Generalised transfinite Turing machines and strategies for games.  

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• Theme: Connections between inductive operators, discrete transfinite machine models of computation, and determinacy.
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• Part I: ITTM description.
• Part II: Fixed points of operators yielding some strategies.
• Part III: Generalising Operators and Machines
Part I: ITTM description

- Allow a standard Turing machine to run transfinitely using one of the usual programs $\langle P_e \mid e \in \mathbb{N} \rangle$.
- Alphabet: $\{0, 1\}$;
- Enumerate the cells of the tape $\langle C_k \mid k \in \mathbb{N} \rangle$.

Let the current instruction about to be performed at time $\tau$ be $I_{i(\tau)}$;
Let the current cell being inspected be $C_{p(\tau)}$.

\[ [HL] \text{Hamkins & Lewis “Infinite Time Turing Machines”, JSL, vol. 65, 2000.} \]
Part I: ITTM description\(^1\)

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  Let the current instruction about to be performed at time \(\tau\) be \(I_{i(\tau)}\);
  Let the current cell being inspected be \(C_{p(\tau)}\).
- Behaviour at successor stages \(\alpha \to \alpha + 1\): as normal.

At limit times \(\lambda\):

(a) we specify cell values by:

\[
C_k(\lambda) = \operatorname{Liminf}_{\beta \to \lambda} C_k(\alpha)
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(where the value in \(C_k\) at time \(\tau\) is \(C_k(\tau)\)).

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At limit times $\lambda$: (a) we specify cell values by:

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(where the value in $C_k$ at time $\tau$ is $C_k(\tau)$).

(b) we also (i) put the R/W to cell $C_{p(\lambda)}$ where

$$p(\lambda) = \operatorname{Liminf}_{\alpha \rightarrow \lambda}^{\ast} \{p(\beta) | \alpha < \beta < \lambda\};$$

(ii) set

$$i(\lambda) = \operatorname{Liminf}_{\alpha \rightarrow \lambda} \{i(\beta) | \alpha < \beta < \lambda\}.$$
Hamkins & Lewis proved there is a universal machine, an $S^m_n$-Theorem, and a Recursion Theorem for ITTM’s, and a wealth of results on the resulting ITTM-degree theory.

We may define halting sets:

$$H = \{(e, x) \mid e \in \mathbb{N}, x \in 2^\mathbb{N} \land P_e(x) \downarrow\}$$

$$H_0 = \{(e, 0) \mid e \in \mathbb{N} \land P_e(0) \downarrow\}$$
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Q. What is $H$ or $H_0$?

Q. How long do we have to wait to discover if $e \in H_0$ or not?

Q. What are the ITTM (semi)-decidable sets of integers? Or reals?
We’d like some type of a “ITTM Normal Form Theorem”:

**Theorem**

*There is a universal predicate $\mathcal{T}$ which satisfies $\forall e \forall x$:*

$$P_e(x) \downarrow z \iff \exists y \in 2^{\mathbb{N}} [\mathcal{T}(e, x, y) \land \text{Last}(y) = z].$$
A Kleene Normal Form Theorem?

We’d like some type of a “ITTM Normal Form Theorem”:

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*However for this to occur we need to know whether the ordinal length of any computation is capable of being output or written by another computation.*
The $\lambda, \zeta, \Sigma$-Theorem\(^2\)

**Theorem**

*Let $\zeta$ be the least ordinal so that there exists $\Sigma > \zeta$ with the property that*

$$L_\zeta \prec_{\Sigma_2} L_\Sigma; \quad (\zeta \text{ is } \text{“$\Sigma_2$-extendible”}.)$$

*(i) Then the universal ITTM on integer input first enters a loop at time $\zeta$.*

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$$L_\lambda \prec_{\Sigma_1} L_\zeta.$$ 

(ii) Then $\lambda = \sup\{\alpha \mid \exists e \ P_e(0) \downarrow \text{in } \alpha \text{ steps}\}$

$$= \sup\{\alpha \mid \exists e \ P_e(0) \downarrow y \in \text{WO} \wedge ||y|| = \alpha\}.$$ 

\footnote{Welch The length of ITTM computations, Bull. London Math. Soc. 2000}
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• As a corollary one derives the Normal Form Theorem and:

Corollary

$$H_0 \equiv \Sigma_1\text{-Th}(L_\lambda).$$

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Part II: Over $\mathbb{N}$ \( \Pi^1_1\text{-IND} = \emptyset \Sigma^0_1 \)

- The *game* quantifier $\emptyset$:

**Definition**
A set $A \subseteq \mathbb{N}$ is $\emptyset \Gamma$ if there is $B \in \mathbb{N} \times \mathbb{R}$ so that:

$$n \in A \iff \text{Player I has a winning strategy in } G_{B_n}$$

where $B_n = \{ x \in \mathbb{R} \mid B(n, x) \}$. 

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**Theorem (Folklore)**
A set $A \subseteq \mathbb{N}$ is (mon.-$\Pi^1_1$)-IND iff it is $\exists \Sigma^0_1$ iff it is $\Sigma_1(L_{\omega_1^{ck}})$. 
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Any $\Sigma^0_1$ game, if it is a win for Player I then she has a (mon.-$\Pi^1_1$)-IND winning strategy. Hence this is also $\Sigma^1_1(L_{\omega^1_{ck}})$. 
Theorem (Solovay)

A set $A \subseteq \mathbb{N}$ is $\Sigma^1_1$-IND iff it is $\varnothing \Sigma_2^0$ iff it is $\Sigma_1(\mathcal{L}_{\sigma^1_1})$.

