Weak systems of determinacy and arithmetical quasi-inductive definitions.

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Abstract

We show that the theories: $\mathbf{\Pi^1_3}$-$\mathbf{CA_0}$, $\mathbf{\Delta^1_3}$-$\mathbf{CA_0}$+$\mathbf{\Sigma^0_3}$-Determinacy, $\mathbf{\Delta^1_3}$-$\mathbf{CA_0}$+**AQI**, and $\mathbf{\Delta^1_3}$-$\mathbf{CA_0}$ are in strictly descending order of strength. (Here **AQI** is the assertion that every arithmetical quasi-inductive definition converges.)

More precisely, we locate winning strategies for various Σ^0_3-games in the L-hierarchy in order to prove the following:

Theorem 1 KP $\vdash \Sigma_2$-Separation $\rightarrow \exists$ *a* β-*model of* $\mathbf{\Delta^1_3}$-$\mathbf{CA_0}$ + $\mathbf{\Sigma^0_3}$-Determinacy .

Alternatively: $\mathbf{\Pi^1_3}$-$\mathbf{CA_0}$ $\vdash$there is a β-model of $\mathbf{\Delta^1_3}$-$\mathbf{CA_0}$ + $\mathbf{\Sigma^0_3}$-Determinacy . The implication is not reversible.

Theorem 2 KP +Δ_2-Comprehension +Σ_2-Collection +**AQI** $\nvdash \mathbf{\Sigma^0_3}$-Determinacy .

Alternatively: $\mathbf{\Delta^1_3}$-$\mathbf{CA_0}$ +**AQI** $\nvdash \mathbf{\Sigma^0_3}$-Determinacy.

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1 Introduction

The work in this paper was, initially at least, motivated by trying to see how the theory of *arithmetical quasi-inductive definitions* (**AQI** as defined below) fits in with other subsystems of second order number theory. We had been working with one example of such a definition, essentially a *recursive* quasi-inductive definition, and had calculated certain ordinals where such definitions reached fixed points or exhibited a looping convergence [15] and [14]. Earlier J. Burgess [2] had in fact distilled from H. Herzberger's notion of a *revision sequence* [5] the notion of arithmetically quasi-inductive, and shown that the same ordinals appeared. (Herzberger's notion was connected with a "truth operator" and thus, strictly speaking is not arithmetic, but just beyond; however this only makes for a trivial difference.) Other examples of constructions involving such quasi-inductive definitions have appeared in the theory of truth [4], and in theoretical computer science: S.Kreutzer in [7] uses essentially arithmetical quasi-inductive definitions to formulate a notion of semantics for partial fixed point logics over structures with infinite domains which separates away this logic from inflationary fixed point logic.

Here however we rather mention some of the possibilities that connect these concepts with potential proof theoretical results on the way to looking at ordinal systems for Π^1_3-CA$_0$: for Π^1_2-CA$_0$, by work of Rathjen [9],[10], we have that this second level of Comprehension is tied up with the theory of arbitrarily long finite Σ_1-elementary chains through the L_α-hierarchy: the first level L_α which is an infinite tower of such models, is the first whose reals form a β-model of Π^1_2-CA$_0$. The same occurs for Π^1_3-CA$_0$: the firstL_β whose reals form a β-model of Π^1_3-CA$_0$ is the union of an infinite tower of models $L_{\zeta_n} \prec_{\Sigma_2} L_{\zeta_{n+1}}$ and presumably one will need to analyse finite chains of such models to get at an ordinal system for this theory. Seeing that **AQI** is connected with levels L_ζ of the Gödel's L-hierarchy, with Σ_2-end extensions, (albeit only chains of length 1) analysing the proof theoretic strength of **AQI** would be a natural stepping stone.

Other notions of inductive definition have been tied to *determinacy.* Positive monotone arithmetical operators have fixed points bounded by the first admissible ordinal ω_1^{ck}, and in turn strategies for recursive open (that is Σ^0_1) games are either in $L_{\omega_1^{\mathrm{ck}}}$ (for player I) or definable over it (for the "closed" player II). Solovay [12] showed that, remarkably, for Σ^0_2-games, strategies for player I in such games occur in L_σ where σ is the closure ordinal of Σ^1_1 monotone inductive definitions (and for Player II they lie in the next admissible set beyond it). Tanaka [13] formulated a subsytem of analysis related to Σ^1_1 monotone inductive definitions, Σ^1_1-MI_0, and showed that over RCA_0 it was equivalent to Σ^0_2-Determinacy. Is there anything at all analogous for **AQI**?

Turning naturally to Σ^0_3-Determinacy there seems little published locating strategies for such games in the L-hierarchy. MARTIN & SOLOVAY (unpublished) have shown that winning strategies for Player I in Σ^0_3-games lie in L_γ where γ is the closure ordinal for $\Game\Sigma^0_3$-monotone inductive definitions. (John in [6] has something that seems similar.) However this does not really yet reveal (at least to us) where such strategies lie. A closer reading of Davis' proof of Σ^0_3-Determinacy shows that it appears provable in $\mathrm{KP}+\Sigma_2$-Separation, and thus that winning strategies appear in the least model which is an infinite tower of the form of a union of a chain of submodels $L_{\zeta_n} \prec_{\Sigma_2} L_{\zeta_{n+1}}$. Whilst we always thought it would be a happy coincidence if Σ^0_3-Determinacy matched up exactly with **AQI** we never really believed it would be so, and the theorems here show this. We have not located the exact ordinal where the winning strategies (for either player) appear, but we have shown that for some games they must appear *after* the first Σ_2-extendible ξ_0 - that is ξ_0 initiates a Σ_2-chain of length 1: $L_{\xi_0} \prec_{\Sigma_2} L_{\xi_1}$ (indeed after the first Σ_2-admissible μso that the reals of L_μ are closed under boldface **AQI**). However for all Σ^0_3 games they must appear before the first γ where the reals of $L_\gamma$are a model of "$\Pi^1_2(\Pi^1_3)$-CA_0" - that is they are closed under instances of Π^1_2 Comprehension, with Π^1_3 (lightface) definable parameters allowed (a precise definition is below). Such an ordinal γ occurs before the beginning of the least Σ_2-chain of length 2: $L_{\zeta_0} \prec_{\Sigma_2} L_{\zeta_1} \prec_{\Sigma_2} L_{\zeta_2}$. It seemed to us that even with extra effort, finding exactly at which level each Σ^0_3 game had a strategy might not be very much more illuminating: Martin and Solovay's result mentioned above seems to indicate that any such precise characterisation of these levels might well be in terms of Σ^0_3 games anyway, rather than some exterior defined subsystem of second order number theory, or of particular properties of certain levels of the L-hierarchy.

