. As in 3.11, β must be regular. Now as β has a γ-stationary subset it must be γ'-stationary reflecting for every $\gamma' < \gamma$, and as we are in L and β is regular by our hypothesis we have β is Σ^1_γ-indescribable. As $\int(L_\nu, p, \kappa)$ is unbounded below β we have $\beta \in \int(L_\nu, p, \kappa)$, so if $\beta \notin \int^{\Pi^1_\gamma}(L_\nu, p, \kappa)$ we must have L_{ν_β} is not Π^1_γ-correct. Thus for some Δ^0_ω formula φ with Π wins $G_\gamma(\kappa, \varphi, X)$ we have Σ wins $G_\gamma(\beta, \varphi, X \cap \beta)$. This means by the remark above that for some $Y \subseteq \beta$ and $\gamma' < \gamma$ we have Π wins $G_{\gamma'}(\beta, \varphi', \langle X \cap \beta, Y\rangle)$. By the $\Pi^1_{\gamma'}$-indescribability of β, $\{\alpha < \beta : \Pi \text{ wins } G_{\gamma'}(\alpha, \varphi', \langle X \cap \alpha, Y \cap \alpha\rangle)\}$ contains an γ'-club. But if Π wins $G_{\gamma'}(\alpha, \varphi', \langle X \cap \alpha, Y \cap \alpha\rangle)$ then Σ wins $G_\gamma(\alpha, \varphi, X \cap \alpha)$ and so $\alpha \notin \int^{\Pi^1_\gamma}(L_\nu, p, \kappa)$.

But this cannot be, as $\int^{\Pi^1_\gamma}(L_\nu, p, \kappa)$ is γ-stationary below β. Thus we must have Π wins $G_\gamma(\beta, \varphi, X \cap \beta)$ and so L_{ν_β} is Π^1_γ-correct[3] and $\beta \in \int^{\Pi^1_\gamma}(L_\alpha, p, \kappa)$. So $\int^{\Pi^1_\gamma}(L_\nu, p, \kappa)$ is γ-stationary closed, and hence γ-club. QED

3.3.2. $\square^\gamma$ *and* $\square^{<\gamma}$. Definition 3.1 (defining a $\square^n$-sequence) can easily be stated for general ordinals γ as follows.

DEFINITION 3.30. Let $\gamma < \kappa$ be ordinals. A $\square^\gamma$*-sequence on* κ is a sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ such that for each α:

1. C_α is a γ-club subset of α and
2. for every $\beta \in d_\gamma(C_\alpha)$ we have $C_\beta = C_\alpha \cap \beta$.

However, this is not the type of $\square$-sequence we shall need to take Theorem 3.2 into the transfinite. The problem here is the limit levels. To continue inductively along the ordinals, we first need to show that, in L, if a regular cardinal is Π^1_n-indescribable for every n, but not Π^1_ω-indescribable, then it does not reflect ω-stationary sets, and similarly for further limit ordinals. To witness this, we need to define a new type of $\square$-sequence. The following definition yields a sequence of an intermediate nature, that for example in the case of $\gamma = \omega$ we are discussing, avoids an ω-stationary set A, and witnesses that A does not reflect.

DEFINITION 3.31. Let $\gamma < \kappa$ be ordinals. A $\square^{<\gamma}$ *sequence* on $\Gamma \subseteq LimOrd \cap \kappa$ is a sequence $\langle (C_\alpha, \eta_\alpha) : \alpha \in \Gamma\rangle$ such that for each α:

1. $\eta_\alpha < \gamma$ and C_α is an η_α-club subset of α
2. for every $\beta \in d_{\eta_\alpha}(C_\alpha)$ we have $\beta \in \Gamma$ with $\eta_\alpha = \eta_\beta$ and $C_\beta = C_\alpha \cap \beta$

We say a $\square^{<\gamma}$-sequence $\langle C_\alpha : \alpha \in \kappa\rangle$ *avoids* $A \subset \kappa$ if for all $\alpha \in \kappa$ we have $A \cap d_{\eta_\alpha}(C_\alpha) = \emptyset$.

We say $S' = \langle (C'_\alpha, \eta'_\alpha) : \alpha \in \Gamma\rangle$ is a *refinement* of $S = \langle (C_\alpha, \eta_\alpha) : \alpha \in \Gamma\rangle$ iff for each α we have $\eta'_\alpha = \eta_\alpha$ and $C'_\alpha \subseteq C_\alpha$.

Equipped with this definition, we can see that a $\square^{<\gamma}$-sequence avoiding some γ-stationary set A witnesses that A does not reflect (*i.e.* $d_\gamma(A) = \emptyset$), even in the case γ is a limit. This is because, for $\alpha \in \kappa$, C_α is an η_α-club avoiding $A \cap \alpha$ and thus A is not $\eta_\alpha + 1$-stationary in α, and as $\eta_\alpha + 1 \leq \gamma$ we have $\alpha \notin d_\gamma(A)$. As this works for both limit and successor γ, we shall use only $\square^{<\gamma}$ in our proof of Theorem 3.32 and thus deal with the limit and successor cases simultaneously. The analogue of Theorem 3.2 for the infinite ordinals is a corollary: restricting the domain of a $\square^{<\gamma+1}$-sequence to the ordinals in $d_\gamma(\kappa)$ gives a $\square^\gamma$-sequence.

[3] The other direction follows by upwards absoluteness of Σ_1 formulae.

3.4. The Main Result: $\square^{<\gamma}$ at a Non-Π^1_γ-Indescribable. We are now ready to state and prove the main theorem of this section:

THEOREM 3.32. *($V = L$) Let $\gamma < \kappa$ be an ordinal and κ be Π^1_η-indescribable for every $\eta < \gamma$ but not Π^1_γ-indescribable, and let $A \subseteq \kappa$ be γ-stationary. Then there is $E_A \subseteq A$ and a $\square^{<\gamma}$-sequence S on κ such that E_A is γ-stationary in κ and S avoids E_A. Consequently κ is not γ-reflecting.*

REMARK. By Cor. 3.24, we could rephrase the antecedent as requiring that κ be Σ^1_γ-indescribable, but not Π^1_γ-indescribable.

COROLLARY 3.33. *($V = L$) The following are equivalent for a regular cardinal $\kappa > \omega$ and ordinals $0 < \gamma < \kappa$:*
(i) κ is Π^1_γ-indescribable;
(ii) there is no $\square^{<\gamma}$ sequence on κ that avoids a γ-stationary $E \subseteq \kappa$;
(iii) κ is γ-reflecting.

PROOF OF COROLLARY FROM THE THEOREM: We only need to show that (i) $\Longrightarrow$ (iii). By Prop.3.21 we have that being γ-stationary is Π^1_γ-expressible over V_κ. If then κ is Π^1_γ-indescribable, then arguing as in Lemma 2.20 we have any two γ-stationary sets in κ reflect simultaneously to some point $\alpha < \kappa$. Hence κ is γ-reflecting. QED

PROOF OF THEOREM: We assume $V = L$. The proof is an induction on γ making repeated use of Lemma 3.29. Thus we fix γ and assume we have that for any $\gamma' < \gamma$, if α is γ'-reflecting and regular then α is $\Pi^1_{\gamma'}$-indescribable.

As before, we produce the $\square^{<\gamma}$-sequence in two steps: first we define

$$S' = \langle (C'_\alpha, \eta_\alpha) : \alpha \in Reg \cap \kappa \rangle$$

which is a $\square^{<\gamma}$-sequence below κ, then we set

$$E_A = \{\alpha \in A \cap Reg : C'_\alpha \cap A \text{ is not } \eta_\alpha\text{-stationary in } \alpha \text{ or } d_{\eta_\alpha}(C'_\alpha) = \emptyset\}$$

and we construct a refinement of S' which avoids E_A. To streamline notation for this proof, we shall abbreviate $\int^{\Pi^1_\gamma}$ to $\int^\gamma$.

Constructing S':

As κ is not Π^1_γ-indescribable we can fix a Δ^0_ω formula $\varphi(v_0, v_1, v_2)$ and $Z \subseteq \kappa$ such that:

$$\Pi \text{ wins } G_\gamma(\kappa, \varphi, Z)$$

but for all $\alpha < \kappa$

$$\Sigma \text{ wins } G_\gamma(\alpha, \varphi, Z \cap \alpha)$$

and so for each $\alpha < \kappa$

$$\exists \eta < \gamma, X \subseteq \alpha \text{ such that } \Pi \text{ wins } G_\eta(\alpha, \varphi', \langle Z \cap \alpha, X \rangle).$$

Let $\alpha \in \kappa$. Let $\bar{\eta}_\alpha$ be be maximal such that $A \cap \alpha$ is $\bar{\eta}_\alpha$-stationary (or 0, if A is not even unbounded), and let D_α be the $<_L$ least $\bar{\eta}_\alpha$-club avoiding $A \cap \alpha$. We have Σ wins $G_\gamma(\alpha, \varphi, Z \cap \alpha)$, so we set Y_α to be $<_L$-least subset of α such that Σ wins with first move Y_α, if γ is odd, and with first move $\langle \eta, Y_\alpha \rangle$ with η minimal, if γ is even. We split into cases and define η_α and X_α:

Case (i): γ is odd, $A \cap \alpha$ is $\gamma - 1$-stationary, and if $\bar{\eta}_\alpha = \gamma - 1$ we have $Y_\alpha <_L D_\alpha$. Set $\eta_\alpha = \gamma - 1$ and $X_\alpha = Y_\alpha$.

Case (ii): If γ is even, let η be least (odd) ordinal such that Σ wins the game $G_\gamma(\alpha, \varphi, Z)$ with first ordinal move η (choosing $\alpha_1 = \eta$), *i.e.* ,

$$\Sigma \text{ wins } G_{\eta+1}(\alpha, \varphi, Z) \text{ but for any for any even } \eta' < \eta, \Pi \text{ wins } G_{\eta'}(\alpha, \varphi, Z).$$

We put α under case (ii) if $A \cap \alpha$ is η-stationary and if $\bar{\eta}_\alpha = \eta$ then $Y_\alpha <_L D_\alpha$. In this case, set $\eta_\alpha = \eta$ and $X_\alpha = Y_\alpha$.

Case (iii): Otherwise: set $\eta_\alpha = \bar{\eta}_\alpha$ and set $X_\alpha = D_\alpha$. Note that if α falls into this case then $\eta_\alpha < \gamma$, as if $\bar{\eta}_\alpha \geq \gamma$ then $A \cap \alpha$ is γ-stationary so the conditions of (i) or (ii) are fulfilled.

We take $\nu_\alpha > \alpha$ to be the least limit ordinal such that L_{ν_α} is $\Pi^1_{\eta_\alpha}$ correct over α and $X_\alpha, A \cap \alpha, Z \cap \alpha \in L_{\nu_\alpha}$. We set $p_\alpha = \{X_\alpha, A \cap \alpha, Z \cap \alpha\}$ and then we set:

$$C'_\alpha = \begin{cases} \int^{\eta_\alpha}(L_{\nu_\alpha}, p_\alpha, \alpha) & \text{if this is } \eta_\alpha\text{-club in } \alpha \\ \text{an arbitrary non-reflecting } \eta_\alpha\text{-stationary set otherwise} \end{cases}$$

This is well defined as we know α is η_α stationary and a cardinal, and so if the trace is not η_α-club we must have that α is not $\Pi^1_{\eta_\alpha}$-indescribable so a non-reflecting set can be found.

We set $S' = \langle (C'_\alpha, \eta_\alpha) : \alpha \in \kappa \cap Reg \rangle$.

CLAIM. *S' is a $\square^{<\gamma}$-sequence.*

PROOF: It is immediate from the definition that each C'_α is η_α-club, so we just need to show that we have the coherence property. So let $\alpha < \kappa$ be regular and suppose C'_α is defined as the trace (otherwise $d_{\eta_\alpha}(C'_\alpha) = \emptyset$ so coherence is trivial). Let $\beta \in d_{\eta_\alpha}(C'_\alpha)$. Let $N_\beta = L_{\nu_\alpha}\{p_\alpha \cup \beta \cup \{\alpha\}\}$ and $\pi : N_\beta \cong L_{\bar{\nu}_\beta}$ be the collapsing map. We need to show $\eta_\beta = \eta_\alpha$, $\pi``p_\alpha = p_\beta$ and $\nu_\beta = \bar{\nu}_\beta$.

Clearly $\pi(A \cap \alpha) = A \cap \beta$ and $\pi(Z \cap \alpha) = Z \cap \beta$. Let $X = \pi(X_\alpha) = X_\alpha \cap \beta$.

If α falls into case (i) then γ is odd and

$$L_{\nu_\alpha} \models \text{“}A \cap \alpha \text{ is } \eta_\alpha\text{-stationary and for any } D <_L X \text{ if } D \text{ is } \eta_\alpha\text{-club then } A \cap d_{\eta_\alpha}(D) \neq \emptyset\text{”}$$

and so by elementarity

$$L_{\bar{\nu}_\beta} \models \text{“}A \cap \beta \text{ is } \eta_\alpha\text{-stationary and for any } D <_L X \text{ if } D \text{ is } \eta_\alpha\text{-club then } A \cap d_{\eta_\alpha}(D) \neq \emptyset\text{”}$$

thus by $\Pi^1_{\eta_\alpha}$ correctness of $L_{\bar{\nu}_\beta}$ we have that we are again in case (i) with $\eta_\beta = \eta_\alpha$.

If α falls into case (ii), we have

$$L_{\nu_\alpha} \models \text{“}\Sigma \text{ wins } G_{\eta_\alpha+1}(\alpha, \varphi, \langle Z \cap \alpha \rangle) \text{ with first move } X_\alpha\text{”}$$

i.e.

$$L_{\nu_\alpha} \models \text{“}\Pi \text{ wins } G_{\eta_\alpha}(\alpha, \varphi', \langle Z \cap \alpha, X_\alpha \rangle)\text{”}.$$

Now by elementarity,

$$\begin{aligned} L_{\bar{\nu}_\beta} \models &\text{“}\Pi \text{ wins } G_{\eta_\alpha}(\beta, \varphi', \langle Z \cap \beta, X \rangle), A \cap \beta \text{ is } \eta_\alpha\text{-stationary} \\ &\text{and for any } D <_L X \text{ if } D \text{ is } \eta_\alpha\text{-club then } A \cap d_{\eta_\alpha}(D) \neq \emptyset.\text{”} \end{aligned}$$

As this is a $\Pi^1_{\eta_\alpha}$ statement we have Π wins $G_{\eta_\alpha}(\beta, \varphi', \langle Z \cap \beta, X \rangle)$ by $\Pi^1_{\eta_\alpha}$-correctness of $L_{\bar{\nu}_\beta}$, and thus Σ wins $G_{\eta_\alpha+1}(\beta, \varphi, Z \cap \alpha)$ with first move X. We also have $A \cap \beta$ is η_α-stationary etc. and so β falls into case (ii) and $\eta_\beta \leq \eta_\alpha$.