(where $\sigma^1_1$ is the closure ordinal of $\Sigma^1_1$-inductive definitions).
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**Theorem (Friedman)**

$Z_2 \not\vdash \Sigma^0_4$-Det.
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Q Are strategies for $\Sigma^0_3$ sets ITTM-semi-decidable? Thus: are they $\Sigma_1(L_{\Sigma})$?
Definition
Let “ITTM” abbreviate: “$\forall X (H^X_0 \text{ exists})$”
(“the complete ITTM-semi-decidable-in-X set exists”).

3Welch “Weak Systems of Determinacy and Arithmetical Quasi-Inductive Definitions”, JSL, June 2011
Definition
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(“the complete ITTM-semi-decidable-in-X set exists”).

Theorem
The theories:
\[ \Pi_3^1-\text{CA}_0, \Delta_3^1-\text{CA}_0 + \Sigma_3^0-\text{Det}, \Delta_3^1-\text{CA}_0 + \text{ITTM}, \Delta_3^1-\text{CA}_0 \]
are in strictly descending order of strength\(^3\).

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Open Questions

Q1 Give another description of the least $\beta$ over which strategies for $\Sigma^0_3$ sets are definable.
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Conjecture 1: Lubarsky’s ‘Feedback-ITTM’s’ are related to this.
Lubarsky’s Feedback ITTM’s

Lubarsky has suggested a variant of the HL-machine which is allowed to make calls to sub-routines to obtain the answer to the question:

“Does the ITTM-computation $P_e$ with the current real $x$ of the scratch tape halt or loop?”.

- In other words one considers ITTM-computations recursive in $H$.
- Clearly such an FITTM may make an infinite chain of such calls in which case Lubarsky calls the computation “freezing”.

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4R. Lubarsky Well founded iterations of Infinite Time Turing Machines, 2010.
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• In other words one considers ITTM-computations recursive in $H$.

• Clearly such an FITTM may make an infinite chain of such calls in which case Lubarsky calls the computation “freezing”.

• The natural Conjecture 1 emerges that any winning strategy for a $\Sigma^0_3$ game which wins for player $I$ can be written by an FITTM.

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Or, Conjecture 2:

- Does $\beta = \text{the closure ordinal of monotone-}\mathcal{E}\Pi_3^0\text{-Inductive Operators}$?
Or, Conjecture 2:

- Does $\beta$ = the closure ordinal of monotone-$\mathcal{E}\Pi^0_3$-Inductive Operators?

Theorem

(i) If $\gamma < \beta$ is least with $L_\gamma \prec_{\Sigma_1} L_\beta$ then $\gamma$ is the closure ordinal of monotone-$\mathcal{E}\Sigma^0_3$-Inductive Operators.
Or, Conjecture 2:

- Does $\beta = $ the closure ordinal of monotone-$\mathcal{O}\Pi_3^0$-Inductive Operators?

**Theorem**

(i) If $\gamma < \beta$ is least with $L_\gamma \prec_{\Sigma_1} L_\beta$ then $\gamma$ is the closure ordinal of monotone-$\mathcal{O}\Sigma_3^0$-Inductive Operators.

(ii) $\Sigma_1$-$Th(L_\gamma)$ is a complete $\mathcal{O}\Sigma_3^0$ set.
Part III: Hypermachines\textsuperscript{5}

- Can we find ‘machines’ that will lift the $\Sigma_2$ “Liminf” of [HL] to a $\Sigma_n$-rule at limit stages?

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**Theorem (S.Friedman-W)**

*For any $n \geq 2$ there is such a $\Sigma_n$ limit rule, so that for a machine running under such a rule, the universal $\Sigma_n$-machine on integer input first enters a loop at the least $\zeta(n)$ s.t. $\exists \Sigma(n) > \zeta(n)$ with

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- Such machines then compute, taken as a whole, all the reals of the least $\beta$-model of analysis $2^\omega \cap L_{\beta_0}$.

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• Such machines then compute, taken as a whole, all the reals of the least $\beta$-model of analysis $2^\omega \cap L_{\beta_0}$.

• Then, *e.g.* using Montalban-Shore, strategies for $n-\Sigma^0_3$ games are computable by the $\Sigma_{n+2}$-machines.
Open Questions continued

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Define semi-decidable sets of reals using the ITTM’s (and $\Sigma_n$-hypermachines) in a standard way; this yields pointclasses $\Gamma_n$ strictly within $\Delta_2^1$. 
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Define semi-decidable sets of reals using the ITTM’s (and $\Sigma_n$ -hypermachines) in a standard way; this yields pointclasses $\Gamma_n$ strictly within $\Delta^1_2$.

Q4 Quantify $\text{Det}(\Gamma_n)$.
A sample theorem of what is known:

**Theorem**

\[ ZFC + Det(\Gamma_2) \Rightarrow \text{There is an inner model with a proper class of strong cardinals}^{6}. \]