Eventually Infinite Time Turing Machine Degrees: Infinite Time decidable reals

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Abstract

We characterise explicitly the decidable predicates on integers of Infinite Time Turing machines, in terms of admissibility theory and the constructible hierarchy. We do this by pinning down ζ, the least ordinal not the length of any eventual output of an Infinite Time Turing machine (halting or otherwise); using this the Infinite Time Turing Degrees are considered, and it is shown how the jump operator coincides with the production of mastercodes for the constructible hierarchy; further that the natural ordinals associated with the jump operator satisfy a Spector criterion, and correspond to the L_ζ-stables. It also implies that the machines devised are "Σ_2 Complete" amongst all such other possible machines. It is shown that least upper bounds of an "eventual jump" hierarchy exist on an initial segment.

1 Introduction:

Hamkins and Lewis in [4] give a notion of computability degree based on Infinite Time Turing Machines. They define:

Definition 1.1 *For* $f, g \in {}^\omega 2$ *write* $f \leq_\infty g \longleftrightarrow \exists p \in \omega\ \phi_p^g(0){\downarrow}f$.

The right hand side here is to be interpreted in the usual manner for Turing computability, except that computations are allowed to take transfinite time. The values of all cells C_i on the tape whilst executing the program with index $p \in \omega$ are calculated according to the usual Turing machine arrangement: only at limit stages ν of time, their values are set to the lim sup of their values at earlier times $\overline{\nu} < \nu$. The head of the machine at stage ν then returns to a special fixed cell, and enters a special limit state for such ν. Such machines can handle infinite sequences of integers and may decide whether such codes a wellfounded relation.

The relation $\phi_p^f(x){\downarrow}$ (in α-steps) ("the p'th program halts on input $x \in {}^\omega 2$ using oracle $f \in {}^\omega 2$" (additionally, "in α-steps")) is Δ^1_2 relation of f, p, x (and α). (We consider input strings $f, x, \ldots$ as reals, or as members of Baire space ${}^{\omega}2$.) Similarly $\phi_p^f(x)\uparrow$ is also Δ_2^1. (It is only necessary to have any or all wellordered sequence(s) of ω-sequences of cell values coding the course of a looping computation to see this *cf.* [4] 2.5.) We showed in [7] that the ordinal lengths of time taken for any computation of the form $\varphi_p^f(0)$ to halt, were themselves all capable of being output by such machines. Both that paper and [4] leave open the question as to what the decidable predicates on integers actually are. In this paper we answer this; we do this in effect by characterising the least ordinal, ζ, *not* the eventual or "longterm" output of any machine (the "eventually writable ordinals" see below) irrespective of the machine's halting. This has a number of consequences. Briefly we show:

- The ordinal ζ is least so that L_ζ has a transitive Σ_2 end-extension (Theorem 2.1 below - we call such an ordinal a "Σ_2-extendible").
- The $\leq_\infty$ -degrees of the model L_ζ: are all Turing (in the ordinary sense) equivalent to those of mastercodes of certain levels $L_{\lambda'}$ of the L-hierarchy below L_ζ (Theorem 1.7 below).
- These levels λ' are also characterisable as (a) the successor L_ζ-stables, and (b) as the natural ordinal associated to the α'th iterate of the weak jump operator of [4] $0^{\nabla\alpha}$ (Lemma 2.3 below).
- In particular this characterises the infinite time decidable sets of integers: they are precisely those reals in the level of the constructible hierarchy L_λ, and thus are the sets of integers that are Σ_1-definable in the least level of the constructible hierarchy that has a transitive Σ_2 end-extension (Lemma 2.3(i)).
- If we define a $\zeta' th$ jump: $\tilde{0} =_{df} 0^{\nabla\zeta}$ as the relevant mastercode of L_ζ, then $\tilde{0}$ is Turing equivalent to the set of indices of computations that, whilst not halting, have in the long run an eventually settled output (Theorem 2.6).
- If we define the notion of relative "eventually infinitely computable", using the operation $f \rightarrow \tilde{f}$ as a jump operator, then the natural hierarchy of degrees of "fast" reals, has length at least as long as the first Σ_2-extendible, which is a limit of such. (Theorem 2.14).
- We show that the machine architecture of [4] is as strong as any other similar machine defined using a uniformly Σ_2 definable operator (Theorem 2.9)).

We conclude the introduction by giving some definitions to flesh out the above, and listing some basic facts concerning the nature of this form of computation. (For any notions involving constructible sets, the reader may find them in [3]. For the basic structure of Turing Machines, see, for example [2]; for admissibility theory either [3] or [1].) We should like to warmly thank Joel Hamkins for his introducing this subject to us, and for his helpful suggestions and patient discussions, and to Benedikt Loewe for his comments on an earlier draft.