By Σ_2-Separationwe mean the usual Axiom of Separation but restricted to formulae that are $\Sigma_2$in the Levy hierarchy. By "KP" we shall mean the usual axioms of Kripke-Platek set theory, but we shall assume these include the Axiom of Infinity. By "KPI_0" we shall mean the conjunction of the axiom of extensionality together with the assertion "For every set x there is a set y with $x \in y \wedge (\mathrm{KP})_y$ ". By "KPI" we shall mean "$\mathrm{KP}+\mathrm{KPI}_{0.}$" By "$\Sigma_2$-KP" we shall mean KP with the Comprehension and Collection Axiom schemes reinforced to allow for Δ_2 and $\Sigma_2$formulae respectively.

We note that KP alone does not prove Σ_1-Separation . The Separation axioms are themselves essentially "boldface" axioms, as they allow parameters into the axiom schemes. If we wish to refer to, e.g., parameter free Σ_2-Separationwe shall call this Σ_2-Separation$_0$. The class of all admissible ordinals other than ω, we shall denote by "ADM". For information on admissibility theory the reader may consult [1]. The reals of a model of KP $+\Sigma_i$-Separationform a model of $\mathbf{\Pi^1_{i+1}}$-CA$_0$ for $i \in \{1, 2\}$. We follow the definitions and development of the theories $\mathbf{\Pi^1_{i+1}}$-CA$_0$ (the latter we take to include the set induction axiom) of [11]. The set of reals belonging to L_α,where α is least such that L_α is a model of KP $+\Sigma_i$-Separation, are those of the minimum β-model of $\mathbf{\Pi^1_{i+1}}$-CA$_0$, $(i > 0)$ (*cf.* [11],VII.5.17). It is well known, and easy to see, that if L_α is a model of KP $+\Sigma_i$-Separation, then L_α is a union of an infinite Σ_i-elementary chain of submodels $L_{\zeta_k} \prec_{\Sigma_i} L_{\zeta_j} \prec_{\Sigma_i} L_\alpha$. The least such α then being the first that is the union of an ω-length such chain. As a consequence KP $+\Sigma_i$-Separation proves the existence of β-models whose reals code Σ_i-elementary chains of length, say 2: $L_{\zeta_0} \prec_{\Sigma_i} L_{\zeta_1} \prec_{\Sigma_i} L_{\zeta_2}$.

1.1 Arithmetical quasi-inductive definitions.

Let $\Gamma : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ be any arithmetic operator (that is "$n \in \Gamma(X)$" is arithmetic; we emphasise that Γneed be neither monotone nor progressive). We define the following iterates of Γ : $\Gamma_0(X) = X; \Gamma_{\alpha+1}(X) = \Gamma(\Gamma_\alpha(X)); \Gamma_\lambda(X) = \liminf_{\alpha\to\lambda} \Gamma_\alpha(X) = \cup_{\alpha<\lambda} \cap_{\lambda>\beta>\alpha} \Gamma_\beta(X)$. Following Burgess we say that $Y \subseteq \omega$is *arithmetically quasi-inductive* if for some such Γ, Y is (1-1) reducible to$\Gamma_{\mathrm{On}}(\varnothing)$. Any such definition has a least countable $\xi = \xi(\Gamma)$ with $\Gamma_\xi(\varnothing) = \Gamma_{\mathrm{On}}(\varnothing)$. If we let ζ denote the supremum of all such $\xi(\Gamma)$, then we have:

Proposition 1 *(Burgess [2] Sect.14) (i) ζ is the least Σ_2-extendible ordinal; that is the least ζ so that there is a $\Sigma > \zeta$ with $L_\zeta \prec_{\Sigma_2} L_\Sigma$.*
(ii) A set Y is arithmetical quasi-inductive iff $Y \in \Sigma_2(L_\zeta)$.

In general we shall stay with the notation that Σ denotes the ordinal height of the least properΣ_2-end extension ofL_ζ.

Proposition 2 *There is a recursive operator Γ with $\xi(\Gamma) = \zeta$.*

Some quasi-inductive definitions may reach a fixed point.

Definition 1 *We say that* Γ reaches a fixed point on X,*if there is* α *so that* $\Gamma_\alpha(X) = \Gamma_{\alpha+1}(X)$; *and if so we call* $\Gamma_\alpha(X)$ *the* fixed point.

Proposition 3 *For any arithmetical operator* Γ*, either* $\xi(\Gamma) = \zeta$*, or else there is an equivalent recursive operator* $\tilde{\Gamma}$ *which reaches* $\Gamma_{\xi(\Gamma)}(\varnothing)$ *as a fixed point, that is there is a recursive operator* $\tilde{\Gamma}$ *with:* $\exists\alpha < \zeta\tilde{\Gamma}_\alpha(\varnothing) = \tilde{\Gamma}_{\alpha+1}(\varnothing) = \Gamma_{\xi(\Gamma)}(\varnothing)$.

The quasi-inductive definitions that reach fixed points (on $\varnothing$, or on some particular input X) form an interesting subclass. Investigation of such is an appealing combination of admissibility theory and reflection properties of ordinals.

Propositions 4 and 6 indicate that in one sense, to study recursive operators, is to study all arithmetic ones: if one has a Σ_n-definable operator, one seemingly only needs to look instead at Π_{n+1}-reflecting ordinals. For such an operator Γas in Proposition 4, it is easy to see from the definition of (ζ, Σ) that $\Gamma_\zeta(\varnothing) = \Gamma_\Sigma(\varnothing)$ and thus we might call (ζ, Σ) a "repeat pair" for Γ and $\varnothing$. Again one may show for such a Γ, that $(\zeta, \Sigma)$is the lexicographic least such repeat pair. We use this to formulate a definition allowing parameters x as starting inputs.