For even $\eta \leq \eta_\alpha$ we have $L_{\nu_\alpha} \models \Pi$ wins $G_\eta(\alpha, \varphi, \langle Z \cap \alpha \rangle)$ and so by elementarity and $\Pi^1_{\eta_\alpha}$-correctness, for such η we have Π wins $G_\eta(\beta, \varphi, \langle Z \cap \beta \rangle)$ and so we must have $\eta < \eta_\beta$. Thus $\eta_\beta = \eta_\alpha$ in this case.

Finally, if α falls into case (iii) then we have $\eta_\alpha = \bar{\eta}_\alpha$ and $X = \pi(D_\alpha)$ so by definition of the case and elementarity,

$$\begin{aligned} L_{\bar{\nu}_\beta} \models &\text{“}A \text{ is } \eta_\alpha\text{-stationary and } X \text{ is } \eta_\alpha\text{-club avoiding } A \text{ and for all } Y <_L X, \\ &\Sigma \text{ does not win } G_{\eta_\alpha+1}(\beta, \varphi, \langle Z \cap \beta \rangle) \text{ with first move } Y.\text{”} \end{aligned}$$

Thus by $\Pi^1_{\eta_\alpha}$-correctness we are in case (iii) and $\eta_\alpha = \eta_\beta$.

Now in all cases $\eta_\beta = \eta_\alpha$, so we set $\eta = \eta_\alpha = \eta_\beta$. To show $X = X_\beta$: we have already seen that β falls into the same case as α. For case (i) or (ii) we have

$$L_{\nu_\alpha} \models \forall U \subseteq \alpha\,[U <_L X_\alpha \rightarrow \Sigma \text{ wins } G_\eta(\alpha, \varphi', \langle Z \cap \alpha, U\rangle)]$$

Then by elementarity we have:

$$L_{\bar{\nu}_\beta} \models \forall U \subseteq \beta\,[U <_L X \rightarrow \Sigma \text{ wins } G_\eta(\beta, \varphi', \langle Z \cap \beta, U\rangle)]$$

and so by absoluteness

$$\forall U \subseteq \beta\,[U <_L X \rightarrow L_{\bar{\nu}_\beta} \models \Sigma \text{ wins } G_\eta(\beta, \varphi', \langle Z \cap \beta, U\rangle)]$$

Thus by Π^1_η-correctness of $L_{\bar{\nu}_\beta}$ we do not have have $X_\beta <_L X$. Also

$$L_{\bar{\nu}_\beta} \models \Pi \text{ wins } G_\eta(\beta, \varphi', \langle Z \cap \beta, X\rangle)$$

and so by Π^1_η-correctness $X_\beta = X$. In case (iii), we can repeat the same argument but instead of φ take the sentence giving us that $A \cap \alpha$ is not $\eta_\alpha + 1$-stationary.

It remains to show that $\nu_\beta = \bar{\nu}_\beta$. We already have that $p_\beta \subseteq L_{\bar{\nu}_\beta}$ and that $L_{\bar{\nu}_\beta}$ is Π^1_η-correct over β, so we only need to show the minimality requirement. Now for each limit ordinal $\gamma > \alpha$ with $\gamma < \nu_\alpha$ we have:

$$\begin{aligned}\Theta(\gamma) :& \big(X_\alpha \notin L_\gamma\big) \vee \big(A \cap \alpha \notin L_\gamma\big) \vee \big(Z \cap \alpha \notin L_\gamma\big)\\ &\vee \exists U \notin L_\gamma\ \exists n \in \omega\ \exists \eta' < \eta\ \big(U \text{ is minimal with } \psi_{\eta'}(n, U)\big)\end{aligned}$$

where $\psi_\eta(.,.)$ is the universal Π^1_η sentence. We finish off as before. By Π^1_η-correctness we have for each $\gamma < \nu_\alpha$ that $L_{\nu_\alpha} \models \Theta(\gamma)$ and thus

$$L_{\nu_\alpha} \models \forall\gamma\Theta(\gamma) \Rightarrow L_{\bar{\nu}_\beta} \models \forall\gamma\Theta(\gamma) \Rightarrow \forall\gamma < \bar{\nu}_\beta\ \ L_{\bar{\nu}_\beta} \models \Theta(\gamma)$$

So by Π^1_η-correctness of $L_{\bar{\nu}_\beta}$ we have $\Theta(\gamma)$ for each limit $\gamma < \bar{\nu}_\beta$, and $\nu_\beta = \bar{\nu}_\beta$. QED

We can now define a part of our $\square^{<\gamma}$-sequence that will satisfy Theorem 3.32. This will be a refinement of S', so the η_α's that were defined above will stay the same. We set

$$\Gamma^1 = \{\alpha \in \kappa : C'_\alpha \cap A \text{ is } \eta_\alpha\text{-stationary in } \alpha\}.$$

If $\alpha \in \Gamma^1$ we set

$$C_\alpha = \begin{cases} d_{\eta_\alpha}(C'_\alpha \cap A) & \text{if } d_{\eta_\alpha}(C'_\alpha \cap A) \text{ is } \eta_\alpha\text{-stationary} \\ \text{an arbitrary non-reflecting } \eta_\alpha\text{-stationary set otherwise.} \end{cases}$$

It is clear that $\langle (C_\alpha, \eta_\alpha) : \alpha \in \Gamma^1\rangle$ is a $\square^{<\gamma}$-sequence on Γ^1. Setting

$$E = E_A = \{\alpha \in A \cap Reg : C'_\alpha \cap A \text{ is not } \eta_\alpha\text{-stationary in } \alpha \text{ or } d_{\eta_\alpha}(C'_\alpha) = \emptyset\}$$

we show that each such C_α avoids E. If $d_{\eta_\alpha}(C_\alpha) \neq \emptyset$ then for $\beta \in d_{\eta_\alpha}(C_\alpha)$ we have $d_{\eta_\alpha}(C'_\alpha \cap A)$ is η_α-stationary below β, so $C'_\beta = C'_\alpha \cap \beta$ with $\eta_\alpha = \eta_\beta$ and $d_{\eta_\beta}(C_\beta) \neq \emptyset$, and by Lemma 2.8 $C'_\alpha \cap A = C'_\beta \cap A$ is $\eta_\alpha = \eta_\beta$-stationary below β.

Now we need to define C_α for $\alpha \in \Gamma^2 = \{\alpha \in \kappa : C'_\alpha \cap A \text{ is not } \eta_\alpha\text{-stationary in } \alpha\} = \kappa \backslash \Gamma^1$. For such α we shall find $C_\alpha \subseteq C'_\alpha$ such that (i) C_α avoids A and hence E and (ii) for $\beta \in d_{\eta_\alpha}(C_\alpha)$ we have $C'_\beta \cap A$ is not η_α-stationary (*i.e.* $\beta \in \Gamma^2$) and $C_\beta = C_\alpha \cap \beta$ (we shall already have $\eta_\beta = \eta_\alpha$ as $C_\alpha \subseteq C'_\alpha$). Once we have (i) and (ii) it is easy to see that $S = \langle C_\alpha : \alpha \in \kappa\rangle$ will satisfy Theorem 3.32, and it will only remain to show that E is γ-stationary.

For $\alpha \in \Gamma^2$ we take $\rho_\alpha \geq \nu_\alpha$ minimal limit ordinal such that $C'_\alpha \in L_{\rho_\alpha}$ and L_{ρ_α} is $\Pi^1_{\eta_\alpha}$-correct over α. Note that if D_α is the $<_L$ least η-club, for some $\eta < \eta_\alpha$, that avoids $A \cap C'_\alpha$ then $D_\alpha \in L_{\rho_\alpha}$ by $\Pi^1_{\eta_\alpha}$-correctness. We now set

$$C_\alpha = \begin{cases} D_\alpha \cap \int^{\eta_\alpha}(L_{\rho_\alpha}, \{C'_\alpha\} \cup p_\alpha, \alpha) & \text{if this is } \eta_\alpha\text{-club in } \alpha \\ \text{an arbitrary non-reflecting } \eta_\alpha\text{-stationary set otherwise.} \end{cases}$$

Then we have $C_\alpha \subseteq d_{\eta_\alpha}(C'_\alpha) \subseteq C'_\alpha$ and it is clear that we have (i): C_α avoids A. For $\beta \in d_{\eta_\alpha}(C_\alpha)$ we have $\beta \in d_{\eta_\alpha}(C'_\alpha)$ hence $C'_\beta = C'_\alpha \cap \beta$. Then we have $A \cap C'_\beta$ is not η_α-stationary in β by elementarity and $\Pi^1_{\eta_\alpha}$-correctness, so $\beta \in \Gamma^2$. Also $D_\beta \cap \int^{\eta_\alpha}(L_{\rho_\beta}, \{C'_\beta\} \cup p_\beta, \beta) = \beta \cap D_\alpha \cap \int^{\eta_\alpha}(L_{\rho_\alpha}, \{C'_\alpha\} \cup p_\alpha, \alpha) = C_\alpha \cap \beta$ so we are in the first case of the definition and $C_\beta = C_\alpha \cap \beta$. This gives (ii) and so we have a $\square^{<\gamma}$-sequence on the regulars avoiding E_A.

We have thus far only given our $\square^{<\gamma}$-sequence on the regulars below κ. To deal with singulars we use Jensen's global $\square$-sequence just as before. It remains to show that E is γ-stationary.

DEFINITION 3.34. We define $H \subseteq A$ by letting $\alpha \in H$ iff $\alpha \in A$ and there is $\mu_\alpha > \alpha$ and q is a parameter from L_{μ_α} such that:

1. L_{μ_α} is $\Pi^1_{\eta_\alpha}$-correct over α
2. $\mu_\alpha < \nu_\alpha$
3. $A \cap \int^{\eta_\alpha}(L_{\mu_\alpha}, q, \alpha) = \emptyset$

LEMMA 3.35. *$H \subseteq E$.*

PROOF: Suppose $\alpha \notin E$, so $C'_\alpha \cap A$ is η_α-stationary and $C'_\alpha = \int^{\eta_\alpha}(L_{\nu_\alpha}, p_\alpha, \alpha)$. Let $\mu < \nu_\alpha$ and $q \in L_\mu$. Then using lemma 3.8 for some $\beta < \alpha$ we have $q \in L_{\nu_\alpha}\{\beta \cup p_\alpha \cup \{\alpha\}\}$ and so $\int^{\eta_\alpha}(L_\mu, q, \alpha)\backslash\beta \supseteq \int^{\eta_\alpha}(L_{\nu_\alpha}, p_\alpha, \alpha) = C'_\alpha$. Thus $\alpha \notin H$.

QED

LEMMA 3.36. *H is γ-stationary.*

PROOF: Let $\eta < \gamma$ and let $C \subseteq \kappa$ be η-club. Take $\mu > \kappa$ minimal such that $C \in L_\mu$ and L_μ is Π^1_η-correct. Set $D = \int^\eta(L_\mu, \{C, Z\}, \kappa)$ and note that $D \subseteq d_\eta(C) \subseteq C$, and D is η-club by Lemma 3.29. Take $\delta = min(D \cap A)$. Set μ_δ to be the ordinal such that $L_\mu\{\{C, Z, \kappa\} \cup \delta\} \cong L_{\mu_\delta}$. We show that $\delta \in H$, with μ_δ and $\{C \cap \delta, Z \cap \delta\}$ witnessing this.

Claim 1: $\eta_\delta = \eta$

First to see $\eta_\delta \geq \eta$. If δ falls into case (i) then $\eta_\delta = \gamma - 1$ so this is trivial. If δ falls into case (iii) then we know that $A \cap \delta$ is not $\eta_\delta + 1$-stationary, but $L_{\mu_\delta} \vDash A \cap \delta$ is η stationary and so by Π^1_η-correctness we must indeed have $A \cap \delta$ is η-stationary and hence $\eta \leq \eta_\delta$.

If δ falls into case (ii) then η_δ is odd and we know Σ wins $G_{\eta_\delta+1}(\alpha, \varphi, Z \cap \delta)$, but for any even ordinal $\eta' \leq \eta$ we have $L_\mu \vDash \Pi$ wins $G_{\eta'}(\kappa, \varphi, Z)$ and so by elementarity and Π^1_η-correctness of L_{μ_δ}, we have Π wins $G_{\eta'}(\delta, \varphi, Z \cap \delta)$. Thus we must have $\eta < \eta_\delta + 1$, *i.e.* $\eta \leq \eta_\delta$.

Now we have to show $\eta_\delta \leq \eta$. First, suppose δ is Π^1_η-indescribable. Then by Lemma 3.29 we have $\int^\eta(L_{\mu_\delta}, \{C \cap \delta, Z \cap \delta\}, \delta)$ is η-club below δ. But $\int^\eta(L_{\mu_\delta}, \{C \cap \delta, Z \cap \delta\}, \delta) = \delta \cap \int^\eta(L_\mu, \{C, Z\}, \kappa) = D \cap \delta$ so we must have A is not $\eta + 1$ stationary below δ. But in each of the three cases A is η_δ-stationary, so $\eta_\delta \leq \eta$.

Now, if δ is not Π^1_η-indescribable we know that δ is not $\eta + 1$-stationary, so again we have that A is not $\eta + 1$-stationary and as above $\eta_\delta \leq \eta$.

Claim 2: $\mu_\delta < \nu_\delta$

We have that for every $X \in \mathcal{P}(\delta) \cap L_{\mu_\delta}$

$$L_{\mu_\delta} \vDash \Sigma \text{ wins } G_\eta(\delta, \varphi', \langle Z \cap \delta, X\rangle).$$

As L_{μ_δ} is Π^1_η-correct this means

$$\forall X \in \mathcal{P}(\delta) \cap L_{\mu_\delta},\ \Sigma \text{ wins } G_\eta(\delta, \varphi', \langle Z \cap \delta, X\rangle).$$

But we know by the choice of ν_δ that

$$\exists X \in \mathcal{P}(\delta) \cap L_{\nu_\delta},\ \Pi \text{ wins } G_\eta(\delta, \varphi', \langle Z \cap \delta, X\rangle).$$

Hence we must have $\nu_\delta > \mu_\delta$

Claim 3: $A \cap \int^{\eta_\alpha}(L_{\mu_\delta}, \{C \cap \delta, Z \cap \delta\}, \delta) = \emptyset$

We have (Lemma 3.9) $A \cap \int^{\eta_\alpha}(L_{\mu_\delta}, \{C \cap \delta, Z \cap \delta\}, \delta) = A \cap \delta \cap \int^{\eta_\alpha}(L_\mu, \{C, Z\}, \kappa) =$

$A \cap \delta \cap \int^{\eta}(L_\mu, \{C, Z\}, \kappa) = A \cap \delta \cap D = \emptyset$.

The last sentence of the Theorem follows exactly as the last sentence of Theorem 3.2 did.

QED (Theorem 3.32)

We can now state Lemma 3.29 without the assumption that for all $\gamma' < \gamma$ any regular cardinal which is γ'-reflecting is $\Pi^1_{\gamma'}$-indescribable, as this is a consequence of Theorem 3.32.