We use the notation $\varphi_p^f(x)^\tau = g$ for the notion "computation $\varphi_p^f(x)$ has real g on its output tape after τ steps". This is then also a Δ_2^1 relation of f, x, g and $\tau \in WO$ (the latter the set of codes of wellorderings). We write "$\phi_p(0)\uparrow x$" for "program p eventually has settled output x on its output tape"; in other words for $\exists\nu\forall\tau \geq \nu\varphi_p(0)^\tau = x$. In this case we write that $x \in \mathcal{EW}$. The following fact and definition name the three ordinals $\lambda < \zeta < \Sigma$ that feature in this paper.

Fact 1.1 *([4] 3.7, 8.3) (i) If* $\mathcal{W} =_{df} \{x \in {}^\omega 2 \mid x$ *is the output of* $\phi_p(0)\downarrow\}$*, then the "writable" ordinals, (those ordinals* α *so that there is* $x \in \mathcal{W} \cap WO$ *with* $||x|| = \alpha$*) form an initial segment of the countable ordinals,* λ*. Similarly if* $\mathcal{EW} =_{df} \{x \in {}^\omega 2 \mid \exists p \phi_p(0)\uparrow x\}$*, then the "eventually writable" ordinals, (those ordinals* α *so that there is* $x \in \mathcal{EW} \cap WO$ *with* $||x|| = \alpha$*) form an initial segment of the countable ordinals,* ζ*.*
(ii) The transitive sets $H(\lambda)$*,* $(H(\zeta))$ *whose elements are coded by reals in* $\mathcal{W}$ *(*$\mathcal{EW}$*) are admissible sets, closed under hyperjump (and more).*

Definition 1.2 *([4]) Let* $\Sigma =_{df} \sup\{||t|| : t \in \mathcal{AW} \cap WO\}$*.* $\mathcal{AW}$ *is the set of "accidentally writable" reals where:*
$\mathcal{AW} =_{df} \{x \in {}^\omega 2 \mid \exists p$ *x appears on any tape at any time of the computation of* $\phi_p(0)\}$*.*

The accidentally writable reals are thus likely to be transient. An equivalent definition is obtained by restricting the appearance of such reals to the output tape alone. Clearly $\mathcal{AW} \supseteq \mathcal{EW} \supseteq \mathcal{W}$.

We use the notation $\mathcal{W}^f, \lambda^f \ldots$ for the notions relativised to a real. They define the *"clockable"* ordinals as those α so that for some p $\phi_p(0)\downarrow$ in α steps. They let $\gamma =_{df} \sup\{\alpha \mid \alpha \text{ clockable}\}$ and show that the order type of the "clockables" is precisely λ. Although there are many gaps in the clockable ordinals, we do have:

Fact 1.2 *([7] Thm.1.1)* $\gamma = \lambda$ *(and by relativisation* $\forall f \in {}^\omega 2 \quad \lambda^f = \gamma^f$*).*

The following is a corollary to the proof of the last fact:

Fact 1.3 *([7] Cor. 3.3)* $H(\lambda) = L_\lambda$*, and for any* $f \in {}^\omega 2 \quad H^f(\lambda^f) = L_{\lambda^f}[f]$*. Also* $H(\zeta) = L_\zeta$ *(Cor.3.5).*

Definition 1.3 *For* $f \in {}^\omega 2$ *the* (weak) jump $f^\nabla =_{df} \{n \in \omega \mid \phi^f_n(0)\downarrow\}$.

It is easy to see that $f \to f^\nabla$ is a Δ^1_2 function. Using Fact 1.2, as $\gamma^f = \lambda^f$, and Fact 1.3, by considering the running of such machines inside $L_{\lambda^f}[f]$ one sees:

Fact 1.4 $f^\nabla \in \Sigma_1(L_{\lambda^f}[f])$.

Indeed, this idea of running the x-computations inside $L_{\lambda^y}[y]$ yields a Spector criterion for such degrees (see Lemma 1.5 below.) For $\alpha < \zeta$ [4] define an α'th iterate of the jump of 0: $0^{\nabla\alpha}$. Hamkins has shown a negative solution to Post's problem for ∞-degrees, below the ζ'th level as follows:

Theorem 1.4 *(Hamkins)[5] The sequence* $\langle 0^{\nabla\alpha} \mid \alpha < \zeta\rangle$ *is a wellordered initial segment of the* ∞*-degrees, with* $0^{\nabla\alpha} = (0^{\nabla\beta})^\nabla$ *if* $\alpha = \beta + 1$*, and for Lim(*α*),* $0^{\nabla\alpha}$ *is defined with* $0^{\nabla\alpha}$ *a minimal upper bound for* $\langle 0^{\nabla\gamma} \mid \gamma < \alpha\rangle$ *.*

1.1 Somewhat slow reals

The characterisations in Facts 1.2, 1.3, 1.4 together alone yield easily the following observations:

Lemma 1.5 *(i)* $x \leq_\infty y \longleftrightarrow x \in L_{\lambda^y}[y]$; *thus (ii)* $x \leq_\infty y \longrightarrow \lambda^x \leq \lambda^y$;
(iii) The assignment $x \longrightarrow \lambda^x$ *then satisfies a Spector criterion:*
$x \leq_\infty y \longrightarrow (x^\nabla \leq_\infty y \longleftrightarrow \lambda^x < \lambda^y)$.