Definition 2 AQI *is the sentence: "For every arithmetic operator* Γ,*for every* $x \subseteq \omega$*, there is a wellordering* W *with a repeat pair* $(\zeta(\Gamma, x), \Sigma(\Gamma, x))$ *in* Field(W)*". If an arithmetic operator* Γ *acting on* x *has a repeat pair, we say that* Γ converges *(with input x).*

Clearly a certain amount of set theory (or analysis) is needed to show that every operator converges. Reformulated using Prop. 3, this is thus:

Lemma 1 KP $\vdash$ **AQI** $\longleftrightarrow \forall x \subseteq \omega\exists\xi, \sigma(L_\xi[x] \prec_{\Sigma_2} L_\sigma[x])$.

We note some facts concerning the pair (ζ, Σ) in L:

Proposition 4 *(i)* ([14] *Thm. 2.1)* L_ζ *is a model of* Σ_2-KP *(and is a union of such).*
(ii) ([15] *Cor.3.4)* L_Σ *is a model of* $\mathrm{KPI}_0 + \Sigma_2$-$\mathrm{Separation}_0$*, but not of* KPI.

Prop. 9 (i) then already shows that **AQI** is stronger than $\mathbf{\Delta^1_3}$-$\mathbf{CA_0}$ since any Σ_2-extendible is already a (union of) β-models of this theory. In Sections 2,3 we prove Theorems 2, and 1 of the abstract respectively.

1.2 Strategies and game trees.

We assume familiarity with the basic notions of two person perfect information games played using integers. We shall follow MARTIN and shall assume that such games are played on *game trees* $T \subseteq^{<\omega} \omega$ where we do allow terminal nodes. We let $G(A;T)$ denote the game with payoff set $A \cap \lceil T \rceil$ where $\lceil T \rceil$ denotes the set of all plays in T. For $q \in T$ we let T_q denote the set of all positions $r \in T$ where $r \supseteq q$.

Lemma 2 *Let A be arithmetic; let M be a transitive model of* KPI$_0$ *with $T \in M$. Then (i) "$G(A;T)$ is not a win for I" is Π_1^M; (ii) if this holds then "p*is a position in II`'s non-losing quasi-strategy for` $G(A;T)$`''` `is` Π_1^M.

Proof "$G(A;T)$ is not a win for I" is equivalent to "$\forall\sigma \in^{<\omega} \omega(\sigma$ is not a winning strategy for I in $G(A;T)$)"; which in turn is equivalent to : "$\forall\sigma$(if σ is a strategy for I in $G(A;T)$ then $\exists r \in^{\omega} \omega\ \sigma * r \notin A \cap \lceil T \rceil$)". The set $\{r|\sigma * r \notin A \cap \lceil T \rceil\}$is then $\Delta_1^1(\sigma, T)$, and hence, if non-empty with $\sigma \in M$, has an element in M. This completes (i). For (ii): let T' be *II*'s non-losing quasi-strategy for $G(A;T)$. Then $p \in T' \Leftrightarrow \mathrm{lh}(p) = k \rightarrow \forall n \leq k(p \restriction n \in T) \wedge \forall 2n+1 < k(q = (p_0, p_1, \ldots, p_{2n+1}) \rightarrow$"$G(A;T_q)$ is not a win for I." □

2 AQI is weaker than Σ_3^0-Determinacy.

Definition 3 *(i) $H_j^\alpha(X)$ denotes the Σ_j-Hull of the set X in L_α. H_j^α abbreviates $H_j^\alpha(\varnothing)$;*
(ii) $T_j^\alpha(X)$ denotes the set of Σ_j formulae, true of parameters from X, in the structure $\langle L_\alpha, \in\rangle$; T_j^α abbreviates $T_j^\alpha(\varnothing)$, the Σ_j-theory of $\langle L_\alpha, \in\rangle$.

For D a class of ordinals we let D^* be the closed class of its limit points.

Definition 4 *Let E_0 be the class of Σ_2-extendible ordinals. If E_k is defined, let E_{k+1} be the class $E_0 \cap E_k^*$. Let $E = \cap_{k\in\omega} E_k$.*

The classes E_k we can think of as having depth k in the "Σ_2-extendible limits of Σ_2-extendible ..." hierarchy: if $\gamma \in E_k$ then there are ordinals $\gamma = \mu_k \leq \mu_{k-1} \leq \ldots \leq \mu_0$ ¡ $\nu_0 < \nu_1 < \ldots < \nu_k$ satisfying $L_{\mu_j} \prec_{\Sigma_2} L_{\nu_j}$ for $j \leq k$.

Definition 5 $\sigma_3 =_{\rm df}$ *the least* σ *so that if any* Σ^0_3 *game is a win for player I, then I has a winning strategy lying in* L_σ*.*

Theorem 3 Σ_2-KP $+\forall\alpha\exists\beta,\gamma(\alpha < \beta < \gamma \wedge L_\beta \prec_{\Sigma_2} L_\gamma) \nvdash \mathbf{\Sigma^0_3}$-Determinacy.

Proof We shall show that the least level of the L-hierarchy that is a model of the antecedent theory is a model $\mathcal{M}_0 = L_\delta$in which $\mathbf{\Sigma^0_3}$-Determinacy fails. The reals of this model form a model of $\mathbf{\Delta^1_3}$-$\mathbf{CA_0} + \mathbf{AQI}$.

In fact we shall prove something slightly stronger in the form of the following Lemma:

Lemma 3 Σ_2-KP $+\forall\alpha\exists\beta(\alpha < \beta \wedge \beta \in E^*) \nvdash \mathbf{\Sigma^0_3}$-Determinacy*.*

We do this, using a technique that goes back to Friedman, by defining certain games G_ψ so that codes for initial segments of the L-hierarchy are recursive in any winning strategy for the game. So henceforth, let $\mathcal{M} = L_\delta$ be the least level of the antecedent theory in the statement of the lemma.