LEMMA 3.37. $(V = L)$ *Let γ be an ordinal and $\kappa > \gamma$ be Π^1_γ-indescribable. Then for any limit $\nu > \kappa$ where L_ν is Π^1_γ-correct over κ, we have $\int^{\Pi^1_\gamma}(L_\nu, p, \kappa)$ is γ-club in κ.*

We finish by deriving a $\square^\gamma$-sequence from a $\square^{<\gamma+1}$-sequence, showing that Theorem 3.2 and its generalisation to infinite γ is indeed an easy corollary of Theorem 3.32.

PROPOSITION 3.38. *Let $\gamma < \kappa$ be ordinals. If $S = \langle (C_\alpha, \eta_\alpha) : \alpha \in \Gamma \rangle$ is a $\square^{<\gamma+1}$-sequence on $\Gamma \subseteq \kappa$, then $\langle C_\alpha : \alpha \in \Gamma \cap d_\gamma(\kappa) \rangle$ is a $\square^\gamma$-sequence.*

PROOF: Let $\alpha \in \Gamma \cap d_\gamma(\kappa)$. Then C_α is η_α-club and $\eta_\alpha \leq \gamma$ and α is γ-stationary, so C_α is γ-club. Suppose $\beta \in d_\gamma(C_\alpha)$. Then as $\eta_\alpha \leq \gamma$, $\beta \in d_{\eta_\alpha}(C_\alpha)$, so $C_\beta = C_\alpha \cap \beta$. Also $\beta \in d_\gamma(\kappa)$, so we have coherence. QED

COROLLARY 3.39. $(V = L)$ *Let $\gamma < \kappa$ and κ be a Π^1_γ-indescribable but not $\Pi^1_{\gamma+1}$-indescribable cardinal, and let $A \subseteq \kappa$ be $\gamma + 1$ stationary. Then there is $E_A \subseteq A$ and a $\square^\gamma$-sequence S on κ such that E_A is $\gamma + 1$-stationary in κ and S avoids E_A.*

PROOF: This follows easily from the preceding proposition - just restricting the domain of the $\square^{<\gamma+1}$-sequence from Theorem 3.32 to $d_\gamma(\kappa)$ gives a $\square^\gamma$-sequence avoiding E_A as defined there. QED

3.5. The γ-club Filters and Non-Threaded $\square^\gamma$-sequences. In this subsection we look in more detail at the γ-club filter and what we can prove from certain assumptions about it. In the first subsection we shall revisit the results of 2.2 and generalise our result from there to γ-stationarity. In the second subsection we generalise the notion of non-threaded $\square$, and show that this $\square^\gamma(\kappa)$ must fail at any $\gamma+1$-reflecting cardinal where the γ-club filter is normal. In the final section we show that with a certain requirement on the generalised club filters γ-stationarity is downward absolute to L. If we make the stronger (and easier to state) assumption that for any ordinal η the η-club filter is normal on any η-reflecting cardinal, then we have that for any ordinal γ, γ-stationarity is downward absoluteness to L at any regular cardinal (Corollary 4.8).

3.5.1. *The γ-Club and Π^1_γ-Indescribability Filters.* Here we look in more detail at the γ-club filter, revisiting the material from 2.2.1 in the light of the definition of Π^1_γ-indescribability given in 3.3.

PROPOSITION 3.40. *If $\mathcal{C}^\gamma(\kappa)$ is normal, then for any $\eta < \gamma$ we have $\mathcal{C}^\eta(\kappa)$ is also normal.*

PROOF: This is because each η-club is γ-club, and so by the normality of $\mathcal{C}^\gamma(\kappa)$, the diagonal intersection of η-clubs must be γ-club and hence η-stationary. The η-stationary closure is automatic. QED

PROPOSITION 3.41. *If γ is a limit ordinal and $\kappa > \gamma$ then $\bigcup_{\eta<\gamma} \mathcal{C}^\eta(\kappa)$ is not γ complete.*

PROOF: Unless κ is γ-stationary $\bigcup_{\eta<\gamma} \mathcal{C}^\eta(\kappa)$ is not even a filter, so suppose κ is γ-stationary. For $\eta < \gamma$ set $C_\eta = d_\eta(\kappa)$. Then each C_η is η-club and $\bigcap_{\eta<\gamma} C_\eta = d_\gamma(\kappa)$. By Proposition 2.10 we have that for each $\eta < \kappa$, $\kappa \backslash d_{\eta+1}(\kappa)$ is $\eta + 1$-stationary and hence $d_\gamma(\kappa) \notin \mathcal{C}^\eta(\kappa)$. Thus $d_\gamma(\kappa) \notin \bigcup_{\eta<\gamma} \mathcal{C}^\eta(\kappa)$. QED

Recall (Definition 3.27) that the Π^1_γ indescribability filter on κ, $\mathcal{F}^\gamma(\kappa)$ is the filter generated by sets of the form $\{\alpha < \kappa : \Pi \text{ wins } G_\gamma(\kappa, \varphi, A) \text{ and } G_\gamma(\alpha, \varphi, A \cap \alpha)\}$.

LEMMA 3.42. *If κ is Π^1_γ-indescribable then the γ-club on κ filter is a subset of $\mathcal{F}^\gamma(\kappa)$ and hence it is normal.*

PROOF: The proof is essentially the same as Lemma 2.21 for finite γ. Firstly, any γ-club is in $\mathcal{F}^\gamma(\kappa)$. Suppose C is γ-club. "C is γ-club" is Π^1_γ and so reflects to a set in the Π^1_γ-indescribability filter on κ. But this is the set of $\alpha < \kappa$ such that $C \cap \alpha$ is γ-club, *i.e.* $d_\gamma(C)$. As $d_\gamma(C) \subseteq C$ we have that $C \in \mathcal{F}^\gamma(\kappa)$. Now suppose $\langle C_\alpha : \alpha < \kappa\rangle$ is a sequence of γ-clubs and set $C = \triangle_{\alpha<\kappa} C_\alpha$. By the normality of $\mathcal{F}^\gamma(\kappa)$ and the fact that each $C_\alpha \in \mathcal{F}^\gamma(\kappa)$, we have that $C \in \mathcal{F}^\gamma(\kappa)$ and hence C is $\gamma + 1$-stationary. γ-closure is easily verified. QED

COROLLARY 3.43. *(Fodor's Lemma for γ-stationary sets) If κ is Π^1_γ-indescribable and $A \subset \kappa$ is γ-stationary, then for any regressive function $f : A \to \kappa$ there is a γ-stationary $B \subseteq A$ such that f is constant on B.*

The following generalisation of Theorem 2.23 is a consequence of Lemma 3.29 and Theorem 3.32.

COROLLARY 3.44. *If $V = L$ the γ-club filter coincides with the Π^1_γ-indescribability filter at any Π^1_γ-indescribable cardinal.*

PROOF: Suppose Π wins $G_\gamma(\kappa, \varphi, X)$. Then $\int^{\Pi^1_\gamma}(L_{\kappa^+}, \{X\}, \kappa)$ is γ-club by Lemma 3.37, and for each $\alpha \in \int^{\Pi^1_\gamma}(L_{\kappa^+}, \{X\}, \kappa)$, we have Π wins $G_\gamma(\alpha, \varphi, X \cap \alpha)$. Thus $\{\alpha < \kappa : \Pi \text{ wins } G_\gamma(\alpha, \varphi, X \cap \alpha)\}$ is in the γ-club filter and in general $F^\gamma(\kappa)$ is included in the γ-club filter. By 3.42 we have the reverse inclusion. QED

3.5.2. *Splitting Stationary Sets.* In this section we extend the results of 2.2.2 to split $\gamma + 1$-stationary sets. We should note here that the methods of section 2.2.2 and what follows do not allow us to show that γ-stationary sets can be split when γ is a limit ordinal. This is because by Proposition 3.41, if γ is a limit ordinal then the filter corresponding to the γ-stationary sets, $\bigcup_{\eta<\gamma} \mathcal{C}^\eta(\kappa)$, is not γ complete, and our results require κ completeness.

LEMMA 3.45. *If $\mathcal{C}^\gamma(\kappa)$ is κ complete then any $\gamma+1$-stationary subset of κ is the union of two disjoint $\gamma + 1$-stationary sets.*

This lemma is proven in the same way as Lemma 2.24, the key points to enable the generalisation being Lemma 3.42 and the fact that a measurable κ is Π^1_γ-indescribable for any $\gamma < \kappa$.

PROOF: Let S be $\gamma + 1$-stationary in κ and suppose S is not the union of two disjoint $\gamma + 1$-stationary sets. Define

$$F = \{X \subset \kappa : X \cap S \text{ is } \gamma + 1\text{-stationary}\}$$

Claim: F is a κ complete ultrafilter.
Upwards closure is clear. Intersection follows by the fact that S cannot be split, as well as $X \in F \Rightarrow \kappa \backslash X \notin F$. That $X \notin F \Rightarrow \kappa \backslash X \in F$ follows by the definition of γ-stationary, and κ completeness follows from the κ completeness of the γ-club filter.

Claim: F is normal.
As we have shown that F is a κ complete ultrafilter on κ, we have that κ is measurable. Now, all measurables are Π^2_1-indescribable, and it is easy to see that Π winning the game $G_\gamma(\kappa, \varphi, A)$ for φ a Δ^0_ω formula and $A \subseteq \kappa$ is Π^2_1 expressible over V_κ, as in describing the game we only quantify over finite sequences of subsets of κ. Hence κ is Π^1_γ-indescribable and so by Lemma

3.42, $\mathcal{C}^\gamma(\kappa)$ is normal. Let $\langle X_\alpha : \alpha < \kappa\rangle$ be a sequence of sets in F. Then each $S\backslash X_\alpha$ is in the non-$\gamma+1$-stationary ideal on κ, so $X_\alpha \cup (\kappa\backslash S) \in \mathcal{C}^\gamma(\kappa)$. Now by the normality of $\mathcal{C}^\gamma(\kappa)$, we have for $X := \triangle_{\alpha<\kappa} X_\alpha \cup (\kappa\backslash S)$ that $X \in \mathcal{C}^\gamma(\kappa)$, and so $X \cap S$ is $\gamma+1$-stationary. But $X = \{\alpha < \kappa : \forall\beta < \kappa\ \alpha \in X_\beta \cup (\kappa\backslash S)\} = \{\alpha < \kappa : \alpha \in \kappa\backslash S \vee \forall\beta < \kappa\ \alpha \in X_\beta\} = (\kappa\backslash S) \cup \triangle_{\alpha<\kappa} X_\alpha$. So $X \cap S = \triangle_{\alpha<\kappa} X_\alpha \cap S$, and thus $\triangle_{\alpha<\kappa} X_\alpha \cap S$ is $\gamma+1$-stationary and hence in F.

Now we have that F is a normal measure, Π^2_1-indescribability of measurables again yields that for any $R \subseteq V_\kappa$ and formula φ that is Π^2_1, if $\langle V_\kappa, \in R\rangle \vDash \varphi$ then

$$\{\alpha < \kappa : \langle V_\alpha, \in, R \cap V_\alpha\rangle \vDash \varphi\} \in F.$$

Setting $R = S$ and $\varphi =$ "S is $\gamma+1$-stationary" we can conclude

$$\{\alpha < \kappa : S \cap \alpha \text{ is } \gamma+1\text{-stationary}\} \in F.$$

Then by definition of F we have $A := \{\alpha \in S : S \cap \alpha \text{ is } \gamma+1\text{-stationary}\}$ is $\gamma+1$-stationary. But by Proposition 2.10 we have $A' := \{\alpha \in S : S \cap \alpha \text{ is not } \gamma+1\text{-stationary}\}$ is $\gamma+1$-stationary. This contradicts our assumption on S as A' and A are two disjoint, $\gamma+1$-stationary subsets of S. QED

THEOREM 3.46. *If κ is weakly compact and $C^\gamma(\kappa)$ is κ-complete then any $\gamma+1$-stationary subset of κ can be split into κ many disjoint $\gamma+1$-stationary sets.*

PROOF: Let $S \subseteq \kappa$ be $\gamma+1$-stationary. We construct a tree T of $\gamma+1$-stationary subsets of S ordered by $\supseteq$, using Lemma 3.45: each $\gamma+1$-stationary set can be split in two. T will have height κ and levels of size $< \kappa$ so as κ has the tree property it has a κ length branch, from which we can construct a partition of S.

We define T inductively such that each level (i) consists of disjoint $\gamma+1$-stationary sets, (ii) has size $< \kappa$, and (iii) is non-empty. Let $T_0 = \{S\}$. Now let $\alpha < \kappa$ and suppose we have defined T_β for each $\beta < \alpha$ such that (i)-(iii) hold. If α is a successor, say $\beta+1$, for each set $A \in T_\beta$ we use Lemma 3.45 to choose $A' \subseteq A$ such that A' and $A\backslash A'$ are both $\gamma+1$-stationary, and take $T_{\beta+1} = \{A', A\backslash A' : A \in T_\beta\}$. Clearly, (i)-(iii) are preserved.

If α is a limit, we define

$$T_\alpha = \{\bigcap b : b \text{ is a branch in } T_{<\alpha} \text{ and } \bigcap b \text{ is } \gamma+1\text{-stationary}\}.$$

Now (i) is clear and as $|\{\bigcap b : b \text{ is a branch in } T_{<\alpha}\}| \leq 2^{|T_{<\alpha}|} < \kappa$ we also have (ii). To demonstrate (iii) first note that by the construction of T we have $S = \bigcup\{\bigcap b : b \text{ is a branch in } T_{<\alpha}\}$ (For each $a \in S$, $\{A \in T_\alpha : a \in A\}$ is clearly a branch although it may have height $< \alpha$). Now we know $|\{\bigcap b : b \text{ is a branch in } T_{<\alpha}\}| < \kappa$ so by κ-completeness of $\mathcal{C}^\gamma(\kappa)$ we must have at least one branch b such that $\bigcap b$ is $\gamma+1$-stationary. But then b must have height α, for if b had height $\beta < \alpha$, then b must have limit height so by definition $\bigcap b \in T_\beta$ which would make b not maximal. So $\bigcap b \in T_\alpha$, and hence (iii) holds (and the definition of T_α makes sense).

Now by the tree property T has a κ branch. Take B to be such a branch. For $A \in B$ let A^+ denote the immediate successor of A in B, and note that $A\backslash A^+$ is disjoint from any set in B succeeding A. Then $\{A \setminus A^+ : A \in B\}$ is a partition of S into κ many disjoint $\gamma+1$-stationary sets. QED

COROLLARY 3.47. *Let κ be Π^1_γ-indescribable, $\gamma \geq 1$. Then any $\gamma+1$-stationary subset of κ can be split into κ many disjoint γ-stationary sets.*

PROOF: A Π^1_γ-indescribable cardinal κ is of course weakly compact, and by Lemma 3.42, the γ-club filter on κ is normal and hence κ-complete. QED

It is also easy to see that if κ is inaccessible and C^γ_κ is κ-complete then for any $\alpha < \kappa$ we can split any $\gamma+1$-stationary set into α many $\gamma+1$-stationary pieces.

3.5.3. *Non-Threaded $\square^\gamma$-sequences.* In this section we generalise a nice folklore result[4] that relates *non-threaded* $\square$ (Definition 3.49) to simultaneous stationary reflection: if we have a non-threaded $\square$-sequence on a cardinal κ then there are two stationary subsets of κ which are not both stationary in any $\alpha < \kappa$. Thus non-threaded $\square$ fails at any 2-stationary cardinal. We start by recalling the definitions.