(For (iii) just note that $x^\nabla \in \Sigma_1(L_{\lambda^x}[x])$ so this is immediate.)

Definition 1.6 *Let* $F_0 = \{x \in {}^\omega 2 \,|\, x \in L_{\lambda^x}\}$.

Then F_0, the set of "very fast" reals, are those that can be computed as being in L, by an ordinal that is the length of time of an x-computation. This is by way of analogy with the largest thin Π^1_1 set of reals, $Q = \{x \in {}^\omega 2 \,|\, x \in L_{\omega_1^{xck}}\}$, the quickly constructible reals. (Although one adjective seems to have overtaken the other here.) It is easily seen that F_0 is a strictly Δ^1_2 and thin set of reals. Those $x \notin F_0$ are correspondingly "somewhat slow".

Theorem 1.7 *(i) For* $\alpha = \beta + 1$, $0^{\nabla\alpha}$ *is of the same (ordinary) Turing degree as* $A^1_{\lambda'}$*, the* Σ_1*-mastercode of* $L_{\lambda'}$*, where* $\lambda' = \lambda^{0^{\nabla\beta}}$ *Moreover* $0^{\nabla\alpha} \equiv_1 A^1_{\lambda'}$.

(ii) For Lim(α), $0^{\nabla\alpha} \equiv_1 \Sigma_1$*-mastercode* $A^1_{\lambda'}$*, where* $\lambda' = \sup\{\lambda^{0^{\nabla\beta}} \,|\, \beta < \alpha\}$*, if* α *is projectible (i.e.* $\rho^1_\alpha < \alpha$*) ; otherwise we may take* $0^{\nabla\alpha} \equiv_1 \Sigma_2$*-mastercode* $A^2_{\lambda'}$ *if* α *is non-projectible.*

Remark: We should point out that we have taken advantage of some ambiguity in the way that [4] define $0^{\nabla\alpha}$ for Lim(α): they take $0^{\nabla\alpha}$ as a recursive union of $\{0^{\nabla\beta} \,|\, \beta < \alpha\}$ along an eventually writable wellordering $y \in \mathcal{EW}$ of type α but without expressing a particular choice of y. (Actually there is a lot of freedom in choosing such a y.) We should make some suitable precise minimal choice, and for explictness we adopt the y of least constructible rank, (which is also in $\mathcal{EW}$ since L_α has an $\mathcal{EW}$ code), then such a y is always to be found recursive in the relevant mastercode of L_α.

Proof: (i) We consider just $\alpha = 1$; (the reader will be able to construct the other successor cases), we need to show that $0^\nabla \equiv_T A =_{df} A^1_\lambda$ where the latter is the Σ_1-mastercode for $L_\lambda = J_\lambda$. We first note that a) $\rho^1_\lambda = 1$ (that is, λ is projectible); b) $p^1_\lambda = \emptyset$. [For a), $\omega\rho^1_\lambda$ is always a Σ_1-cardinal of L_λ, and so if it were any larger than ω one could argue inside L_λ that all computations would have halted before λ, whereas we know that λ is the supremum of all clockable ordinals by Fact 1.2 above; for b) this is similar: if ν were the largest ordinal of p then ν would satisfy $J_\nu \prec_{\Sigma_1} J_\lambda$ and again this would imply all computations have halted before λ.] Thus A may be defined as $\{\langle i, x\rangle \in \omega \times [\omega]^{<\omega} \,|\, L_\lambda \models \varphi_i(x)\}$ (where $\langle \varphi_i \,|\, i < \omega\rangle$ is some fixed recursive enumeration of the the Σ_1 formulae of $\mathcal{L}_{\{\dot{\in},\dot{=}\}}$ in one free variable). Hence A is a subset of $\omega \times [\omega]^{<\omega}$, and can be coded as a set of integers. As $0^\nabla \in \Sigma_1(L_\lambda)$ (by Fact 1.4) and the mastercode codes Σ_1-truth, 0^∇ is rudimentary in A, in fact there is i_0 so that $p \in 0^\nabla \longleftrightarrow \langle i_0, p\rangle \in A$. Hence $0^\nabla \leq_1 A$. For the converse, let $B = \{\alpha < \lambda \,|\, \rho^1_\alpha = 1 \wedge p^1_\alpha = \emptyset\}$. It is easy to see that B is unbounded in λ (in fact in the case $\alpha = 1$ we are considering $B = \lambda$; for other successor α, B contains a final segment of λ'). Hence $A = \bigcup_{\alpha \in B} A^1_\alpha$, where the A^1_α are defined for $\alpha \in B$ just as A was defined from λ. Now define $n : \omega \times [\omega]^{<\omega} \to \omega$ as follows: $n(i, x)$ is to be the code number of the program that performs the following computation.