Let $\Psi = \{\psi | \psi \in \Sigma_1 \wedge L_\delta \models \psi\}$ be the Σ_1-theory of L_δ.[1]

(1) "$\mathbf{\Sigma^0_3}$-Determinacy" is $\Sigma^{\rm KP}_1$.

Proof

$\forall n \in \omega[\text{if } A_n \text{ is the } n' \text{th} \Sigma^0_3 \text{set, then } \exists\sigma(\sigma \text{is } a \text{ winning strategy for } a \text{ player in} G(A_n;^{<\omega}\omega))]$

The expression in curved brackets is $\Pi^1_1(\sigma)$, and thusoverall is $\Sigma^{\rm KP}_1$.

□

Hence, were $\mathbf{\Sigma^0_3}$-Determinacyto hold in L_δ it would be equivalent to some $\psi \in \Psi$. For any $\psi \in \Psi$ we define: $\alpha_\psi =$the least β so that $L_\beta \models {\rm KP} + \psi$. The following is straightforward:

(2) Let $\alpha = \sup\{\alpha_\psi | \psi \in \Psi\}$. Let $\alpha' =$ the least $\beta(L_\beta \prec_{\Sigma_1} L_\delta)$. Then $\alpha' = \alpha$.

We shall show for every $\psi \in \Psi$ there is a game G_ψ with a Π^0_3 payoff set, but without a winning strategy in L_{α_ψ}. In view of the comment just before (2), this will suffice.

Fix for the rest of the argument $\psi \in \Psi$. Let α denote α_ψ. We consider the following game $G = G_\psi$.

I (Ulrich) plays $m_0,m_1,\ldots,m_i$ $x = (m_0,m_1,\ldots,m_i,\ldots)$

II (Agathe) plays $n_0,n_1,\ldots,n_i$ $y = (n_0,n_1,\ldots,n_i,\ldots)$

[1]Essentially we show that Ψ is a $\Game\Sigma^0_3$set of integers: see the Corollary 1 below.

in the usual way, playing in the i'th round integers (m_i, n_i). Let $z = x \oplus y$.

Rules for I.

Let Tbe the theory KP $+V = L + \psi$. x must be a code for the complete Σ_1-theory of an ω-model of $T+$"there is no set model of T".

We denote by $\langle M, E\rangle$ the model I constructs if he obeys this rule. We may regard also as part of the rule that x as given by I should be specified simply by I stating "$k \in A^1_M$" or "$k \not\in A^1_M$" where A^1_M is the standard Σ_1-code for the appropriate level of the L-hierarchy. (This amounts to no more than requiring the standard fixed recursive ordering of Σ_1-sentences $\langle \psi_k | k \in \omega\rangle$ used in enumerating such Σ_1-codes, be used here - really any prior fixed enumeration will do.)

Note 1 If $\langle M, E\rangle$ is wellfounded then it is isomorphic to $\langle L_{\alpha_\psi}, \in\rangle$.

Note 2 Every $x \in L_{\alpha_\psi}$ is Σ_1-definable by some parameter free term t_x.

Amongst the codes for sentences that I plays are those of the form

$\ulcorner t_m \in \mathrm{On} \wedge t_n \in \mathrm{On} \wedge t_m < t_n \urcorner$

These we shall use to formulate rules for player II. So far the Rules for I amount to a Π^0_2 condition on x and so on z. Let $r : \omega \longrightarrow \omega \times \omega$ be a recursive enumeration of ω^2 in which each (i, j) appears infinitely often.

Rules for II.

At round k: if $(i, j) = r(k)$ and $n_k \neq 0$, then $\exists k' < k$ with $n_k = m_{k'}$, and

(a) *if*

$B = \{k' < k | (i, j) = r(k') \wedge n_{k'} \neq 0\} \neq \varnothing$

then *if* $l = \max B \wedge n_l = m_{l'} = \ulcorner t_r \in \mathrm{On} \wedge t_s \in \mathrm{On} \wedge t_r < t_s \urcorner$ *then*

(b) $\exists u \in \omega m_{k'} = \ulcorner t_u \in \mathrm{On} \wedge t_r \in \mathrm{On} \wedge t_u < t_r \urcorner$;

(c) *if* $B = \varnothing$ then $n_k = m_{k'}$ must simply be of the form given at (b) for some u, r.

The winning conditions. II wins if I fails to obey his rules, or both players obey their respective rules and additionally

$\exists(i, j)[\{n_k | r(k) = (i, j) \wedge n_k \neq 0\}$ is infinite].

This is a Σ^0_3 winning condition for II on z. Hence G_ψ has a Π^0_3 payoff set.

Remark 1 In other words, if I obeys his rules, II can win if for some (i, j), r^{-1}"(i, j)in effect picks out an infinite descending chain through the ordinals of the model $\mathcal{M}$ that I reveals *via* the gödel numbers of the $\Sigma_1$sentences that are true there.

Remark 2 *II* is not allowed to point out that $t_s < t_r$ until *I* has asserted this at some earlier stage. *II* is thus not predicting what the model will look like below t_r; she is merely adverting to the fact that *I* has revealed that $t_s < t_r$.

Lemma 4 *I has a winning strategy.*

Proof *I* plays out all "$k \in A$" for all $k \in A^1_\alpha$, and "$k \notin A$" for all $k \notin A^1_\alpha$. Obviously then $\langle M, E\rangle \simeq \langle L_{\alpha_\psi}, \in\rangle$ and *II* has no chance to pick out any infinite descending chains. □

The point is the following:

Lemma 5 *Let τ be any winning strategy for I. Let $x = A^1_\alpha$;then $x \leq_T \tau$.*

From this the theorem then follows as $x \notin L_\alpha$, being essentially the latter's Σ_1-truth set.

Proof of Lemma 5 We argue that, with *II* only playing constantly $n_k = 0$ for all k, that *I* is forced to play for x a list of all the correct facts "$k \in / \notin A$" for $A = A^1_\alpha$.The point is to show that if at any time *I* deviates from this course of action, then he will lose - and hence the purported strategy τ is not a winning one.