DEFINITION 3.48. A $\square^\gamma$*-sequence* (for $\gamma \geq 0$) on κ is a sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ such that for each α:

1. C_α is a γ-club subset of α
2. for every $\beta \in d_\gamma(C_\alpha)$ we have $C_\beta = C_\alpha \cap \beta$

We say a $\square^\gamma$-sequence $\langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ *avoids* $A \subset \kappa$ if for all α we have $A \cap d_\gamma(C_\alpha) = \emptyset$.

The generalisation of non-threaded $\square$ is straight-forward:

DEFINITION 3.49. $\square^\gamma(\kappa)$ is the statement that there is a $\square^\gamma$-sequence on κ which has no thread, *i.e.* there is no $C \subseteq \kappa$, such that C is γ-club and for all $\alpha \in d_\gamma(C)$ we have $C_\alpha = C \cap \alpha$. Such a sequence is called a $\square^\gamma(\kappa)$-sequence.

THEOREM 3.50. *Suppose κ is a regular γ-reflecting cardinal and the γ-club filter on κ is normal. Let $S = \langle C_\alpha : \alpha \in d_\gamma(\kappa)\rangle$ be a $\square^\gamma$-sequence. Then the following are equivalent:*

1. *S is a $\square^\gamma(\kappa)$-sequence, i.e. S has no thread.*
2. *For any $\gamma+1$-stationary set T there are $\gamma+1$-stationary S_0, $S_1 \subseteq T$ such that for any $\alpha \in d_\gamma(\kappa)$ we have $d_\gamma(C_\alpha) \cap S_0 = \emptyset$ or $d_\gamma(C_\alpha) \cap S_1 = \emptyset$*

Thus $\square^\gamma(\kappa)$ implies κ is not $\gamma+1$-reflecting.

We give the proof for $\gamma > 0$, following the proof given in [22], though the generalisation is not straight-forward. For $\gamma = 0$ see [22] Proposition 27 or the following but adding a "-1-club", where $C \subseteq \kappa$ is -1-club if C is an end-segment of κ.

PROOF:

We start with $(2) \to (1)$, so let S_0 and S_1 satisfy (2) and assume for a contradiction that C is an γ-club subset of κ which threads S. Then we have $\alpha < \beta < \delta$ in $d_\gamma(C)$ such that $\alpha \in S_0$ and $\beta \in S_1$. Now $C_\delta = C \cap \delta$ so we have $\alpha \in C_\delta \cap S_0$ and $\beta \in C_\delta \cap S_1$ - but this is a contradiction.

Now suppose S has no thread and $T \subseteq \kappa$ is $\gamma+1$-stationary. We split into two cases.
Case 1: There is an $\eta < \gamma$ and an η-club set D such that $\{\alpha < \kappa : d_\gamma(C_\alpha) \cap D = \emptyset \text{ or } \alpha \notin T\}$ contains an γ-club.

If we define $T' = \{\alpha \in T \cap D : d_\gamma(C_\alpha) \cap D = \emptyset\}$ then T' is $\gamma+1$-stationary as it is the intersection of a γ-club with T. Firstly, suppose for each $\alpha \in d_\gamma(\kappa)$ we have $|d_\gamma(C_\alpha) \cap T'| \leq 1$. Then for any pair S_0 and S_1 of disjoint $\gamma+1$-stationary subsets of T' and any α we must have $d_\gamma(C_\alpha) \cap S_0 = \emptyset$ or $d_\gamma(C_\alpha) \cap S_1 = \emptyset$, so we're done.

If this is not the case, we show that S avoids T' and hence we can use Lemma 3.45 to split T' into 2 disjoint $\gamma+1$-stationary sets, which will give (2). So suppose for some $\alpha \in d_\gamma(\kappa)$ that $|d_\gamma(C_\alpha) \cap T'| \geq 2$. Take $\beta_0 < \beta_1$ both in $d_\gamma(C_\alpha) \cap T'$. Then we have $C_{\beta_1} \cap \beta_0 = C_\alpha \cap \beta_0 = C_{\beta_0}$, so $\beta_0 \in d_\gamma(C_{\beta_1})$ - but this is a contradiction with the definition of T' since $\beta_1 \in T'$ but $\beta_0 \in D$.

Case 2: For any $\eta < \gamma$ and any η-club D we have $\{\alpha \in T : d_\gamma(C_\alpha) \cap D \neq \emptyset\}$ is $\gamma+1$-stationary.

For $\alpha \in d_\gamma(\kappa)$ we set

$$S_0^\alpha = \{\beta \in T \backslash \alpha : \alpha \notin d_\gamma(C_\beta)\}$$

[4] With thanks to Philipp Lücke for pointing out this result as something which may generalise.

and

$$S_1^\alpha = \{\beta \in T\backslash\alpha : \alpha \in d_\gamma(C_\beta)\}.$$

Then it is clear that for any given α either S_0^α or S_1^α is $\gamma+1$-stationary.

Claim There is some $\alpha \in d_\gamma(\kappa)$ such that S_0^α and S_1^α are both $\gamma+1$-stationary.

PROOF OF CLAIM: Suppose not. Then for each α we have either S_0^α is not $\gamma+1$-stationary or S_1^α is not $\gamma+1$-stationary. Set $A = \{\alpha \in \kappa : S_0^\alpha$ is not $\gamma+1$-stationary$\}$. We claim that A is γ-stationary. So let D be η-club for some $\eta < \gamma$. Then as we are in Case 2, $\{\alpha \in T : d_\gamma(C_\alpha) \cap D \neq \emptyset\}$ is $\gamma+1$-stationary. Define a regressive function on T by $\alpha \mapsto min(d_\gamma(C_\alpha) \cap D)$. By normality of the γ-club filter, we can fix $T' \subseteq T \cap D$ and α such that for any $\beta \in T'$ we have $min(d_\gamma(C_\beta) \cap D) = \alpha$. Then $S_1^\alpha \supseteq T'$ so S_1^α is $\gamma+1$-stationary and so we have $\alpha \in A \cap D$. As $\eta < \gamma$ and D were arbitrary and we have shown that $A \cap D \neq \emptyset$ we can conclude that A is γ-stationary.

Now let $\alpha < \alpha'$ be elements of A and D^α, $D^{\alpha'}$ by γ-clubs witnessing, respectively, that S_0^α and $S_0^{\alpha'}$ are not $\gamma+1$-stationary. Fix $\beta \in D^\alpha \cap D^{\alpha'} \cap T$. Then $\alpha \in d_\gamma(C_\beta)$ so $C_\alpha = C_\beta \cap \alpha$. Similarly, $C_{\alpha'} = C_\beta \cap \alpha'$ so we have $C_\alpha = C_{\alpha'} \cap \alpha$. Setting $C = \bigcup_{\alpha\in A} C_\alpha$ we have that C is γ-club: as each C_α is γ-closed C is γ-closed and as the γ-stationary union of γ-stationary sets, C must be stationary. But then C is a thread through S - contradiction. QED (*Claim*)

Now let α be such that S_0^α and S_1^α are both $\gamma+1$-stationary. Set $S_0 = S_0^\alpha$ and $S_1 = S_1^\alpha$ and note that for any $\beta \in S_0 \cup S_1$ we have $\beta > \alpha$. We show these sets satisfy (2). Suppose $\delta \in d_\gamma(\kappa)$. If $\beta \in d_\gamma(C_\delta) \cap S_0$ and $\beta' \in d_\gamma(C_\delta) \cap S_1$ then $\alpha \in C_\beta$ and $\alpha \notin C_{\beta'}$ by definition of S_0 and S_1, but $C_\beta \cap \beta' = C_\delta \cap \beta \cap \beta' = C_{\beta'} \cap \beta$. As $\alpha < \beta \cap \beta'$ this gives us a contradiction. Hence C_δ must avoid either S_0 or S_1.

QED (Theorem 3.50)

COROLLARY 3.51. $(V = L)$. *Suppose κ is Π^1_γ-indescribable. Then the following are equivalent: (i) κ is not $\Pi^1_{\gamma+1}$-indescribable; (ii) $\square^\gamma(\kappa)$.*

PROOF: Our supposition yields the hypotheses of the last theorem. (ii) implies that κ is not $\gamma+1$-reflecting, and so not $\Pi^1_{\gamma+1}$-indescribable by Cor.3.33. Thus (i). In turn (i) yields *via* Thm.3.32 for any $\gamma+1$-stationary T, the $\gamma+1$-stationary E_T which avoids the given $\square^\gamma$-sequence the theorem delivers. Hence by *2* of the last theorem, with $E_T = S_0 = S_1$ we have (i) as required. QED

§**4. Downward Absoluteness.** The downward absoluteness to L of a cardinal being 1-reflecting was proven by Magidor in [24] §1 (the theorem was stated there for $\kappa = \omega_2$, but the proof is the same for any regular κ). There, the only assumption needed on κ was regularity. To generalise this and get γ-reflecting cardinals in L, we shall require some extra assumptions, as we see below. The proof works inductively, and we shall need a slightly stronger statement than the downward absoluteness of κ being 1-reflecting: we shall require the downward absoluteness of 2-stationarity for sets in L. At higher levels of stationarity the first assumption we require is the normality of the η-club filter - in the case of 1-reflecting cardinals the club filter was guaranteed to be normal by assuming κ is regular. We shall also need a further assumption that "many" cardinals below κ have the properties guaranteeing downward absoluteness of lower levels of stationarity.

The proof will be split into three cases - limit ordinals, successors of limit ordinals, and double successors. The proof for double successors case is based on Magidor's proof (which essentially gives downward absoluteness of 2-stationarity) though there is more work to be done for the higher levels as we do not have absoluteness of γ-clubs for $\gamma > 0$. This will be seen particularly in the last part of the proof. The limit stages are straightforward, but for the successors of limits we need a slight variation on the notion of normality. The following definition gives us the appropriate notion.

DEFINITION 4.1. For $\gamma > 0$ we say that a cardinal κ is γ-*normal* if κ is γ-reflecting and for any $\gamma+1$-stationary $S \subseteq \kappa$ and regressive function $f : S \to \kappa$, f is constant on a γ-stationary set.

For successor ordinals this reduces to normality of the η-club filter:

PROPOSITION 4.2. *If $\gamma > \eta$ then κ is γ-normal implies $\mathcal{C}^\eta(\kappa)$ is normal. If $\gamma = \eta+1$ and κ is γ-reflecting then we have the converse: $\mathcal{C}^\eta(\kappa)$ being normal implies κ is $\eta+1$-normal.*

Thus κ is 1-normal iff the club filter is normal iff κ is regular.

PROOF: Suppose $\gamma > \eta$ and κ is γ-normal. Let $\langle C_\alpha : \alpha < \kappa \rangle$ be a sequence of η-club subsets of κ. Then $\triangle_{\alpha<\kappa} C_\alpha$ is η-stationary closed so if $\triangle_{\alpha<\kappa} C_\alpha$ is not η-club then it is not η-stationary. Let $\eta' < \eta$ and C be a η'-club avoiding $\triangle_{\alpha<\kappa} C_\alpha$. Then setting $C'_\alpha = C_\alpha \cap C$ we have C'_α is η-club and $\triangle_{\alpha<\kappa} C'_\alpha = \emptyset$. Define $f : \kappa \to \kappa$ by $f(\alpha)$ is the least β such that $\alpha \notin C'_\beta$. Then as $\triangle_{\alpha<\kappa} C'_\alpha = \emptyset$, f is regressive on κ and so by the γ-normality of κ there is some $\beta < \kappa$ such that $f^{-1}(\beta)$ is a γ-stationary set. But this contradict C'_β being η-club. QED

Note that for a limit ordinal γ, the domain of f must be $\gamma+1$-stationary, so this requirement is weaker than $\bigcup_{\eta<\gamma} \mathcal{C}^\eta$ being normal (which is always false for limit γ, see Proposition 3.41), but stronger than each $\mathcal{C}^\eta$ being normal (the latter can occur when κ is not γ-reflecting). The notion gets stronger as γ-increases:

PROPOSITION 4.3. *If $\gamma > \eta$ and κ is γ-normal then κ is η-normal.*

PROOF: Fix $\gamma > \eta$ and suppose κ is γ-normal. By Proposition 4.2 we have $\mathcal{C}^\eta(\kappa)$ is normal, and hence by Fodor's Lemma, for any $\eta+1$-stationary S any regressive $f : S \to \kappa$ is constant on an $\eta+1$-stationary, and hence an η-stationary, set. QED

PROPOSITION 4.4. *If κ is Π^1_γ-indescribable then*

1. *κ is γ-normal, and*
2. *$\{\lambda < \kappa : \lambda$ is η-normal for all $\eta < \gamma\}$ is $\gamma+1$-stationary in κ.*

PROOF: If κ is Π^1_γ-indescribable then $\mathcal{C}^\gamma(\kappa)$ is normal by Lemma 3.42, so κ is γ-normal. We have that "$\forall \eta < \gamma(\kappa$ is Π^1_η-indescribable)" is Π^1_γ expressible (by Cor. 3.26) so

$$\{\lambda < \kappa : \forall \eta < \gamma(\lambda \text{ is } \Pi^1_\eta\text{-indescribable})\}$$

is in the Π^1_γ indescribability filter, and so by Lemma 3.42 is $\gamma+1$-stationary. However, by part (1),

$$\{\lambda < \kappa : \forall \eta < \lambda(\lambda \text{ is } \Pi^1_\eta\text{-indescribable})\} \subseteq \{\lambda < \kappa : \lambda \text{ is } \eta\text{-normal for all } \eta < \gamma\}$$

and thus the latter is $\gamma+1$-stationary. QED

We now state our main theorem.

THEOREM 4.5. *Let γ be an ordinal and $\kappa > \gamma$ a regular cardinal such that*

1. *for all $\eta < \gamma$ κ is η-normal and*
2. *setting*

$$A_\gamma = \{\alpha < \kappa : \text{for all } \eta \text{ with } 1 < \eta+1 < \gamma, \alpha \text{ is } \eta\text{-normal}\}$$

we have that A_γ is γ-stationary in κ.

Then $(\kappa$ *is* γ*-stationary*$)^L$ *and hence* $(\kappa$ *is* Σ^1_γ*-indescribable*$)^L$*. Furthermore, for any* $S \subseteq \kappa$ *such that* $S \in L$ *and* $S \cap A_\gamma$ *is* γ*-stationary we have* $(S$ *is a* γ*-stationary subset of* $\kappa)^L$*. Consequently, if* A_γ *is* η*-club for some* $\eta < \gamma$ *then* γ*-stationarity at* κ *is downward absolute to* L*.*

A few notes before we begin the proof. Assumption (2) is needed so that there are enough $\alpha < \kappa$ where we can apply the inductive hypothesis. For $\gamma = \eta + 2$, $A_\gamma = \{\alpha < \kappa : \alpha \text{ is } \eta\text{-normal}\}$ and if η is a limit ordinal and $\gamma = \eta$ or $\gamma = \eta + 1$ then $A_\gamma = \{\alpha < \kappa : \text{for all } \eta' < \eta\ \alpha \text{ is } \eta'\text{-normal}\}$. For $\gamma = 2$ requirement (1) reduces to regularity and (2) is vacuous. For $\gamma = 3$ requirement (1) is just the 1-club filter is normal, and (2) that the regular cardinals are 3-stationary - but this is a consequence of (1) as we shall see below.