We consider the universal machine $\mathcal{U}$, which we think of as simulating all the computations $\varphi_p(0)$ simultaneously (by recursively splitting up the three given tapes into ω many infinite pieces); when a simulated computation $\phi_p(0)\downarrow z$ halts on one of our simulated tapes of $\mathcal{U}$, z is checked to see if it is the code of a wellfounded model M of the form "$V = J_\alpha$" with $\alpha \in B$; if so, and $M \models \varphi_i(x)$ then the program halts. Clearly n can be taken as a recursive function, and $n(i,x) \in 0^\nabla \longleftrightarrow \langle i, x\rangle \in A$.

(ii) For $Lim(\alpha)$: we can split into the projectible/non-projectible cases. In each case $0^{\nabla\alpha}$ is either $\Sigma_1(L_{\lambda'})$ or $\Sigma_2(L_{\lambda'})$. For $\rho^1_\alpha = 1$ the argument is very similar to the above: we have a code for the wellordering of type α of least constructible rank recursive in $A_{\lambda'}$; and we use this wellordering to organise an argument similar to (i). For the $\rho^1_\alpha = \alpha$ case the argument is similar to Theorem 2.6 to come, and again we shall let the reader construct the argument from that. **Q.E.D.**

We now remark that no eventually writable real is somewhat slow:

Lemma 1.8 $\mathcal{EW} \subseteq F_0$.

Proof: Let $x \in \mathcal{EW}$ be arbitrary. By Fact 1.3 above $x \in H(\zeta) = L_\zeta$. Let ρ be the constructible rank of x; as $\rho < \zeta$, so let $r \in \omega$ be such that $\phi_r(0)\uparrow t$ where $t \in WO$ is a code for ρ. Consider the relativized x-computation that simulates $\phi_r(0)$, and checks at each stage whether the real currently written to output by $\phi_r(0)$ contains a code for a wellorder τ with $x \in L_\tau$; if it does it then halts itself with a code for τ on the output tape. Hence $\rho \leq \tau < \lambda^x$. Hence $x \in F_0$. **Q.E.D.**

As a corollary to the above results we shall show:

Corollary 1.9 *(i)* $\{z \,|\, z =_\infty 0^{\nabla\alpha}\} = \{z \in L_{\lambda^{0^{\nabla\alpha}}} \,|\, z \notin L_{\lambda^{0^{\nabla\beta}}}$ *for any* $\beta < \alpha\}$.

(ii) $L_\zeta \models$ *" the* $\leq_\infty$ *-degrees are simply those of* $\{0^{\nabla\alpha} \,|\, \alpha < \zeta\}$*, and* $Lim(\alpha) \longrightarrow 0^{\nabla\alpha}$ *is the least upper bound of* $\{0^{\nabla\beta} \,|\, \beta < \alpha\}$*."*

Proof: (i) We use the fact that the last lemma implies that $z \in L_{\lambda^z} = L_{\lambda^z}[z]$ for any $z \in L_\zeta$.

First suppose $z =_\infty 0^{\nabla\alpha}$. Then we know (Lemma 1.5(i)) that $z \in L_{\lambda^{0^{\nabla\alpha}}}[0^{\nabla\alpha}] = L_{\lambda^{0^{\nabla\alpha}}}$. Clearly if $\beta < \alpha \wedge z \in L_{\lambda^{0^{\nabla\beta}}}$ then $z \leq_\infty 0^{\nabla\beta} <_\infty 0^{\nabla\alpha}$. For the converse clearly any z in the right hand side satisfes $z \leq_\infty 0^{\nabla\alpha}$, and hence $\lambda^z \leq \lambda^{0^{\nabla\alpha}}$. But then by the last lemma $z \in L_{\lambda^z}$. Let $\lambda' = \sup\{\lambda^{0^{\nabla\beta}} \,|\, \beta < \alpha\} < \lambda^{0^{\nabla\alpha}}$. Then $0^{\nabla\alpha}$ is definable over $L_{\lambda'}$. By assumption on z, $\lambda' < \lambda^z$ and so a code for $L_{\lambda'+1}$, and hence for $0^{\nabla\alpha}$, is $\leq_\infty z$.
(ii) is immediate from (i). **Q.E.D.**

We shall be able to characterise the ordinals $\lambda^{0^{\nabla\alpha}}$ once we have investigated the ordinal ζ a little more closely.

2 Eventually infinite degrees

Theorem 2.1 *(i)* ζ *is the least ordinal* ξ *such that* L_ξ *has a transitive* Σ_2*-end extension.*

(ii) L_ζ *is* Σ_2*-admissible, but* $L_\zeta \not\models \Sigma_2$*-separation.*

We shall call an ordinal as in (i) a "Σ_2-extendible" ordinal. By "Σ_2-admissible" we mean an ordinal ξ such L_ξ is a model of the usual admissible set axioms, with Δ_2-Comprehension and Σ_2-Collection.