II plays $n_k = 0$ until such a point, if ever, when *I* asserts $k \in A$ or $k \notin A$ whereas in reality$k \notin A^1_{\alpha_\psi}$ or $k \in A^1_{\alpha_\psi}$. At this point *II* knows that *I*'s eventual model $\langle M, E\rangle$ will be illfounded, and so she must act to discover a descending chain. In this case we shall denote by $\beta = \beta_M =_{\mathrm{df}} \mathrm{On} \cap \mathrm{WFP}(M)$. However she will not yet know, and in fact will not at any move know where β_M lies. As $(\mathrm{KP})_M$ by the Truncation Lemma (*cf.*[] [1]) $\beta_M \in \mathrm{ADM}$. By our requirements on the theory T, and upwards persistence of Σ_1 formulae, we must have $\beta_M \leq \alpha_\psi$.

Definition 6 *Let $F : \omega \twoheadrightarrow \mathrm{ADM} \cap \alpha_\psi + 1$ be some fixed surjection.*

The idea is that at rounds k where $r(k) = (i, j)$ *II* will be making the working assumption that the ordinal height of the wellfounded part of M, β_M, is precisely $F(i)$, and will be trying to find an illfounded chain through On^M above β_M. She will be working simultaneously on all such possible β_M. We shall prove that if *I* deviates from enumerating $A^1_{\alpha_\psi}$, knowing that one of them is the correct assumption, she can be successful and win the game G_ψ; thus *I* is forced to play only the truth concerning $A^1_{\alpha_\psi}$.

We assume then that I has played an untruth. We concentrate on a fixed i and hence on $\beta = \beta_M = F(i)$,and describe how II can move in rounds k with $r(k) = (i, j)$.

(3) $H_2^\beta = L_\beta$.

Proof Otherwise we should have $H_2^\beta = L_\mu$ for a $\mu < \beta$. Then $\mu$is a Σ_2-admissible. But as $L_\beta \models \mathrm{KP}$ we shall be able to argue that:

Claim $\mu \in (E^*)^*$.

Proof This will follow by elementary reflection arguments, once we establish that there is a club set $D \subseteq (\mu, \beta)$ of ordinals ν satisfying $L_\mu \prec_{\Sigma_2} L_\nu$. Let $T = T_2^\beta = T_2^\mu$. Then $T \in L_\beta$. For $\sigma \in T \cap \Sigma_2$, define $f_\sigma(\xi) \simeq \mu\zeta > \xi[L_\zeta \models \neg\sigma]$. The sequence $\langle f_\sigma | \sigma \in T \cap \Sigma_2\rangle$ is then $\Delta_1^{L_\beta}(\{T\})$ and so by admissibility we can find a club $D \subseteq \beta$ of closure points ν with f_σ"$\nu \subseteq \nu$ for all $\sigma \in T \cap \Sigma_2$. As $H_2^\mu = L_\mu$, this shows that $L_\mu \prec_{\Sigma_2} L_\nu$ for any $\nu \in D$ and is more than enough to show that $\mu \in (E^*)^*$. □

This contradicts our smallness assumption on δ as the least such of this type. □

(4) *Claim* $\exists a \notin \mathrm{WFP}(M) \forall b < a(b \notin \mathrm{WFP}(M) \to T_2^b \not\subset T = T_2^\beta$.

Proof If this failed then $\forall a \notin \mathrm{WFP}(M) \exists b < a(b \notin \mathrm{WFP}(M) \wedge T_2^b \subset T$. For such b we must have that $H_2^b = L_{\gamma_b}$ for a $\gamma_b < \beta$. To see this consider the following. Let $\langle t_k^2 | k \in \omega\rangle$ be a recursive enumeration of all parameter free Σ_2-skolem functions. Consider the Σ_2 sentences: $\sigma_{k,l} \equiv \exists x, y(x = t_k^2 \wedge y = t_l^2 \wedge x < y \in \mathrm{On})$. If $H_2^b = L_g$ for a $g \notin \mathrm{WFP}(M)$ we could not have that all such $\sigma_{k,l}$ true in L_b^M are in T, as otherwise we should have an illfounded chain of the ordinals below β coded into T!

Hence for such a b we have $(L_{\gamma_b} \prec_{\Sigma_2} L_b)_M$. However the supposition implies there is an infinite descending chain of such b in the illfounded part of M. This implies that we have an infinite nested sequence of $\Sigma_2$intervals: there exists $\langle b_n | n < \omega\rangle$, $\langle \gamma_n | n < \omega\rangle$ with $(\gamma_n \leq \gamma_{n+1} \leq \ldots < b_{n+1} < b_n)$, and with $(L_{\gamma_n} \prec_{\Sigma_2} L_{b_n})_M$, for $n < \omega$. This implies that each $\gamma_n \in E$, and in fact in $E^*, (E^*)^*, \ldots$thus contradicting our smallness hypothesis. □

In $\langle M, E\rangle$, every set is given by a Σ_1 parameter free skolem term. Let $\langle t_k | k \in \omega\rangle$ be our priorly fixed recursive enumeration of the Σ_1-skolem functions. II makes the additional working assumption, or guess if you will, that $t_j^M = a_0$, where a_0 is a witness for a to the truth of the last Claim. (Again the point is that II does not know in advance which term in M will denote such a_0.) As I reveals more and more facts about his model, he must, if $M$is not to be isomorphic to L_α, at some point reveal a Σ_1-fact which is true in M but false in L_α. There really is then such an M-ordinal a_0. *II* will, in effect, place this $a_0 = t_j^M$ at the head of her putative descending chain, and set $r_0 = j$. In order to choose the next element of the chain *II* considers the set $T = T_2^\beta$. Set $T_0 = (T_2^{t_{r_0}})_M$

II now waits until *I* asserts that some σ_0 is in T_0, but *II* sees is not in T. (If II is wrong in her guess about t_j of course, then she may fruitlessly wait for ever...)