PROOF: We prove this by induction, so suppose the theorem is true for any $\eta < \gamma$ and κ satisfies the assumptions (1) and (2) of the theorem. First suppose γ is a limit ordinal . Then

$$\begin{aligned} A_\gamma &= \{\alpha < \kappa : \textit{for every } \eta + 1 < \gamma\ \alpha \textit{ is } \eta\textit{-normal}\} \\ &= \{\alpha < \kappa : \textit{for every } \eta < \gamma\ \alpha \textit{ is } \eta\textit{-normal}\} \\ &= \bigcap_{\eta < \gamma} A_\eta \end{aligned}$$

and so each A_η is γ-stationary and hence η-stationary. Thus we have κ satisfies, for all $\gamma' < \gamma$ (1) for all $\eta < \gamma'$ κ is η-normal and (2) $A_{\gamma'}$ is γ-stationary. Also, if $S \subseteq \kappa$ is such that $S \cap A$ is γ-stationary, then for any $\eta < \gamma$, $S \cap A_\eta$ is γ- (and hence η-) stationary. Therefore for any such S the inductive hypothesis gives that, for each $\eta < \gamma$, $(S$ is η-stationary$)^L$. But then $(S$ is γ-stationary$)^L$ by definition.

Now suppose $\gamma = \eta + 1$ and η is a limit ordinal. By (1), κ is η-normal. Suppose we have $S \subseteq \kappa$ with $S \in L$ such that $S \cap A_\gamma$ is γ-stationary. To show that S is $\eta + 1$-stationary in L, let $B \subseteq \kappa$ with $B \in L$ such that for every $\alpha \in S$

$$L \models B \cap \alpha \text{ is not } \eta\text{-stationary}.$$

For each $\alpha \in S$ set η_α to be the least ordinal such that $B \cap \alpha$ is not η_α stationary. As η is a limit ordinal, $\eta_\alpha < \eta$ for all $\alpha \in S \cap A_\gamma$. Therefore, by the η-normality of κ there is some $\delta < \eta$ such that setting $X_\delta = \{\alpha \in S : \eta_\alpha = \delta\}$ we have $X_\delta \cap A_\gamma$ is η-stationary. Fix such a δ, and note $X_\delta \in L$.

As $\delta < \eta$, by the inductive hypothesis κ is δ-reflecting in L and X is $\delta + 1$-stationary in L. Now working in L, suppose B were δ-stationary. Then $d_\delta(B)$ would be δ-club below κ, so we would have some $\alpha \in d_\delta(B) \cap X$. But this is a contradiction as for any $\alpha \in X$, $B \cap \alpha$ was not δ-stationary. So we have B is not δ-stationary in L. Thus

$$L \models B \text{ is not } \eta\text{-stationary in } \kappa$$

and so S is $\eta + 1 = \gamma$-stationary in L.

Finally, if γ is not a limit ordinal or a successor of a limit ordinal, we have more work to do. The proof will proceed roughly as Magidor's proof for the case $\gamma = 2$ ($\gamma = 1$ is the downward absoluteness of stationarity, which is obvious). There will be several points of departure from Magidor's proof, as we shall have to take into account the non-absoluteness of η-clubs and η-stationary sets. To simplify the presentation we shall assume that $\gamma > 2$.

Let $\gamma = \eta + 2$ with $\eta \geq 1$. Let κ be regular and satisfy (1) and (2) of the theorem and $S \subseteq \kappa$ be such that $S \cap A_\gamma$ is γ-stationary with $S \in L$. Then (1) gives us that $\mathcal{C}^\eta(\kappa)$ is normal.

CLAIM. *The regular cardinals are* 1*-club below* κ*.*

PROOF: As $\eta \geq 1$ and we have assumed the η-club filter on κ is normal, $\mathcal{C}^1(\kappa)$ must be normal. Suppose the singulars were 2-stationary below κ. Let $f : Sing \cap \kappa \to \kappa$ with $f(\alpha) = cof(\alpha)$. Then f is regressive so the normality of $\mathcal{C}^1(\kappa)$ would give a 2-stationary set A of ordinals all having the same cofinality. But this is impossible, as for any $\alpha \in A$ taking a club of order type

$cf(\alpha)$ its limit points would all have smaller cofinality, so $A \cap \alpha$ could not be stationary. QED

We now show that we can assume for each α in S, we have that in L, α is regular and η-reflecting.

Now let $S' = \{\alpha \in S : (\alpha \text{ is regular and } \eta\text{-reflecting})^L\}$. Clearly $S' \in L$. So we just need to show $S' \cap A_\gamma$ is γ-stationary in V. We have $S \cap A_\gamma$ is $\eta+2$-stationary and as κ is $\eta+1$-reflecting $d_{\eta+1}(A_\gamma)$ is $\eta+1$-club. Also, by the above claim, the regulars are 1-club, so setting

$$S^* = S \cap A_\gamma \cap d_{\eta+1}(A_\gamma) \cap Reg$$

we have S^* is $\eta + 2 = \gamma$-stationary. We show $S^* \subseteq S'$, so let $\alpha \in S^*$. Then α is regular and η-normal. Also as $\alpha \in d_{\eta+1}(A_\gamma)$ we have A_γ, and hence A_η, is $\eta + 1$-stationary in α. Thus α satisfies the assumptions of the theorem for $\eta + 1$ and so by the inductive hypothesis α is η-reflecting and regular in L.

From now on we assume $S = S'$, so for all $\alpha \in S$, $(\alpha \text{ is regular and } \eta\text{-reflecting})^L$, and $S^* = S \cap A_\gamma \cap d_{\eta+1}(A_\gamma) \cap Reg$ is a γ-stationary subset of S. Furthermore, for any $\alpha \in S^*$ we have by the inductive hypothesis that for any $T \subseteq \alpha$ such that $T \cap A_\gamma$ is $\eta + 1$-stationary, $(T \text{ is } \eta + 1\text{-stationary})^L$.

We now want to show that S is $\eta + 2$-stationary in L, so we let $B \subseteq \kappa$ with $B \in L$ such that for every $\alpha \in S$

$$L \vDash B \cap \alpha \text{ is not } \eta + 1\text{-stationary}.$$

we shall show

$$L \vDash B \text{ is not } \eta + 1\text{-stationary in } \kappa.$$

Now working in L: For $\alpha \in S$ we have α is η-reflecting and B is not $\eta + 1$-stationary, so we can find an η-club $D \subseteq \alpha$ which avoids B. Let D_α be the minimal such set in the canonical well-ordering of L. Now we can take ν_α to be minimal such that $B \cap \alpha, D_\alpha \in L_{\nu_\alpha}$ and L_{ν_α} is Π^1_η-correct for α. (In Magidor's proof η-stationary correctness was not required, because there it was 0-stationary correct; and a set being unbounded is absolute for transitive models.) It is clear that $|\nu_\alpha| = |\alpha|$. Also, if α is regular then any L_β for $\beta > \alpha$ will be correct about d_η below α, and thus D_α can be uniformly defined within any level of L which is Π^1_η-correct for α, contains $B \cap \alpha$ and sees that $B \cap \alpha$ is not $\eta + 1$-stationary.

Set $M_\alpha = \langle L_{\nu_\alpha}, \in, \alpha, B \cap \alpha, D_\alpha \rangle$. Now following Magidor's proof, as the structure M_α is no bigger than α in L it is isomorphic to a structure $N_\alpha = \langle \alpha, E_\alpha, \mu_\alpha, B'_\alpha, D'_\alpha \rangle$. We take N_α minimal and let h_α be the inverse collapse, $h_\alpha : M_\alpha \to N_\alpha$. Set $f_\alpha = h_\alpha \restriction \alpha$. Now we go back to working in V.

LEMMA 4.6. *There is an η-club $G \subseteq \kappa$ such that for any $\alpha, \beta \in G \cap S^*$ if $\alpha < \beta$ then $\langle N_\alpha, f_\alpha \rangle \prec \langle N_\beta, f_\beta \rangle$.*

PROOF: Let $\mathcal{L}_N$ be the language $\mathcal{L}_\in$ with added constant symbols μ, B, D and function symbol f. First we show that for any particular formula $\varphi \in \mathcal{L}_N$ with k free variables and any $\alpha_1, \ldots, \alpha_k$ ordinal parameters from κ we have (where μ, B, D and f are interpreted in the obvious way as $\mu_\beta, B'_\beta, D'_\beta$ and f_β) that either

$$X_{\varphi(\vec{\alpha})} = \{\beta < \kappa : \langle N_\beta, f_\beta \rangle \vDash \varphi(\vec{\alpha}) \text{ or } \beta \notin S^*\}$$

or

$$X_{\neg\varphi(\vec{x})} = \{\beta < \kappa : \langle N_\beta, f_\beta \rangle \vDash \neg\varphi(\vec{\alpha}) \text{ or } \beta \notin S^*\}$$

contains an η-club. The lemma will then follow by taking the diagonal intersection over parameters and formulae.

Fix φ etc. Working in V, assume for contradiction that setting

$$A_1 := \{\beta \in S : \alpha_1, \ldots, \alpha_k < \beta, N_\beta \vDash \varphi(\vec{\alpha})\}$$

$$A_2 := \{\beta \in S : \alpha_1, \ldots, \alpha_k < \beta, N_\beta \vDash \neg\varphi(\vec{\alpha})\}$$

we have $A_1 \cap S^*$ and $A_2 \cap S^*$ are both $\eta+1$-stationary, and $A_1, A_2 \in L$. As S^* is $\eta+2$-stationary, we can find a regular $\alpha \in S^*$ such that $A_1 \cap S^*$ and $A_2 \cap S^*$ are both $\eta+1$-stationary in α. Thus we have α satisfies the assumptions (1) and (2) for $\eta+1$, and so by the inductive hypothesis α is Π^1_η-indescribable in L, and as $A_{\eta+1} \supseteq A_\gamma \supseteq S^*$ we have $A_1 \cap \alpha, A_2 \cap \alpha$ are also $\eta+1$-stationary in L. Without loss of generality suppose $N_\alpha \vDash \varphi(\vec{\alpha})$. We now work in L and show that on an η-club below α we have $N_\beta \vDash \varphi(\vec{\alpha})$ and hence we have a contradiction with A_2 being $\eta+1$-stationary.

So, let $\rho > \nu_\alpha$ be an admissible ordinal such that L_ρ is Π^1_η-correct for α. Now α is Π^1_η-indescribable in L so by Lemma 3.37, we have

$$\int^{\Pi^1_\eta}(L_\rho, \{B\cap\alpha, D\cap\alpha, \alpha_1, \ldots, \alpha_k\}, \alpha)$$

is η-club below α. Let $\beta \in \int^{\Pi^1_\eta}(L_\rho, \{B\cap\alpha, D\cap\alpha, \alpha_1, \ldots, \alpha_k\}, \alpha)$, let δ be such that $L_\delta \cong L_\rho\{\beta \cup \{\alpha, B\cap\alpha, D\cap\alpha, \alpha_1, \ldots, \alpha_k\}\}$, and π be the collapsing map. It is clear that $\pi(B\cap\alpha) = B\cap\beta$ and δ is admissible. As the D_β's were uniformly definable from $B\cap\beta$ at Π^1_η-correct levels of L, we have $\pi(D_\alpha) = D_\beta$.

As $\rho > \nu_\alpha$, we have

$$L_\rho \vDash \exists\gamma\ L_\gamma \text{ is } \Pi^1_\eta\text{-correct for } \alpha \text{ and } B\cap\alpha, D_\alpha \in L_\gamma.$$

Thus we have

$$L_\delta \vDash \exists\gamma\ L_\gamma \text{ is } \Pi^1_\eta\text{-correct for } \beta \text{ and } B\cap\beta, D_\beta \in L_\gamma$$

and because L_δ is Π^1_η-correct this statement hold in L, so we have $\nu_\beta < \delta$. Thus $\pi(N_\alpha) = N_\beta$, and so $N_\beta \vDash \varphi(\vec{\alpha})$. Hence, on an η-club (the Π^1_η trace) below α we have $N_\beta \vDash \varphi(\vec{\alpha})$ and hence we have a contradiction with A_2 being $\eta+1$-stationary in α. Thus we must have either $A_1 \cap S^*$ is not $\eta+1$-stationary in κ or $A_2 \cap S^*$ is not $\eta+1$-stationary in κ. But this means either $X_{\varphi(\vec{\alpha})}$ or $X_{\neg\varphi(\vec{\alpha})}$ contains an η club, which is what we wanted to show.

Now we conclude by taking the diagonal intersection. Let $\langle \varphi_n : n \in \omega\rangle$ list all the formulae in the language $\mathcal{L}_\mathcal{N}$. Let $\langle \vec{\alpha}_\delta : \delta < \kappa \wedge lim(\delta)\rangle$ enumerate ${}^{<\omega}\kappa$ in order type κ. On a club of C we have $\langle \vec{\alpha}_\delta : \delta < \alpha\rangle$ lists all of ${}^{<\omega}\alpha$. Then for limit $\delta < \kappa$ we can set $C_{\delta+n} = \kappa$ if φ does not have $lh(\vec{\alpha}_\delta)$ free variables. Otherwise we set $C_{\delta+n} = X_{\varphi_n(\vec{\alpha_\delta})}$ if this contains an η-club and if not $C_{\delta+n} = X_{\neg\varphi_n(\vec{\alpha_\delta})}$, which must then contain an η-club. Then $C \cap \triangle_{\beta<\kappa} C_\beta$ contains an η-club, and for any $\alpha < \beta$ with $\alpha, \beta \in S^* \cap C \cap \triangle_{\beta<\kappa} C_\beta$, and for any $\varphi \in \mathcal{L}_N$ and $\vec{\alpha} \in {}^{<\omega}\alpha$ we have that

$$\langle N_\alpha, f_\alpha\rangle \vDash \varphi(\vec{\alpha}) \text{ iff } \langle N_\beta, f_\beta\rangle \vDash \varphi(\vec{\alpha})$$

and thus $\langle N_\alpha, f_\alpha\rangle \prec \langle N_\beta, f_\beta\rangle$. (Although each $X_{\varphi(\vec{x})} \in L$, as containing a η-club is not absolute between V and L, the appropriate sequence of $X_{\varphi(\vec{x})}$ to take the diagonal intersection of and obtain G need not be in L, and hence we cannot guarantee $G \in L$ at this stage.) QED

To finish the proof we want to show B is not $\eta+1$-stationary in L, so we shall produce an η-club set D with $D \cap B = \emptyset$. For each $\alpha < \beta \in S^* \cap G$ we have $M_\alpha \prec M_\beta$. Let M be the direct limit $\varinjlim \langle M_\alpha\rangle_{\alpha \in G\cap S^*}$. As each $M_\alpha \vDash V = L$ we have $M \vDash V = L$, and thus M is just some L_ρ. Also setting $D = \bigcup_{\alpha \in S^*\cap G} D_\alpha$, it is clear that $M = \langle L_\rho, \in, \kappa, B, D\rangle$ (remember $B = \bigcup_{\alpha\in S} B_\alpha$). we shall show that in L, $d_\eta(D) \cap B = \emptyset$ before checking that D is stationary.