Proof: (i)
We first show:

(1) $L_\zeta \prec_{\Sigma_2} L_\Sigma$

Proof: Let $L_\Sigma \models \varphi(x) \longleftrightarrow \exists w \forall z \psi(w,z,x)$ where $\psi \in \Sigma_0$, $x \in L_\zeta$. Let $\mathcal{U}$ again be the "universal" machine that runs on its tapes copies of all the computations of the form $\phi_p(0)$. We use the definition:

Definition 2.2 *Let* $\sigma(\nu) =_{df}$
$= \Sigma\{||x|| : x \in \mathcal{AW} \cap WO \wedge x$ *is on one of* $\mathcal{U}$*'s tapes at stage* $\nu\}$.

Recall from Fact 1.3 that $L_\zeta = H(\zeta)$ is the class of sets with $\mathcal{EW}$-writable codes. As then $x \in \mathcal{EW}$ we may write a program that searches for the $L_{\sigma(\nu)}$-least witness w' so that $L_{\sigma(\nu)} \models \forall z \psi(w', z, x)$ writing it to an output tape. But if $L_\Sigma \models \forall z \psi(w,z,x)$, and w is L_Σ-least witnessing this, with $w \in L_\gamma$ say, then eventually as time ν increases, $\sigma(\nu) \geq \gamma$ and w is $L_{\sigma(\nu)}$-least also, and hence w is written to the tape; as $\forall z \psi(w,z,x)$ is Π_1, this w need never be changed; hence w must be in $\mathcal{EW}$. Again the Π_1 statement goes down to L_ζ. Hence $L_\zeta \models \varphi(x)$. **Q.E.D.(1)**

From (1) it follows by straightforward methods that:

(2) L_ζ is Σ_2-admissible (and is a limit of such).

(1) & (2) show that ζ is a candidate for ξ. Now let ξ be the least Σ_2-extendible ordinal with some $\sigma > \xi$ with $L_\xi \prec_{\Sigma_2} L_\sigma$.

(3) $\zeta \leq \xi$

Proof: Suppose for a contradiction that $\xi < \zeta$. Now run the argument of the main proposition of [7] that for any computation $\phi_p(0)$, any cell C_i that has stabilized at time σ has in fact done so by time ξ; this entails that the snapshot (meaning the complete picture of the machine, its states, the values of its cells C_i etc.) at time ξ is exactly that at time σ. (We can do this by appealing now to $L_\xi \prec_{\Sigma_2} L_\sigma$ which was all that the cited argument required). But we saw that there was an index $q_0 \in \omega$ (Cor.3.4 (ii) of [7]) for which the stage ζ was the very first stage at which the snapshot of $\phi_{q_0}(0)$ was at the beginning of the infinitely looping cycle. This can only mean that $\zeta \leq \xi$! This contradicts our supposition. **Q.E.D.(i)**

Proof: of (ii): We have already remarked on Σ_2 admissibility at (2) above. Suppose $L_\zeta \models \Sigma_2$-Separation. Then there are unboundedly many $\xi < \zeta$ with $L_\xi \prec_{\Sigma_2} L_\zeta$. But this contradicts the leastness of ζ! **Q.E.D.(Theorem 2.2)**

This in turn will yield an immediate characterisation of the ordinal λ (as the least L_ζ-stable), and indeed of the ordinals in $\Lambda =_{df} \{\lambda^{0^{\nabla\alpha}} \mid \alpha < \lambda\}$: they are just the successors in the enumeration of the L_ζ-stables. Let Λ^* be the set of limit points of Λ. Then let $\langle \lambda_\alpha \mid \alpha < \zeta \rangle$ enumerate continuously and monotonically $\{\lambda' \mid L_{\lambda'} \prec_{\Sigma_1} L_\zeta\} \cup \{0\}$.

Lemma 2.3 *(i)* $\lambda = \lambda_1$, i.e. λ *is the least* $\overline{\lambda}$ *such that* $L_{\overline{\lambda}} \prec_{\Sigma_1} L_\zeta$.
(ii) The set of L_ζ*-stables is precisely the set of* $\lambda' \in \Lambda \cup \Lambda^*$. *In fact* $\lambda_{\alpha+1} = \lambda^{0^{\nabla\alpha}}$.

Proof: (ii) Let $L_{\overline{\lambda}} \prec_{\Sigma_1} L_\zeta$. Suppose $\overline{\lambda}$ is a L_ζ-stable but is not a limit of Λ; then for some $\alpha < \zeta$, $\overline{\lambda} \leq \lambda^{0^{\nabla\alpha}}$ whilst $\delta =_{df} \sup\{\lambda^{0^{\nabla\gamma}} \,|\, \gamma < \alpha\} < \overline{\lambda}$. Suppose for a contradiction $\overline{\lambda} < \lambda^{0^{\nabla\alpha}}$. Note that δ is the constructible rank of $0^{\nabla\alpha}$. Then $\overline{\lambda} \in \mathcal{W}^{0^{\nabla\alpha}}$. Hence $\exists\varphi_p^{0^{\nabla\alpha}}(0){\downarrow}l$ where $l \in WO$ and codes $\overline{\lambda}$. Then the following statement holds in L_ζ: "$\exists p \in \omega \exists$ a course of computation witnessing that $\varphi_p^{0^{\nabla\alpha}}(0){\downarrow}$", and it is $\Sigma_1^{L_\zeta}(\{\delta\})$. This is absurd as it would go down to the admissible set $L_{\overline{\lambda}}$ whereas the output of $\varphi_p^{0^{\nabla\alpha}}(0)$ is a code for $\overline{\lambda}$. **Q.E.D.**