(5) Suppose $M \models$"$a_1 < a_0$ is least so that $\forall b \leq a_0(b \geq a_1 \rightarrow (\sigma_0)_{L_b})$." Then $a_1 \notin \mathrm{WFP}(M)$

Proof Were $a_1 \in L_\beta$ then we should have $\sigma_0 \in T$. □

II may thus wait until *I* asserts that some such $\sigma \in T_0 \backslash T$ and additionally that some term t_{r_1} names the ordinal a_1 defined in (5) above. At some round l then, Ulrich must play the number $m_l = \ulcorner t_{r_1} \in \mathrm{On} \wedge t_{r_0} \in \mathrm{On} \wedge t_{r_1} < t_{r_0} \urcorner$; once all these facts have been gathered together, Agathe may at the next appropriate round k with $r(k) = (i, j)$, set $n_k = m_l$.

II now has two elements of a descending chain in the illfounded part of M. Now she watches out for assertions that *I* makes about $T_1 = (T_2^{t_{r_1}})_M$, waiting for some σ_1 asserted by him to be in T_1 but which does not lie in T. By exactly the same considerations that held at (5) some a_2, t_{r_2}, are definable, and so she can continue. By the end of the game, *if* this working assumption about $\beta_M$and t_j was the correct one, the chain so defined by continuation of this process will be infinite, and she will have won.

If *I* deviates from playing the correct truth set, then at least one of *II*'s assumptions will turn out to be a correct one, and hence she will be assured of winning.

QED(**Lemma 5 & Theorem 3**) □

Corollary 1 *$\alpha \leq \sigma_3$.*

Proof Let α_ψ *etc.* be defined as above. Suppose for a contradiction that $\alpha > \sigma_3$. Suppose ψ is chosen with α_ψ least greater than σ_3.We adjust the games played above. Let $G_{\psi,\varphi}$ be the game described in the last theorem, except that for $\varphi \in \Sigma_1$, *I* must now play a code x for a model of $T+$"there is no set model of T" $+\varphi$. Everything else remains the same: *II*'s task is still to find an infinite descending chain through the ordinals of *I*'s model. Note that if $\varphi \in T_1^{\alpha_\psi}$ *I* again has a winning strategy: just play out the correct master code $A^1_{\alpha_\psi}$. But consider the case where $\varphi \notin T_1^{\alpha_\psi}$: then, if I obeys his rules, and x codes an ω-model M of this theory, then M is not wellfounded, and either is an end-extension of L_{α_ψ}, or has $\mathrm{WFP}(M) \cap \mathrm{On} < \alpha_\psi$.

Claim. If $\varphi \notin T_1^{\alpha_\psi}$ then II has a winning strategy $\tau_{\psi,\varphi} \leq_T A^1_{\alpha_\psi}$.

This is essentially what was shown above (or a modification thereof): II consults $A^1_{\alpha_\psi}$ as to what the correct theory of L_{α_ψ} actually is, and discovers an infinite descending chain through I's model M. Hence

$S =_{\mathrm{df}} \Pi_1 \,\mathrm{Th}(L_{\alpha_\psi}) = \{\neg\varphi | \varphi \in \Sigma_1,\ II \text{ has a winning strategy for } G_{\psi,\varphi}\}$.

But by our assumption on σ_3 there is a set $H \in L_{\alpha_\psi}$ containing winning strategies for all Π^0_3-games that are a win for player II. Hence membership of $\neg\varphi$ in S is determined by searching through H for a winning strategy for II; this is a Σ_1-search. Hence $S \in \Sigma_1^{L_{\alpha_\psi}}(\{H\})$. But $S \in \Sigma_1^{L_{\alpha_\psi}}$ being the Σ_1-theory of L_{α_ψ}; but then by admissibility, $S \in L_{\alpha_\psi}$, which is absurd as S codes the complete Σ_1-Theory of this model by Note 2 above. $\square$

Remark 3 Both the theorem and so the corollary can be improved a little, thus pushing σ_3 up.

3 (Boldface)$\boldsymbol{\Sigma}^0_3$-Determinacy is weaker than Σ_2-Separation.

We shall closely follow Martin's account of Davis' proof ([3]) of Σ^0_3-Determinacy. That account is performed within $\mathrm{ZC}^- + \Sigma_1$-Replacement, but we shall pay attention to definability considerations. As remarked above $\mathrm{KP} + \Sigma_2$-Separation proves the existence of models M of $\mathrm{KP} + V = L$ with $\gamma_0 < \gamma_1 \in \mathrm{ON}^M$ and $(L_{\gamma_0} \prec_{\Sigma_2} V \wedge L_{\gamma_0} \prec_{\Sigma_1} L_{\gamma_1} \prec_{\Sigma_1} V)_M$. Elementary considerations show that if this holds, so does:

(1)$(L_{\gamma_0} \prec_{\Sigma_2} L_{\gamma_1})_M$.

Notice that if M is a model as described, then *parameter-free* Σ^M_2-definable subsets are all in M, and are in fact so-definable over $(L_{\gamma_0})_M$. Moreover if a is such a subset, then any $\Sigma^M_1(\{a\})$-definable subset of ω is so definable over $(L_{\gamma_1})_M$, and hence also belongs to M. This is what we meant when we spoke of models of "$\Pi^1_2(\Pi^1_3)$-CA_0": M is such. We shall show for such an M:

Theorem 4 $L^M_{\gamma_0} \models \boldsymbol{\Sigma^0_3}$-Determinacy.

This will complete the first theorem of the abstract (as well as the non-reversibility of its implication, since by taking γ_0, γ_1least with such properties we have that then Σ_2-Separation fails in L_{γ_0}).

Proof We shall assume that $V = M$ where $M$is a model with the above properties. We shall thus drop the subscript M throughout the proof. Let Abe $\Sigma^0_3(x)$ for some real $x \in L_{\gamma_0}$. We shall also drop the parameter x henceforth, as all the arguments relativise uniformly. Suppose $T \in L_{\gamma_0}$ is a game tree. We shall show $G(A;T)$ is determined, with a winning strategy in L_{γ_0}. We suppose that player I, Ulrich, has no such winning strategy in L_{γ_0} and shall prove that II, Agathe, does.