It is clear that for any $\alpha < \beta$ from $G \cap S^*$ we have D_β is an end extension of D_α. So if D is η-stationary below some $\alpha < \kappa$ then for some (any) $\beta > \alpha$ with $\beta \in S^* \cap G$ we have $D_\beta \cap \alpha = D \cap \alpha$ so by definition of D_β we have $\alpha \notin B$ as required. It remains to show that D is η-stationary. We want to use the fact that an η-stationary union of η-stationary sets is η-stationary - but to apply this we need to find an η-stationary set $H \in L$ with $D_\alpha = D \cap \alpha$ for each $\alpha \in H$. (Note we cannot do this in V and use the inductive hypothesis to show D is η-stationary in L, because the D_α's need not be η-stationary in V.) The obvious candidate for this is $G \cap S^*$, but as noted above, we needn't have $G \cap S^* \in L$.

CLAIM. *There is $H \in L$ with $G \cap S^* \subseteq H$ and for each $\alpha \in H$, $M_\alpha \prec M$.*

PROOF:
Let $H = S \cap \int^{\Pi^1_\eta}(L_\rho, \{B, D\}, \kappa)$. Note that we do not assume L_ρ is Π^1_η-correct, so we do not yet know that $H \neq \emptyset$.

First we show that for α in H we have $L_\rho\{\alpha, B, D\} \cong L_{\nu_\alpha}$ with $\pi(B) = B \cap \alpha$ and $\pi(D) = D_\alpha$ and hence $M_\alpha \prec M$. Fix $\alpha \in H$ and let δ be such that $L_\delta \cong L_\rho\{\alpha \cup \{B, D\}\}$, and π be the collapsing map. Clearly $\pi(B) = B \cap \alpha$ and as D was defined from B in the same way that D_α was defined from $B \cap \alpha$, and L_δ is Π^1_η-correct and sees that $B \cap \alpha$ is not $\eta + 1$-stationary, we must have $\pi(D) = D_\alpha$. Thus to show $\delta = \nu_\alpha$ we only need to verify minimality of L_δ. Let $\Theta(\beta)$ be the following statement, where ψ_γ is the universal Π^1_γ formula.

$$\begin{aligned}\Theta(\beta): \qquad & \forall\gamma > \beta\big[\big(B \cap \beta \notin L_\gamma\big) \vee \big(D_\beta \notin L_\gamma\big) \\ & \qquad \vee \exists U \notin L_\gamma\ \exists n \in \omega\ \exists \eta' < \eta\ \big(U \text{ is minimal with } \psi_{\eta'}(\beta, n, U)\big)\big].\end{aligned}$$

Then if L_ν is Π^1_η-for β and $L_\nu \models \Theta(\beta)$ we must have L_ν is the minimal Π^1_η-correct level containing $B \cap \beta$ and D_β.

Now for any $\beta \in G \cap S^*$

$$L_{\nu_\alpha} \models \Theta(\beta)$$

so by elementarity

$$L_\rho \models \Theta(\kappa)$$

and again by elementarity

$$L_\delta \models \Theta(\delta).$$

Thus δ is minimal so $\delta = \nu_\alpha$ and $M_\alpha \prec M$.

Now assume $\alpha \in S^* \cap G$. We show $L_{\nu_\alpha} \cong L_\rho\{\alpha \cup \{B, D, \kappa\}\}$ and so as L_{ν_α} is Π^1_η-correct $\alpha \in H$. Let $j : M_\alpha \prec M$ By Lemma 3.10 $L_{\nu_\alpha}\{\{B \cap \alpha, D_\alpha\} \cup \alpha + 1\} = L_{\nu_\alpha}$ and thus

$$L_{\nu_\alpha} \cong L_\rho\{j``L_{\nu_\alpha}\} = L_\rho\{\{j(B \cap \alpha), j(D_\alpha)\} \cup j``\alpha + 1\} = L_\rho\{\{B, D, \kappa\} \cup \alpha\}$$

and we're done. QED

Now as $H \supseteq S^* \cap G$ we have in V that H is η-stationary in κ, and hence by the induction $L \models H$ is η-stationary in κ. Also, for every $\alpha \in H$ we have $M_\alpha \prec M$ and hence $D_\alpha = D \cap \alpha$ and D_α is η-stationary. Then in L, D is the η stationary union of η-stationary sets, so $L \models D$ is η-stationary in κ. As $d_\eta(D)$ avoids B we have

$$L \models B \text{ is not } \eta + 1\text{-stationary in } \kappa$$

and so we're done. QED

This gives us the following equiconsistency result, showing that assumption on the γ-club filters can be as strong as Π^1_γ-indescribability.

COROLLARY 4.7. *Fix an ordinal γ. It is consistent that there is a regular cardinal $\kappa > \gamma$ such that*

1. *for all $\eta < \gamma + 1$, κ is η-normal and*
2. $\{\alpha < \kappa : \text{for all } \eta \text{ with } 0 < \eta < \gamma, \alpha \text{ is } \eta\text{-normal}\}$ *is $\gamma + 1$-stationary in κ.*

if and only if it is consistent that there is a Π^1_γ-indescribable cardinal.

PROOF: Assuming κ satisfying (1) and (2) as above, Theorem 4.5 tells us that in L, κ is $\Sigma^1_{\gamma+1}$-indescribable and hence Π^1_γ-indescribable. Thus if the existence of a regular cardinal satisfying (1) and (2) is consistent with ZFC so is the existence of a Π^1_γ-indescribable cardinal. Proposition 4.4 gives the converse. QED

If we make a stronger assumption on κ we can get the following much simpler (though weaker) statement of downward absoluteness:

COROLLARY 4.8. *Assume that for any ordinal $\gamma < \kappa$ with κ a γ-reflecting regular cardinal, that the γ-club filter on κ is normal. Then if $S \subseteq \kappa$ is γ-stationary with $S \in L$ we have $(S$ is γ-stationary in $\kappa)^L$.*

We shall see in the next section that we need the stronger statement to get the downward absoluteness of γ-ineffability.

§**5. Some applications: Ineffability and $\diamondsuit$ Principles.** In this section we give generalisations of ineffability and $\diamondsuit$ principles using γ-stationary sets. As we shall see, many of the old results follow through in this context. We shall also detail the relations between the different levels of these generalised principles. In this section γ is an ordinal less than the cardinal κ unless otherwise stated. We first recall the definition of ineffability:

DEFINITION 5.1. A regular, uncountable cardinal κ is *ineffable* iff whenever $f : [\kappa]^2 \to 2$ there is a stationary set $X \subseteq \kappa$ such that $|f``[X]^2| = 1$.

5.1. γ-Ineffables. We start by defining a new, natural generalisation of ineffability, and exploring its basic properties.

DEFINITION 5.2. A regular, uncountable cardinal κ is γ-ineffable for $\gamma < \kappa$ iff, whenever $f : [\kappa]^2 \to 2$, there is a γ-stationary set $X \subseteq \kappa$ such that $|f``[X]^2| = 1$.

REMARK. It is clear that γ-ineffability implies β-ineffability if $\beta < \gamma$. Note that 1-ineffable reduces to the ordinary definition of ineffability, and 0-ineffable is weakly compact.

From now on we shall assume $\gamma \geq 1$. The following theorem gives a useful characterisation of γ-ineffability, well-known for $\gamma = 1$.

THEOREM 5.3. *Let κ be a regular, uncountable cardinal. Then κ is γ-ineffable iff whenever $\langle A_\alpha : \alpha < \kappa \rangle$ is such that for all α, $A_\alpha \subseteq \alpha$, there is a set $A \subseteq \kappa$ such that $\{\alpha < \kappa : A_\alpha = A \cap \alpha\}$ is γ-stationary in κ.*

PROOF: ($\Rightarrow$) Suppose κ is γ-ineffable and let $\langle A_\alpha : \alpha < \kappa \rangle$ be as above. Define a function $h : [\kappa]^2 \to 2$ by $h(\{\alpha, \beta\}) = 0$ iff, assuming $\alpha < \beta$, there is some δ such that $A_\alpha \cap \delta \subsetneq A_\beta \cap \delta$. By γ-ineffibility, let X be γ-stationary such that $|h``[X]^2| = 1$. Suppose first $h``[X]^2 = \{0\}$. Then for $\alpha, \beta \in X$ with $\alpha < \beta$ we have $A_\alpha \cap \delta \subsetneq A_\beta \cap \delta$ for some δ.

For each $\nu < \kappa$, let α_ν be least in X such that $\alpha_\nu \geq \nu$ and for all $\beta \in X$ with $\beta > \alpha_\nu$, we have $A_\beta \cap \nu = A_{\alpha_\nu} \cap \nu$. This is possible as if $\langle \beta_i : i < \kappa \rangle$ enumerates $X \setminus \nu$ we have $sup\{\alpha < \kappa : A_\nu \cap \alpha \subseteq A_{\beta_i} \cap \alpha\}$ is strictly increasing, so will pass ν. Then after this $A_{\beta_i} \cap \nu$ is subset-increasing, so as κ is regular it must eventually be constant.

Let $C = \{\delta \in \kappa : \forall \nu < \delta \ \alpha_\nu < \delta\}$, which is closed unbounded. Thus $Y = X \cap C \cap LimOrd$ is γ-stationary in κ. Now for $lim(\nu)$ we have α_ν is the least $\alpha \in X$ such that $\alpha \geq sup_{\eta < \nu} \alpha_\eta$, so if $\nu \in Y$ then $\alpha_\nu = \nu$. Hence for any $\alpha \in Y$ and $\beta \in X$ with $\beta > \alpha$ we have $A_\beta \cap \alpha = A_\alpha$. Thus setting $A = \bigcup_{\alpha \in Y} A_\alpha$ we have $Y \subseteq \{\alpha \in \kappa : A \cap \alpha = A_\alpha\}$ so we're done.

Now suppose $h``[X]^2 = \{1\}$. Then for $\alpha < \beta$ in X we have $A_\alpha = A_\beta$ or $A_\alpha \cap \delta \supsetneq A_\beta \cap \delta$ for some δ. For each $\nu < \kappa$ we can define α_ν exactly as before, for similar reasons, and the construction goes through in the same way.

($\Leftarrow$) Let $f : [\kappa]^2 \to 2$ be given. For $\alpha < \kappa$ set

$$A_\alpha = \{\beta < \alpha : f(\{\beta, \alpha\}) = 1\}$$

Then by assumption there is $A \subseteq \kappa$ such that

$$X = \{\alpha < \kappa : A \cap \alpha = A_\alpha\}$$

is γ-stationary. It is easy to see that $f``[A \cap X]^2 = \{1\}$ and $f``[X \setminus A]^2 = \{0\}$. But $X = (X \cap A) \cup (X \setminus A)$, so one of these must be γ-stationary and we're done. QED

We cannot fully generalise the implication from ineffability to Π^1_2-indescribability, the reason being that γ-club filter may not coincide with the Π^1_γ-indescribability filter outside of L, although we cannot provide an example of these filters being different. We do however have that any γ-ineffable is $\gamma+1$-reflecting (Theorem 5.5 below).

THEOREM 5.4. *If $V = L$ or $\gamma \in \{1, 2\}$, and κ is γ-ineffable then κ is $\Pi^1_{\gamma+1}$-indescribable.*

The case for $\gamma = 1$ is given in [14] VII.2.2.3. We do an induction essentially following that proof.

PROOF: Assume the theorem holds for any $\eta < \gamma$. Suppose for a contradiction that φ is Δ^0_ω, $A \subseteq \kappa$ and

$$\forall X \subseteq \kappa \, \Sigma \text{ wins } G_\gamma(\kappa, \varphi, \langle A, X\rangle)$$

but for any $\alpha < \kappa$

$$\exists X \subseteq \alpha \, \Pi \text{ wins } G_\gamma(\alpha, \varphi, \langle A \cap \alpha, X\rangle)$$

For each $\alpha < \kappa$ fix some $X_\alpha \subseteq \alpha$ such that Π wins $G_\gamma(\alpha, \varphi, \langle A, X_\alpha\rangle)$. By γ-ineffability of κ, take $X \subseteq \kappa$ such that $S := \{\alpha < \kappa : X_\alpha = X \cap \alpha\}$ is γ-stationary. Now we have Σ wins $G_\gamma(\kappa, \varphi, \langle A, X\rangle)$, so we can fix $\eta < \gamma$ and $Y \subseteq \kappa$ such that Π wins $G_\eta(\kappa, \varphi', \langle A, X, Y\rangle)$. Now inductively we have that κ is Π^1_η-indescribable, so if $V = L$ or $\eta = 0, 1$, by Theorem 2.23 and Lemma 3.44 this statement reflects to an η-club C. Let $\alpha \in C \cap S$. Then as $\alpha \in S$ we have Π wins $G_\gamma(\alpha, \varphi, \langle A, X_\alpha\rangle)$, *i.e.* for any $\gamma' < \gamma$ and any $Y' \subseteq \alpha$, Σ wins $G_{\gamma'}(\alpha, \varphi', \langle A, X \cap \alpha, Y'\rangle)$. But $\alpha \in C$ so Π must win $G_\eta(\alpha, \varphi', \langle A \cap \alpha, X \cap \alpha, Y \cap \alpha\rangle)$ so we have a contradiction. QED

Outside of L, we can still obtain the following:

THEOREM 5.5. *If κ is γ-ineffable, then κ is $\gamma+1$-reflecting.*

PROOF: Suppose κ is γ-ineffable, $S \subseteq \kappa$ with S not $\gamma+1$-stationary in any $\alpha < \kappa$. We show S is not $\gamma+1$-stationary. For each $\alpha \in \kappa$ take $C_\alpha \subseteq \alpha$ to be γ_α-club avoiding S, for some $\gamma_\alpha \leq \gamma$. By γ-ineffability, let C and γ' be such that $X = \{\alpha < \kappa : C_\alpha = C \cap \alpha, \gamma' = \gamma_\alpha\}$ is γ-stationary. We claim C is γ'-club and avoids S. C is γ'-stationary as $X \subseteq C$: to see this, let $\alpha < \beta \in X$. Then C_α is γ'-stationary in α by definition of C_α. Now as $\alpha, \beta \in X$, $C_\beta \cap \alpha = C_\alpha$, and as C_β is γ' stationary closed, thus $\alpha \in C_\beta$ and so $\alpha \in C$. Also, C is γ'-stationary closed as each C_α with $\alpha \in X$ is. So C is γ' club and avoids S, and we're done. QED

In order to prove that γ-ineffability is downwards absolute to L, we now look in more detail at the theory of γ-ineffables. The following shows that if κ is $\gamma+1$-ineffable then the γ-club filter on κ is normal.