We have characterised both the ordinals of the form $\lambda^{0^{\nabla\alpha}}$ and the reals of the degrees $\mathbf{0}^{\nabla\alpha}$ for $\alpha < \zeta$, and a number of questions suggest themselves. One of which is simply: is this hierarchy continuable? What, if anything, would be a ζ'th jump? We use the characterisation of the ordinal ζ obtained above, and look at the natural candidate that presents itself: the mastercode of L_ζ.

As L_ζ is not a model of Σ_2-separation by Theorem 2.1 above, we have $\rho^2_\zeta = \omega$, and so $A^2_\zeta \subseteq \omega \times [\omega]^{<\omega}$ is essentially a set of integers. This is our candidate for $0^{\nabla\zeta}$, and we now abbreviate this as $\tilde{0}$ (this is specified more precisely at (6) below in Theorem 2.6).

Lemma 2.4 $\tilde{0}$ *is the* $\leq_\infty$ *-l.u.b. of* $\{0^{\nabla\gamma} \,|\, \gamma < \zeta\}$.

Proof: Suppose z is another upper bound of $\{0^{\nabla\gamma} \,|\, \gamma < \zeta\}$. If $\lambda^z > \zeta$, then clearly $\tilde{0} \leq_\infty z$. So suppose $\lambda^z = \zeta$. We know by the relativised 2.3 that

(1) $L_{\lambda^z}[z] \prec_{\Sigma_1} L_{\lambda^{\nabla z}}[z] \prec_{\Sigma_1} L_{\zeta^z}[z] \prec_{\Sigma_2} L_{\Sigma^z}[z]$.

But Σ^z cannot be greater than Σ since then the Σ_1 statement that "there is a pair (ζ, Σ) with $L_\zeta \prec_{\Sigma_2} L_\Sigma$" would go down to $L_{\lambda^z}[z]$ contradicting ζ's leastness. Hence $\Sigma^z = \Sigma$. One may now verify that for no ζ' with $\zeta < \zeta' < \Sigma$ does $L_{\zeta'} \prec_{\Sigma_2} L_\Sigma$ hold. This would contradict (1). **Q.E.D.**

Definition 2.5 *Let* $S = \{p \,|\, \phi_p(0){\downarrow} \text{or } \phi_p(0){\uparrow}x \text{ with } x \in \mathcal{EW}\}$

S is thus the set of program numbers that halt or have, eventually at least, a settled real on their output tapes. We shall show up to Turing equivalence, the mastercode $\tilde{0}$ and S are the same set.

Theorem 2.6 $\tilde{0} \equiv_T S$*; in fact* $\tilde{0} \equiv_1 S$.

Proof:
(1) $S \in \Sigma_2(L_\zeta)$.

$p \in S \longleftrightarrow L_\zeta \models$ "$\exists\nu\forall$ times τ later than ν $\phi_p(0)^\nu = \phi_p(0)^\tau$".

(2) $S \notin L_\zeta$.
Proof: Suppose otherwise for a contradiction. Then $S \in L_\zeta$ and so $S \in \mathcal{EW}$. So for some

r, $\phi_r(0)\uparrow S$. For $\nu < \Sigma$ and for T any set of integers, let $\sigma^T(\nu) =_{df} \Sigma\{||x|| : x \in WO \wedge p \in T \wedge x = \phi_p(0)^\nu\}$.

Let q be the index of the program that at a stage ν computes (using the universal machine $\mathcal{U}$), $\sigma^{\phi_r(0)^\nu}(\nu)$ Then by definition of r, $\phi_q(0)\uparrow\sigma^{\phi_r(0)^\nu}(\nu) = \sigma^S(\zeta)$ for all $\nu \geq \zeta$. Hence $\sigma^S(\zeta) \in \mathcal{EW}$. But $\sigma^S(\zeta) = \zeta$. Contradiction! **Q.E.D.(2)**

(3) $\lambda^S > \zeta$; in fact for any $T \in L_\Sigma \backslash L_\zeta$ $\lambda^T > \zeta$.
Proof: Pick such a T. Run the universal T-machine $\mathcal{U}^T$ until a τ is found with $T \in L_\tau$; output τ; hence $\tau < \lambda^\tau$. And $\zeta < \tau$. **Q.E.D.(3)**

This already yields:
(4) $\tilde{0} \leq_\infty S$.