Lemma 6 *Let $B \subseteq A \subseteq \lceil T \rceil$ with $B \in \Pi^0_2$. If $(G(A;T)$ is not a win for $I)_{L_{\gamma_0}}$; then there is a quasistrategy $T^* \in L_{\gamma_0}$ for II with the following properties:*

(i) $\lceil T^* \rceil \cap B = \varnothing$;

(ii) $G(A;T^*)$ is not a win for I.

Proof of Lemma. Under the assumption I has no winning strategy in L_{γ_0} and by Σ_2 reflection, he does not have one in V $(= M)$ either. If T' is II's non-losing quasi-strategy for $G(A;T)$, then membership in T' is not only Π_1,but also $\Pi_1^{L_{\gamma_0}}$ due to the same Σ_2 reflection. In short $T' \in V$. We thus have that for every $p \in T'$, $G(A, T'_p)$ is not a win for I.

(2) $T' \in L_{\gamma_0}$

Proof Elementary reflection arguments show that there are arbitrarily large $\nu < \gamma_0$ satisfying $L_\nu \prec_{\Sigma_1} L_{\gamma_0}$. Thus the same$T'$ is $\Pi_1(T)$ definable here also for a sufficiently large such ν with $T \in L_{\nu.}$ □

Following closely the original argument, we call a position$p \in T'$*good* if there is a quasi-strategy T^* for II in T'_p so that the following hold:

(i) $\lceil T^* \rceil \cap B = \varnothing$;

(ii) $G(A;T^*)$ is not a win for I.

We are thus trying to prove that the starting position $\varnothing$ is good, since if such a quasi-strategy T^* exists, then such will also exist in L_{γ_0} by Σ_2 reflection. As (ii) is $\Pi_1(T^*)$ by Lemma 2 we have:

(2) "p is good" is $\Sigma_2(T)$.

Let $H \subseteq^{<\omega} \omega$ be the class of p that are good. Then

(3) $H \in \Sigma_2^{L_{\gamma_0}}(T)$ and hence H is a set in V.

We define the function $t : H \longrightarrow L_{\gamma_0}$ by:

$t(p) = L$-least quasistrategy (p) witnessing (i) and (ii) that p is good.

Then t is $\Sigma_1(H, T') \cap \Sigma_2(T)$ and $t \in V$ as a function.

Let $B = \cap_{n\in\omega} D_n$ with each D_n recursively open. Define

$$E_n = A \cup \{x \in \lceil T' \rceil \mid (\exists p \subseteq x(\lceil T'_p \rceil \subseteq D_n \wedge p \,\text{is not good})\}.$$

(4) "$x \in E_n$" is $\Delta_1(H, T')$.

The proof proceeds by showing

$$(+) \qquad \exists n \in \omega(G(E_n; T') \,\text{is not}\, a \,\text{win for}\, I).$$

We first suppose this shown and see how to devise a T^* demonstrating that $\varnothing$ is good and thus that the lemma holds. Fix such an $n$witnessing that (+)holds .

(5) "q is in T'', II's non-losing quasi-strategy for $G(E_n; T')$" is $\Pi_1(H, T')$. Hence $T'' \in V$.

Proof $T'' = \{q \in T' | \forall p \subseteq q(G(E_n, T'_p)$ is not a win for $I)\}$. That this is Π_1 then follows from (4) and Lemma 2. By Σ_1-reflection it is then $\Pi_1^{L_{\gamma_1}}(H, T')$ and hence is a set. □

Firstly we shall put $q$into T^* if (a) $q \in T''$ and for all positions $p \subseteq q$ $\lceil T'_p \rceil \not\subseteq D_n$; otherwise (b) there is a shortest initial segment $p \subseteq q$ with $p \in T''$ and $\lceil T'_p \rceil \subseteq D_n$. In this latter case, by definition of T'' p is good. If the subsequent moves in q are consistent with $t(p) = \hat{T}(p)$, then we also put q into T^*. Otherwise $q \notin T^*$.

(6) "q *" is $\Pi_1^{L_{\gamma_1}}(H, T')$

Proof $H \in L_{\gamma_0+1}$, and t is definable over $L_{\gamma'}$ where γ' is the L-rank of H. The complexity of $T^*$is thus that of T''. □

(7) T^* witnesses that $\varnothing$ is good.

Proof If $x \in \lceil T^* \rceil$, then either $x \notin D_n$ or $x \in \lceil \hat{T}(p) \rceil$. In the latter case (i) ensures that $x \notin B$. Thus $\lceil T^* \rceil \cap B = \varnothing$. We must show that $G(A; T^*)$ is not a win for Ulrich . Suppose for a contradiction that σ were a winning strategy for I for this game. Note that there cannot be a position p consistent with σ so that $\lceil T'_p \rceil \subseteq D_n$:for otherwise for this p we have $T^*_p = \hat{T}(p)$; but $\hat{T}(p)$ has property (ii) and so $G(A; T^*_p)$ is not a win for$I$. But then σ cannot after all be a winning strategy for I in $G(A; T^*)$. As there is no such position p like this, we must have that $\forall x\ \sigma * x \in \lceil T'' \rceil$. But T'' is II's non-losing quasi-strategy for $G(E_n; T')$. Hence I has no winning strategy for $G(E_n; T'')$. (Suppose that τ_0 were a winning strategy for$I$ in this game. The usual argument is that τ_0

can be used to find a winning strategy for I in $G(E_n;T') : I$ plays using τ_0 until, if ever, II departs from T'' at some position p; then, as $p \notin T''$ I may play using a winning strategy for $G(E_n;T'_p)$. That this yields a strategy *in* V, and so a contradiction, is because there is a set *in* V of the L-least winning strategies for $G(E_n;T'_p)$ for those $p \in T' \backslash T''$.) In particular σ itself cannot be such a winning strategy; hence there is a play x consistent with σ satisfying $x \notin E_{n.}$ As $E_n \supseteq A$, we thus have $x \notin A$. But σ was originally assumed to be a winning strategy for Iin $G(A;T^*)$. This is a contradiction! □

(8) There is such a T^* witnessing that $\varnothing$ is good with $T^* \in L_{\gamma_0}$.