LEMMA 5.6. *If κ is γ-ineffable then κ is γ-normal.*

PROOF: Let κ be γ-ineffable. First we show that any $f : \kappa \to \kappa$ which is regressive is constant on a γ-stationary set. So let f be such a function and for each $\alpha < \kappa$ set $A_\alpha = \{f(\alpha)\}$. By ineffability, there is some $A \subseteq \kappa$ such that

$$X = \{\alpha < \kappa : A \cap \alpha = \{f(\alpha)\}\}$$

is γ-stationary. But then for $\alpha, \beta \in X$ we must have $f(\alpha) = f(\beta)$.

Now suppose for a contradiction that S is $\gamma+1$-stationary and $f : S \to \kappa$ is regressive such that for any $\alpha < \kappa$, $f^{-1}(\alpha)$ is not γ-stationary. For each $\alpha < \kappa$ take $\eta_\alpha < \gamma$ and $C_\alpha \subseteq \kappa$ such that C_α is η_α-club and avoids $f^{-1}(\alpha)$. Then as each C_α is γ-stationary closed, $\triangle_{\alpha<\kappa} C_\alpha$ is γ-closed. Also, as each C_α avoids $f^{-1}(\alpha)$ and f is regressive, $\triangle_{\alpha<\kappa} C_\alpha$ must avoid S. Thus we must have that $\triangle_{\alpha<\kappa} C_\alpha$ is not γ-stationary. Let $\eta < \gamma$ and C be a η-club avoiding $\triangle_{\alpha<\kappa} C_\alpha$. Then setting $C'_\alpha = C_\alpha \cap C$ we have C'_α is $max\{\eta, \eta_\alpha\}$-club and $\triangle_{\alpha<\kappa} C'_\alpha = \emptyset$. Let $f : \kappa \to \kappa$ be defined by $f(\alpha)$ is the least β such that $\alpha \notin C'_\beta$. As each $\alpha \notin \triangle_{\alpha<\kappa} C'_\alpha$ we have that f is well defined and regressive. By above, f is constant on a γ-stationary set X, so there is $\delta < \kappa$ such that for each $\alpha \in X$, $\alpha \notin C'_\delta$. But this contradicts C'_δ being η_δ-club. QED

DEFINITION 5.7. We say $S \subseteq \kappa$ is *γ-ineffable in κ* iff whenever $f : [S]^2 \to 2$, there is $X \subseteq S$ such that $|f``[X]^2| = 1$ and X is γ-stationary in κ.

REMARK. It is easy to see that if some $S \subseteq \kappa$ is γ-ineffable in κ then κ is γ-ineffable.

It is clear that to be γ-ineffable a subset of κ must be γ-stationary, but we shall see that such a set must in fact be $\gamma + 2$-stationary. There are always proper subsets of κ which are γ-ineffable:

PROPOSITION 5.8. *If κ is γ-ineffable and C is η-club for some $\eta < \gamma$ then C is γ-ineffable.*

PROOF: Extend $f : [C]^2 \to 2$ to $g : [\kappa] \to 2$ arbitrarily. By γ-ineffability of κ there is a γ-stationary set $X \subseteq \kappa$ such that g is constant on $[X]^2$. As X is γ-stationary, $X \cap C$ is also γ stationary, and as g extended f, we must have f is constant on $X \cap C$. QED

We also have the analogue of Theorem 5.3 for γ-ineffable subsets:

THEOREM 5.9. *Let κ be a regular, uncountable cardinal. Then $S \subseteq \kappa$ is γ-ineffable iff whenever $\langle A_\alpha : \alpha \in S\rangle$ is such that for all $\alpha \in S$, $A_\alpha \subseteq \alpha$, there is a set $A \subseteq \kappa$ such that $\{\alpha \in S : A_\alpha = A \cap \alpha\}$ is γ-stationary in κ.*

PROOF: The proof of Theorem 5.3 works in exactly the same way relativised to S.

QED

PROPOSITION 5.10. *If $S \subseteq \kappa$ is γ-ineffable then S is $\gamma + 2$-stationary.*

PROOF: This is essentially the same argument as Theorem 5.5. Suppose S is γ-ineffable but not $\gamma + 2$-stationary. Let $T \subseteq \kappa$ be $\gamma + 1$-stationary with $d_{\gamma+1}(T) \cap S = \emptyset$. Then for each $\alpha \in S$, $T \cap \alpha$ is not $\gamma + 1$-stationary in α so we can find $C_\alpha \subseteq \alpha$ which is γ-club and avoids $T \cap \alpha$. Now, using the γ-ineffability of S we can find $C \subseteq \kappa$ such that $\{\alpha \in S : C \cap \alpha = C_\alpha\}$ is γ-stationary. Then C is γ-club and avoids T - but this contradicts T being $\gamma + 1$-stationary. QED

LEMMA 5.11. *If κ is γ-ineffable then the set*

$$E_\gamma = \{\alpha < \kappa : \alpha \textit{ is } \eta\textit{-ineffable for every } \eta < \gamma\}$$

is γ-ineffable.

PROOF: Let κ be γ-ineffable. Suppose $\langle B_\alpha : \alpha \in E_\gamma\rangle$ is such that $B_\alpha \subseteq \alpha$ for each $\alpha \in E_\gamma$. We show that there is a $B \subseteq \kappa$ such that $\{\alpha \in E_\gamma : B \cap \alpha = B_\alpha\}$ is γ-stationary. For $\alpha \in E_\gamma$ set $A_\alpha = \{\beta + 1 : \beta \in B_\alpha\}$. For each $\alpha \notin E_\gamma$ let $\eta_\alpha < \gamma$ be such that α is not η_α-ineffable and let $B_\alpha = \langle B^\beta_\alpha \subseteq \beta : \beta < \alpha\rangle$ be a sequence witnessing that α is not η_α-ineffable. Let $A_\alpha \subseteq \alpha$ code the sequence B_α and η_α such that $0 \in A_\alpha$.

By γ ineffability we have a $A \subseteq \kappa$ such that

$$X = \{\alpha < \kappa : A \cap \alpha = A_\alpha\}$$

is γ-stationary. Now if $0 \notin A$ we have $X \subseteq E_\gamma$ and so we're done - setting $B = \{\alpha < \kappa : \alpha + 1 \in A\}$ we have for limit α that $A \cap \alpha = A_\alpha$ iff $B \cap \alpha = B_\alpha$, so B is as required. If $0 \in A$ then $X \cap E_\gamma = \emptyset$. We show that this leads to a contradiction.

So suppose each $\alpha \in X$ is not η-ineffable for some $\eta < \gamma$. Then A codes $B = \langle B^\beta \subseteq \beta : \beta < \kappa \rangle$ and η such that for each $\alpha \in X$, $B \restriction \alpha = B_\alpha$ and $\eta_\alpha = \eta$.. Using the γ-ineffability of κ again, let $C \subseteq \kappa$ be such that

$$Y = \{\alpha < \kappa : C \cap \alpha = B^\alpha\}$$

is γ-stationary. As κ is η-reflecting $d_\eta(Y)$ is η-club. But then for $\alpha \in d_\eta(Y) \cap X$ we have $\{\beta < \alpha : C \cap \beta = B^\beta_\alpha\} = \{\beta < \alpha : C \cap \beta = B^\beta\} = Y \cap \alpha$ and must therefore be η-stationary in α. But this contradicts our assumption that $\langle B^\alpha_\beta : \beta < \alpha \rangle$ was a witness that α was not η_α-ineffable, as $C \cap \alpha$ correctly guesses B_β on a $\eta = \eta_\alpha$-stationary subset of α. So we have a contradiction and E_γ is γ-ineffable. QED

We can now show that γ-ineffability is downward absolute to L. Recall that ([14] VII.2.2.5) if κ is ineffable then $(\kappa \text{ is ineffable})^L$. To generalise this proof we shall use Theorem 4.5.

THEOREM 5.12. *If κ is γ-ineffable then $(\kappa$ is γ-ineffable$)^L$.*

PROOF: Let κ be γ-ineffable. First we show that κ satisfies (1) and (2) of Theorem 4.5. By Lemma 5.6 we have that κ is γ-normal and hence η-normal for any $\eta < \gamma$. By Lemma 5.11 we have $\{\alpha < \kappa : \text{for all } \eta < \gamma, \alpha \text{ is } \eta\text{-ineffable}\}$ is γ-ineffable. But if α is η-ineffable then α is η-normal so setting $E = \{\alpha < \kappa : \text{for all } \eta < \gamma, \alpha \text{ is } \eta\text{-normal}\}$ then E is γ-ineffable in κ and hence $\gamma + 2$-stationary. Thus

$$A_\gamma = \{\alpha < \kappa : \textit{for all } \eta \textit{ with } 1 < \eta + 1 < \gamma \ \alpha \textit{ is } \eta\textit{-normal}\} \supseteq E$$

and so A_γ is γ-stationary.

Now, using the characterisation of γ-ineffability from Theorem 5.3, let $\langle A_\alpha : \alpha < \kappa \rangle$ be a sequence in L with each $A_\alpha \subseteq \alpha$. This is clearly such a sequence in V, so as E is γ-ineffable we can find a set $A \subseteq \kappa$ such that

$$X = \{\alpha < \kappa : A_\alpha = A \cap \alpha \text{ and for every } \eta < \gamma, \alpha \text{ is } \eta\text{-normal}\}$$

is γ-stationary. Then by the weak compactness of κ, as each $A_\alpha \in L$ and X is unbounded in κ, we have $A \in L$. Setting $X' = \{\alpha < \kappa : A_\alpha = A \cap \alpha\}$, we have $X' \in L$ and $X \cap X' = X$ is γ-stationary. Thus by Theorem 4.5 X' is γ-stationary in L, and hence $(\kappa \text{ is } \gamma\text{-ineffable})^L$. QED

5.2. Diamond Principles. We now turn to generalising diamond ($\diamondsuit$) principles, and relate them to our generalised ineffability. Like $\square(\kappa)$, $\diamondsuit_\kappa$ asserts the existence of a sequence of sets $S_\alpha \subseteq \alpha$ for $\alpha < \kappa$ - in the case of $\diamondsuit$ the sequence must "guess" any subset of κ sufficiently often. For the original $\diamondsuit$, "sufficiently" is "stationarily", so by altering this to γ-stationarity we can define a new notion:

DEFINITION 5.13. $\diamondsuit^\gamma_\kappa$ is the assertion that there is a sequence $\langle S_\alpha : \alpha < \kappa \rangle$ such that for any $S \subseteq \kappa$ we have $\{\alpha < \kappa : S_\alpha = S \cap \alpha\}$ is γ-stationary in κ.

The original principle is thus $\diamondsuit^1_\kappa$.

REMARK. As with ineffability, these principles get stronger as γ increases: for $\beta < \gamma$ we have $\diamondsuit^\gamma_\kappa \Rightarrow \diamondsuit^\beta_\kappa$.

THEOREM 5.14. *If κ is γ-ineffable then $\diamondsuit^\gamma_\kappa$ holds.*

PROOF: Define by recursion a sequence $\langle (S_\alpha, C_\alpha, \eta_\alpha) : \alpha < \kappa \rangle$: let (S_α, C_α) be any pair of subsets of α with $\eta_\alpha < \gamma$ such that C_α is η_α-club in α and for any $\beta \in C_\alpha$ we have $S_\alpha \cap \beta \neq S_\beta$. If there is no such pair, set $S_\alpha = C_\alpha = \eta_\alpha = \emptyset$. We now use γ-inneffability to see that $\langle S_\alpha : \alpha < \kappa \rangle$ is a $\diamondsuit^\gamma_\kappa$-sequence.

By the characterisation of γ-ineffability in Theorem 5.3 and some simple coding (noting that successor levels are irrelevant), we can find $S, C \subseteq \kappa$ and $\eta < \gamma$ such that

$$A = \{\alpha < \kappa : S \cap \alpha = S_\alpha \wedge C \cap \alpha = C_\alpha \wedge \eta_\alpha = \eta\}$$

is γ-stationary.

Let $X \subseteq \kappa$ and suppose that $B := \{\alpha < \kappa : S_\alpha = X \cap \alpha\}$ is not γ-stationary in κ. Take D γ'-club in κ witnessing this, i.e $\gamma' < \gamma$ and for any $\alpha \in D$, $X \cap \alpha \neq S_\alpha$. By γ-stationarity of A we can pick $\alpha < \beta$ both in $A \cap d_{\gamma'}(D)$. Then $C_\beta \cap \alpha = C \cap \beta \cap \alpha = C \cap \alpha = C_\alpha$. Now as $(X \cap \alpha, D \cap \alpha, \gamma')$ works for the definition of $(S_\alpha, C_\alpha, \eta_\alpha)$ we must have that C_α is η-club in α. Hence $C_\beta \cap \alpha = C_\alpha$ is η-stationary in α, and so $C_\beta \neq \emptyset$ and we have by η-stationary closure that $\alpha \in C_\beta$. Then by definition of $(S_\beta, C_\beta, \eta_\beta)$ we must have $S_\alpha \neq S_\beta \cap \alpha$. But this is a contradiction as $\alpha, \beta \in A$ so $S_\alpha = S \cap \alpha = S_\beta \cap \alpha$. QED

The $\diamondsuit$ principle itself is not a large cardinal notion, and in fact within the constructible universe $\diamondsuit_\kappa$ holds at every regular uncountable cardinal κ (see [14] Chapter IV). Our new $\diamondsuit_\kappa^\gamma$ principles also hold in L, with minimal assumptions on κ, as we shall now see.

As γ stationarity is Π^1_γ expressible we have, as a corollary to Lemma 3.37, the following:

LEMMA 5.15. *If $V = L$ and κ is Π^1_γ-indescribable then for any limit $\nu > \kappa$ with L_ν Π^1_γ-correct over κ we have $\int^\gamma(L_\nu, p, \kappa)$ is γ-club in κ.*

DEFINITION 5.16. A model $\langle M, \in\rangle$ is γ-stationary correct at κ if for any $S \in \mathcal{P}(\kappa) \cap M$, $M \models$ "S is γ-stationary in κ" iff S is γ-stationary in κ.

PROPOSITION 5.17. *If $V = L$ and κ is a γ-stationary regular cardinal then $\diamondsuit_\kappa^\gamma$ holds.*

PROOF: By recursion we define, for each $\alpha < \kappa$, an ordinal $\eta_\alpha < \gamma$ and a pair of subsets of α (S_α, C_α) (the S_α's will form our $\diamondsuit^\gamma$-sequence).

Assume we have defined $\langle S_\beta : \beta < \alpha\rangle$. Let $\psi(\alpha, \eta, C, S)$ be the statement that η is an ordinal below γ, and (S, C) is a pair of subsets of α with C η-club in α and for any $\beta \in C$, $S \cap \beta \neq S_\beta$.

If there are η, S and C such that $\psi(\alpha, \eta, S, C)$ holds, we take η_α to be the least such η, and then (S_α, C_α) the $<_L$ least pair with $\psi(\alpha, \eta_\alpha, S_\alpha, C_\alpha)$. If no such η exists set $S_\alpha = C_\alpha = \eta_\alpha = \emptyset$.