(5) $S \leq_1 \tilde{0}$.
Proof: $\tilde{0}$ as a mastercode (see (6)) is a complete $\Sigma_2(L_\zeta)$ set of integers. There is thus an $i_0 \in \omega$ so that $n \in S \longleftrightarrow \langle i_0, n\rangle \in \tilde{0}$. **Q.E.D.(5)**

(6) $\tilde{0} \leq_1 S$.

Proof: We code all $\Sigma_2(L_\zeta)$ facts into S by finding a recursive function $n : \omega \times [\omega]^{<\omega} \to \omega$ so that $\langle i, x\rangle \in \tilde{0} \longleftrightarrow n(i,x) \in S$. We first note that p^2_ζ (the standard Σ_2-parameter of ζ) is empty: if $\nu \in p^2_\zeta$ then ν would be an L_ζ-cardinal; but this is impossible as otherwise all computations have settled snapshot sequences at some L_ζ-countable ordinal stage $\nu' < \nu$. Secondly, by (2), $\rho^2_\zeta = \omega$. Hence if $\langle \varphi_i \,|\, i < \omega\rangle$ enumerates all Σ_2-formulae with one free variable, we may consider $\tilde{0} = \{\langle i, x\rangle \in \omega \times [\omega]^{<\omega} \,|\, L_\zeta \models \varphi_i(x)\}$. Suppose $\varphi_i(w) \equiv \exists u \forall v \psi_i(u, v, w)$ where $\psi_i \in \Sigma_0$. Now let $\langle i, x\rangle \in \omega \times [\omega]^{<\omega}$ be arbitrary; let $n(i, x)$ be the program number of the computation that does the following: at stage ν $\sigma(\nu)$ is calculated, and a check is performed on the current contents, t say, of the output tape to see if it is (a code for) the $L_{\sigma(\nu)}$-least witness u to $L_{\sigma(\nu)} \models \forall v \psi(u, v, x)$. If it is, then t is allowed to remain on the tape; if it is not, then t will be overwritten. If such a code y say, of a witness $u \in L_{\sigma(\nu)}$ exists, then t is overwritten by y; if there is no u, then it is simply overwritten by a code for $\sigma(\nu)$. The following Claim completes (6) and the theorem.
Claim $L_\zeta \models \varphi_i(x) \longleftrightarrow n(i, x) \in S$.
Proof: $(\rightarrow)$ If $L_\zeta \models \varphi_i(x)$ then $\exists u_0 \in L_\zeta \quad L_\zeta \models \forall v \psi_i(u_0, v, x) \Rightarrow \forall \xi \geq \zeta \; L_\xi \models \forall v \psi_i(u_0, v, x)$, by using $L_\zeta \prec_{\Sigma_1} L_\sigma$. Consequently for such ξ, the L_ξ-least such u^* satisfies $\forall v \psi_i(u^*, v, x)$ has $u^* \in L_\zeta$, and once $\sigma(\nu) \geq \zeta$, then the u^* written to the output tape is never changed. Hence $n(i, x) \in S$.
$(\leftarrow)$: if $L_\zeta \models \neg\varphi(x)$ then using $L_\zeta \prec_{\Sigma_2} L_\Sigma$ one may check that for any $u \in L_\Sigma$ then for all sufficiently large $\xi < \Sigma \quad L_\xi \models \neg\forall v \psi_i(u, v, x)$. Consequently as $\sigma(\nu)$ increases beyond ζ, any code of u written to the output tape at stage ν must be overwritten at a later time. If $L_{\sigma(\nu)}$ finds no *local* u witnessing $\forall v \psi_i(u, v, x)$ then $\varphi_{n(i,x)}$ writes $\sigma(\nu)$ to the tape, but again, only to have it overwritten later. Hence $n(i, x) \notin S$. **Q.E.D.**

We remark now that Theorem 2.1 shows the machine architecture to be more powerful than might be thought ostensibly the case. The Infinite Time Turing Machines have something in the nature of a non-monotonic inductive operator: a set of integers may be input, and at various stages there is a set of integers present on the output tape. If the p'th machine has an index in S, then the output after running the operator $\phi_p(0)$ settles down at some ordinal $< \zeta$. It turns out that the ∞-Time Machines are as good as any Σ_2 definable machines. To make this precise we make the following definitions. If $s_\tau = s_\tau(p,y) = \langle C_i^{p,y}(\tau) \mid i < \omega \rangle$ enumerates the contents of all the cells C_i on the tapes of $\varphi_p(y)$ at time τ, [4] have defined the operation

$$\Gamma_0(s_\tau) = s_{\tau+1}$$

as being determined by the underlying Turing machine program at successor stages, whilst

$$\Gamma_0(\langle s_\tau | \tau < \nu \rangle) = s_\nu = \langle \lim_{\tau \to \nu} \sup_{\tau < \tau' < \nu} C_i^{p,y}(\tau') \mid i < \omega \rangle$$

One could clearly devise a machine that performs differently at limit stages (or even at any stage), and so that the value of $C_i^{p,y}(\nu)$ is determined not just by the previous values in that cell alone, but by the whole course of computation so far. More generally still, let us call any function $\Gamma : {}^{<On}({}^{\omega}2) \to ({}^{\omega}2)$ defined on sequences of reals "appropriate", and let us call a "Γ-machine", one that uses the operator Γ at all stages.