Proof By (6) and (7) we see that a T^* witnessing the requisite Σ_2 formula can be constructed definably over L_{γ_1} in the parameters T' and H. But the formula only mentions T'. Hence by Σ_2-reflection there is then such a $T^* \in L_{\gamma_0}$. □

We thus have to show that (+) above holds. We showed that $\varnothing$ is good unless for all $n \in (E_n;T')$ is a win for I. If we define:

$$E_n^p = A \cup \{x \in \lceil T' \rceil \mid (\exists q \subseteq x(p \subseteq q \wedge \lceil T'_q \rceil \subseteq D_n \wedge q \,\text{is not good})\}$$

then the same argument shows that:

(9) $\forall p \in T'$ (p is not good $\longrightarrow \forall n \in \omega$ ($G(E_n^p;T'_p)$ is a win for I)).

We suppose the lemma false and obtain a contradiction by building a winning strategy σ for I for the game $G(A;T')$ (recall that T' was Agathe's non-losing quasi-strategy in $G(A;T)$).

We define the function $s : \overline{H} \times \omega \longrightarrow V$ defined by: $s(p,n) = L$-least winning strategy for I in $G(E_n^p;T'_p)$. By (9) this function is well defined, and total, on $\overline{H} \times \omega$ and moreover is $\Sigma_1(H,T')$. As $(\text{KP})_V$ we have that $s \in V$. Let $\sigma_0 = s(\varnothing,0)$. Then σ_0 is a winning strategy for I in $G(E_0;T')$. σ agrees with σ_0 until a first, if such occurs, position p_0 is reached with $\lceil T'_{p_0} \rceil \subseteq D_0$ but p_0 is not good . If so, then we use the strategy $\sigma_1 = s(p_0,1)$ for I in $G(E_n^{p_0};T'_{p_0})$. σ now agrees with $\sigma_1$until, if ever, a position p_2 is reached with $\lceil T'_{p_1} \rceil \subseteq D_1$ but p_1 is not good . The play continues using $\sigma_2 = s(p_2,2)$. If $q = \cup_{n\in\omega} p_n$ is a non-terminal position, we let σ be some arbitrary but canonical choice on positions extending q. By our closure assumptions on V we have that the strategy σ so defined is a set.

If for some play x we have that for some n p_n is undefined, this implies that $x \in E_n^{p_{n-1}}$ (or E_0 if $n = 0$). But additionally there is no initial position $p \subseteq x$ with $(a)p_{n-1} \subseteq p$(ifn ¿ 0); (b) $\lceil T'_p \rceil \subseteq D_n$, and (c) p not good. This means that $x \in A$ by the definition of E_n. On the other hand, if all the p_n are defined, then we shall have that $x \in \cap_{n\in\omega} D_n \subseteq B \subseteq A$. Either way we have shown that any play x arising from following the strategy σ lies in A. However this contradicts the assumption that I has no winning strategy for $G(A; T')$. This finishes the proof of the Lemma. $\square$

The proof of the theorem now follows MARTIN [8] pretty much verbatim but again paying attention to definability issues. We repeatedly apply the Lemma with $A = \cup_{n\in\omega} A_n$ and each $A_n \in \Pi^0_2$, acting in turn as an instance of B in the Lemma. This is a Σ_2-recursion defining a strategy τ for II over L_{γ_0} since all the relevant quasi-strategies given by the Lemma lie in this model. These details now follow.

One applies the lemma with $B = A_0$ obtaining a quasi-strategy for II: $T^*(\varnothing)$. By Σ_2-reflection the L-least such lies in L_{γ_0}, and we shall assume that $T^*(\varnothing)$ refers to it. For any position $p_1 \in T$ with $\mathrm{lh}(p_1) = 1$, let $\tau(p_1)$be some arbitrary but fixed move in $T'(\varnothing)$, II's non-losing quasi-strategy for the game $G(A, T^*(\varnothing))$. The relation "$p \in T'(\varnothing)$" is $\Pi_1^{L_{\gamma_0}}(\{T^*(\varnothing)\})$ and hence "$y = T'(\varnothing)$" $\in \Delta_2^{L_{\gamma_0}}(\{T^*(\varnothing)\})$ and thus $T'(\varnothing)$ also lies in L_{γ_0}. For definiteness we let $\tau(p_1)$ be the numerically least move. For any play, p_2 say, of length 2 consistent with the above definition of τ so far, we apply the lemma again with $B = A_1$ and with $(T^*(\varnothing))_{p_2}$ replacing T. This yields a quasi-strategy for II, call it $T^*(p_2)$, which is definable in a Σ_2 way over L_{γ_0}, in the parameter $(T^*(\varnothing))_{p_2}$. Let $T'(p_2) \in L_{\gamma_0}$ be II's non-losing quasi-strategy for $G(A, T^*(p_2))$, this time with "$y = T'(p_2)$" $\in \Delta_2^{L_{\gamma_0}}(\{T^*(p_2)\})$. Again for $p_3 \in T^*(p_2)$ any position of length 3, let $\tau(p_3)$be some arbitrary but fixed move in $T'(p_2)$. Now we consider appropriate moves p_4 of length 4, and reapply the lemma with $B = A_2$ and $(T^*(p_2))_{p_4}$. Continuing in this way we obtain a strategy τ for II so that $\tau \restriction^{2k+1} \omega$, for $k < \omega$, is defined by a recursion that is $\Sigma_2^{L_{\gamma_0}}(\{T\})$. As $L_{\gamma_0} \models \Sigma_2$-KP, we have that $\tau \in L_{\gamma_0}$. If x is any play consistent with τ, either x is finite, in which case $x \in T'(\varnothing)$, and thus $T'(\varnothing)_x \not\subseteq A$, and hence $x \notin A$; or else x is infinite. In this latter case, for every n, by the defining properties of $T^*(p_{2n})$given by the relevant application of the lemma, $x \in \lceil T^*(x \restriction 2n) \rceil \subseteq \neg A_{n.}$Hence $x \notin A$, and in both cases τ is a winning strategy for II as required. QED(**Theorem)**

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