Suppose $\langle S_\alpha : \alpha < \kappa\rangle$ is not a $\diamondsuit_\kappa^\gamma$-sequence and take η and (S, C, η) to be the least witness to this, i.e $\psi(\kappa, \eta, S, C)$ holds and if $\eta' < \eta$ or $(S', C') <_L (S, C)$ then $\neg\psi(\kappa, \eta', S', C')$. Note that all this can be carried out in L_{κ^+}, and that as $\eta < \gamma$ and we are in L, κ is Π^1_η-indescribable and hence by Lemma 3.37 the η trace forms an η-club.

Suppose $\alpha \in \int^\eta(L_{\kappa^+}, \kappa, \{S, C, \eta\})$ with $\alpha > \eta$ and $L_\nu \cong L_{\kappa^+}\{\alpha, \{S, C, \eta\}\}$.

$L_\nu \models$ "η is the least ordinal such that there exists a pair (C', S') with $\psi(\alpha, \eta, S', C,)$ and $(C \cap \alpha, S \cap \alpha)$ is the $<_L$ least such pair"

As α is in the η trace, we have L_ν is η-stationary correct so $\psi(\alpha, \eta, S \cap \alpha, C \cap \alpha)$ must hold and $\psi(\alpha, \eta', S', C,)$ fail for $\eta' < \eta$ or $(S', C') <_L (S \cap \alpha, C \cap \alpha)$. But this was the definition of S_α so $S_\alpha = S \cap \alpha$, contradiction. QED

A stronger principle than $\diamondsuit$ is $\diamondsuit^*$, which requires sets to be guessed on a club set of α, but allows for more guesses at each α. Unlike $\diamondsuit$, $\diamondsuit^*$ is incompatible with ineffability (see [5]). The original principle is $\diamondsuit^{*1}$ in the following definition.

DEFINITION 5.18. $\diamondsuit_\kappa^{*\gamma}$ is the assertion that there is a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ such that $A_\alpha \subseteq \mathcal{P}(\alpha)$ and $|A_\alpha| \leq |\alpha|$ for each $\alpha < \kappa$, and for any $X \subseteq \kappa$ there is some $\gamma' < \gamma$ such that $\{\alpha < \kappa : X \cap \alpha \in A_\alpha\}$ is in the γ'-club filter on κ.

REMARK. In contrast to the case for $\diamondsuit$, here we have that at γ-reflecting cardinals κ, $\diamondsuit_\kappa^{*\gamma'}$ implies $\diamondsuit_\kappa^{*\gamma}$ for $\gamma' < \gamma$.

As for the original principle, $\diamondsuit_\kappa^{*\gamma}$ cannot hold at a γ-ineffable cardinal κ:

THEOREM 5.19. *If κ is γ-ineffable then $\diamondsuit_\kappa^{*\gamma}$ fails.*

PROOF: Suppose κ is γ-ineffable and let $\langle A_\alpha : \alpha < \kappa\rangle$ be a sequence such that $A_\alpha \subseteq \mathcal{P}(\alpha)$ and $|A_\alpha| \leq |\alpha|$ for each $\alpha < \kappa$. We find $B \subseteq \kappa$ such that $\{\alpha < \kappa : B \cap \alpha \notin A_\alpha\}$ is γ-stationary.

For each $\alpha < \kappa$ let $B_\alpha \subseteq \alpha$ be a set different from each set in A_α - we can find such a set as $|A_\alpha| = \alpha$. Now by γ-ineffability there is $B \subseteq \kappa$ such that $X = \{\alpha < \kappa : B_\alpha = B \cap \alpha\}$ is γ-stationary. But then B is not guessed by $\langle A_\alpha : \alpha < \kappa\rangle$ on X, so $\langle A_\alpha : \alpha < \kappa\rangle$ cannot be a $\diamondsuit^{*\gamma}$-sequence. As $\langle A_\alpha : \alpha < \kappa\rangle$ was arbitrary, $\diamondsuit^{*\gamma}_\kappa$ fails.

QED

In fact, the notion of $\diamondsuit^\gamma_\kappa$ is only really of interest for a successor ordinal $\gamma < \kappa$:

PROPOSITION 5.20. *If γ is a limit ordinal and $\diamondsuit^\gamma_\kappa$ holds iff there is some $\gamma' < \gamma$ such that $\diamondsuit^{\gamma'}_\kappa$.*

PROOF: Suppose γ is a limit ordinal and $\diamondsuit^\gamma_\kappa$ holds. Let $\mathcal{A} = \langle A_\alpha : \alpha < \kappa\rangle$ a sequence such that $A_\alpha \subseteq \mathcal{P}(\alpha)$ and $|A_\alpha| \leq |\alpha|$ for each $\alpha < \kappa$. Assume that $\mathcal{A}$ is constructibly closed, in that for each α, $A_\alpha = \mathcal{P}(\alpha) \cap L_\nu[A_\alpha]$ for some limit ordinal ν with $\alpha < \nu < \alpha^+$ - clearly we can always expand $\mathcal{A}$ to satisfy this condition, and $\mathcal{A}$ will remain a $\diamondsuit^{*\gamma}_\kappa$-sequence. Suppose for each $\gamma' < \gamma$, $\mathcal{A}$ is not a $\diamondsuit^{*\gamma'}_\kappa$-sequence, and take $B_{\gamma'}$ to witness this, setting $X_{\gamma'} = \{\alpha < \kappa : B_{\gamma'} \cap \alpha \notin A_\alpha\}$, which is γ'-stationary. Set $X = \bigcup X_{\gamma'}$ and let B code (definably and uniformly) each of the $B_{\gamma'}$'s. Then X is γ-stationary and we claim $X \subseteq \{\alpha < \kappa : B \cap \alpha \notin A_\alpha\}$. Suppose $\alpha \in X$ and $B \cap \alpha \in A_\alpha$. Then as the coding was definable, each $B_{\gamma'} \cap \alpha \in A_\alpha$ - contradiction. QED

COROLLARY 5.21. *If γ is a limit ordinal and κ is γ'-ineffable for every $\gamma' < \gamma$ then $\diamondsuit^{*\gamma}_\kappa$ fails.*

In L we have that, given the precondition of γ-stationarity, the failure of $\diamondsuit^{*\gamma}_\kappa$ characterises the γ-ineffables - but only for successor ordinals γ.

THEOREM 5.22. *Assume $V = L$ and let κ be a regular cardinal that is γ-stationary with γ a successor ordinal. Then κ is not γ-ineffable iff $\diamondsuit^{*\gamma}_\kappa$ holds.*

PROOF: ($\Rightarrow$) Let κ be a regular uncountable cardinal which is not γ-ineffable. Let $\langle A_\alpha : \alpha < \kappa\rangle$ be the $<_L$ least sequence such that $A_\alpha \subseteq \alpha$ for each α and for any $A \subseteq \kappa$ we have $\{\alpha < \kappa : A_\alpha = A \cap \alpha\}$ is not γ-stationary. For each $\alpha < \kappa$ set M_α to be the least $M \prec L_\kappa$ such that $\alpha + 1 \subseteq M_\alpha$ and $\langle A_\alpha : \alpha < \kappa\rangle \in M$. Let $\sigma_\alpha : M_\alpha \cong L_{\nu_\alpha}$.

Set $S_\alpha = \mathcal{P}(\alpha) \cap L_{\nu_\alpha}$, and note $|L_{\nu_\alpha}| = |\nu_\alpha| = |\alpha|$, so $|S_\alpha| \leq |\alpha|$. we shall show that if $\eta + 1 = \gamma$ then for any $X \subseteq \kappa$, setting $C = \int^\eta(L_{\kappa^+}, \{X\}, \kappa)$ we have C is η-club and $X \cap \alpha \in S_\alpha$ for all $\alpha \in C$. Fix such an X and take $\alpha \in \int^\eta(L_{\kappa^+}, \{X\}, \kappa)$. Let $\pi : N_\alpha \cong L_\mu$. Then $\pi\restriction\alpha = \alpha$, $\pi(\kappa) = \alpha$ and $\pi(X) = X \cap \alpha$, so $X \cap \alpha \in L_\beta$. Thus we are done if we can show $\mu \leq \nu_\alpha$. Suppose to the contrary that $\mu > \nu_\alpha$. As $\langle A_\alpha : \alpha < \kappa\rangle$ is definable in L_{κ^+} and $\alpha \in M_\alpha$ we have $\langle A_\gamma : \gamma \leq \alpha\rangle \in M_\alpha$. Then $\sigma_\alpha(\langle A_\gamma : \gamma \leq \alpha\rangle) = \langle A_\gamma : \gamma \leq \alpha\rangle$ because $\alpha + 1 \subseteq M_\alpha$, so $\langle A_\gamma : \gamma \leq \alpha\rangle \in L_{\nu_\alpha} \subseteq L_\beta$. Setting $E = \{\gamma < \alpha : A_\gamma = A_\alpha \cap \gamma\}$ we have $E \in L_\beta$ and $\pi^{-1}(E) = \{\gamma < \kappa : A_\gamma = \gamma \cap \pi^{-1}(A_\alpha)\}$. First suppose $L_\beta \vDash$ "E is γ-stationary in α". Then $\pi^{-1}(E)$ is γ-stationary in κ by elementarity. Also we have (in L_{κ^+} and hence in L) that $\pi^{-1}(A_\alpha) \subseteq \kappa$ and $\pi^{-1}(E) = \{\beta < \kappa : A_\beta = \beta \cap \pi^{-1}(A_\alpha)\}$. But this is a contradiction as $\langle A_\alpha : \alpha < \kappa\rangle$ was chosen to witness κ not being γ-ineffable.

So we must have $L_\beta \vDash$ "E is not γ-stationary in α". Thus for some $C \subseteq \alpha$, $L_\beta \vDash$ "C is η-club in α and $C \cap E = \emptyset$". Then inverting the collapse we get $C' = \pi^{-1}(C)$ is η-club in κ and $C' \cap \pi^{-1}(E) = \emptyset$". As L_β is in the η-trace it is η-stationary correct for α. Now $C' \cap \alpha = C$ and by η-stationary correctness, we have that $C' \cap \alpha$ is η-stationary, and hence $\alpha \in C'$. Thus by the definition of E we have $A_\alpha \neq \alpha \cap \pi^{-1}(A_\alpha)$. But this is a contradiction as $\pi\restriction\alpha = id\restriction\alpha$. Thus we must have $\mu \geq \nu_\alpha$ and we're done.

The converse direction ($\Leftarrow$) follows from Theorem 5.19. QED

In fact this characterisation can fail for limit γ: Assuming there is an ω-ineffable we can take κ to be the least cardinal which is n-ineffable for every $n < \omega$. Then κ is not Π^1_ω-indescribable as being n-ineffable is Π^1_{n+2} over L_κ, so being n-ineffable for every $n < \omega$ is Π^1_ω. Hence (as we are in L), κ is not ω-reflecting, and so κ is not ω-ineffable. Now, for each $n < \omega$, as κ is n-ineffable $\diamondsuit^{*n}$ fails, so by Proposition 5.20 $\diamondsuit^{*\omega}$ also fails. We can however give a complete description of when $\diamondsuit^{*\gamma}_\kappa$ holds in L:

COROLLARY 5.23. *If $V = L$ then $\diamondsuit^{*\gamma}_\kappa$ holds iff κ is γ-stationary and one of the following:*

1. *γ is a successor and κ is not γ-ineffable*
2. *γ is a limit and κ is not γ'-ineffable for some $\gamma' < \gamma$.*

PROOF: For γ a successor this is just Theorem 5.22. For γ a limit by Proposition 5.20 if $\diamondsuit^{*\gamma}_\kappa$ holds then for some $\gamma' < \gamma$, $\diamondsuit^{*\gamma'}_\kappa$ holds so by Theorem 5.19 κ is not γ'-ineffable. The converse direction is immediate from Theorem 5.22, remembering that for $\gamma' < \gamma$, $\diamondsuit^{*\gamma'}_\kappa$ implies $\diamondsuit^{*\gamma}_\kappa$. QED

Now we turn to the relationship between $\diamondsuit$ and $\diamondsuit^*$. In the classical case Kunen showed that $\diamondsuit^*_\kappa$ implies $\diamondsuit_\kappa$ (see [26], 5.38). If the γ-club filter on κ is normal then we can generalise this: $\diamondsuit^{*\gamma+1}_\kappa$ implies $\diamondsuit^{\gamma+1}_\kappa$. The proof goes *via* a weaker principle, $\diamondsuit^{-\gamma}$:

DEFINITION 5.24. $\diamondsuit^{-\gamma}_\kappa$ is the assertion that there is a sequence $\langle A_\alpha : \alpha < \kappa\rangle$ such that $A_\alpha \subseteq \mathcal{P}(\alpha)$, $|A_\alpha| \leq |\alpha|$ for each $\alpha < \kappa$, and for any $X \subseteq \kappa$ the set $\{\alpha < \kappa : X \cap \alpha \in A_\alpha\}$ is γ-stationary in κ.

This is a clear weakening of both $\diamondsuit^*$ and $\diamondsuit$. The following proof is essentially based on one of Kunen: see [19], Theorem 7.14.

THEOREM 5.25. *If the γ-club filter on κ is normal then $\diamondsuit^{\gamma+1}_\kappa \leftrightarrow \diamondsuit^{-\gamma+1}_\kappa$, and hence $\diamondsuit^{*\gamma+1}_\kappa \to \diamondsuit^{\gamma+1}_\kappa$.*

PROOF: It is clear that $\diamondsuit^{\gamma+1}_\kappa \to \diamondsuit^{-\gamma+1}_\kappa$. Suppose the γ-club filter on κ is normal and $\diamondsuit^{-\gamma+1}_\kappa$ holds. Let $\langle \mathcal{A}_\alpha : \alpha < \kappa\rangle$ be a $\diamondsuit^{-\gamma+1}_\kappa$-sequence such that each $A \in \mathcal{A}_\alpha$ codes a subset of $\alpha \times \alpha$. Let $\{A^\beta_\alpha : \beta < \alpha\}$ enumerate these coded sets in $\mathcal{A}_\alpha$.

For a given α and $\beta < \alpha$, set $S^\beta_\alpha = \{\nu \in \alpha : (\nu, \beta) \in A^\beta_\alpha\}$. We shall see that for some $\beta < \kappa$, $\langle S^\beta_\alpha : \alpha < \kappa\rangle$ is a $\diamondsuit^{\gamma+1}_\kappa$-sequence. Suppose this is not the case. For each $\beta < \kappa$ take $X^\beta \subseteq \kappa$ and C^β γ-club such that for $\alpha \in C^\beta$, $X^\beta \cap \alpha \neq S^\beta_\alpha$. Set

$$X = \bigcup_{\beta<\kappa} X^\beta \times \{\beta\}$$

and

$$C = \triangle_{\beta<\kappa} C^\beta.$$

By the normality of the γ-club filter, C is γ-club. Then for $\alpha \in C$, if $\beta < \alpha$ then $X^\beta \neq S^\beta_\alpha$ and so $X \cap (\alpha \times \alpha) \neq A^\beta_\alpha$. Thus $X \notin \mathcal{A}_\alpha$. But this contradicts $\langle \mathcal{A}_\alpha : \alpha < \kappa\rangle$ being a $\diamondsuit^{-\gamma+1}_\kappa$-sequence. QED

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