For this to conform to any kind of machine-like picture, we need some constraints on this definition, for example that a machine either halts or loops, and thus we have to say when such a generalised machine is considered to be "looping".

Definition 2.7 *An appropriate Γ is a* suitable Σ_n-operator *if there is a Σ_n formula $\varphi_\Gamma(v_0, v_1, v_2)$ so that for all ν, y φ_Γ defines over $L_\nu[y]$ the predicate*

$$P(\nu, i, p, y) \iff C_i^{p,y}(\nu) = 0$$

Thus we have uniformly in y and ν that $P(\nu, i, p, y)$ if and only $L_\nu[y] \models \varphi_\Gamma(i, p, y)$. For φ_Γ as above, suppose $\varphi_\Gamma(y, p, i) \longleftrightarrow \exists x \psi_\Gamma(x, y, p, i)$ where ψ_Γ is Π_{n-1}. Let

$$x_i(\gamma) = x_i^{p,y}(\gamma) = \begin{cases} \text{the } L_\gamma[y]\text{-least witness } x \text{ so that } L_\gamma[y] \models \psi_\Gamma(x, y, p, i) \text{ if such exists;} \\ 0 \text{ otherwise.} \end{cases}$$

Then $\langle x_i(\gamma) | i < \omega \rangle$ is a sequence of witnesses to the values of $C_i^{p,y}(\gamma)$ for $i < \omega$. Let us say that a Γ machine is *seen to be looping at limit stages* $\nu < \tau$ if (i) $s_\nu = s_\tau$; (ii) $\langle x_i(\nu) \rangle_i = \langle x_i(\tau) \rangle_i$. Thus, both the "snapshots" at time ν and τ are the same, but also that the witnesses to the zero cell values are unchanged. If this happens, and the machine has not already halted, then we say that the computation is not halting, and write $\varphi_p^\Gamma(y)\uparrow$.

Let $\mathcal{W}_\Gamma$, $\mathcal{EW}_\Gamma \ldots$ denote the classes of reals arising when we use such a machine.

Definition 2.8 *If Γ is a suitable Σ_n-operator, then Γ-machines are* Σ_n-complete, *if for any other suitable Σ_n-operator $\overline{\Gamma}$, $\mathcal{EW}_\Gamma \supseteq \mathcal{EW}_{\overline{\Gamma}}$.*

Our definitions ensure that the following is fairly immediate.

Theorem 2.9 *(i) Γ_0 is Σ_2-complete. (ii) For any suitable Σ_2-operator Γ, $\mathcal{W} \supseteq \mathcal{W}_\Gamma$.*

Proof: Let Γ be any suitable Σ_2 operator. Then our definitions of this, and of "$\varphi_p^\Gamma(0)\uparrow$" are such that, taking into account that $L_\zeta \prec_{\Sigma_2} L_\Sigma$, we have that for any p, if $\varphi_p^\Gamma(0)\uparrow$, then for this computation $s_p^\Gamma(\zeta) = s_p^\Gamma(\Sigma)$ and $\langle x_i(\zeta)\rangle_i = \langle x_i(\Sigma)\rangle_i$. Hence any $x \in WO \cap \mathcal{EW}_\Gamma$ must have rank $< \zeta$. But then any $x \in \mathcal{EW}_\Gamma$ is in L_ζ. (ii): because we can simulate the course of computation of a Γ-machine inside L_ζ, if $\varphi_p^\Gamma(0)\downarrow$ then it does so by stage λ, by appealing to $L_\lambda \prec_{\Sigma_1} L_\Sigma$. **Q.E.D.**

One might consider 2.1 and 2.4 as some justification for the claim that the ordinal ζ is somehow more intrinsic to these notions of computation than λ. The set S is thus a complete $\Sigma_2(L_\zeta)$ set of integers. We eventually get around to considering an *eventually infinite* degree hierarchy of the title of this paper:

Definition 2.10 $f \leq_{e\infty} g \longleftrightarrow f \in \mathcal{EW}^g$.

By analogy with Π^1_1 and hyperdegrees, any real strictly below (in $\leq_{e\infty}$) the complete $\Sigma_2(L_\zeta)$ set of integers has $e\infty$-degree $\mathbf{0}$. (Although the better analogy is perhaps with Δ^1_2-degrees.) We define a jump:

Definition 2.11 *$\tilde{x}$ is the set of indices $\{p \in \omega \mid \exists y \in \mathcal{EW}^x\ \varphi_p^x(0)\uparrow y\}$.*

Results corresponding to those above are:

Lemma 2.12 *(i) $x \leq_{e\infty} y \longleftrightarrow x \in L_{\zeta^y}[y]$; thus (ii) $x \leq_{e\infty} y \longrightarrow \zeta^x \leq \zeta^y$;*
(iii) The assignment $x \longrightarrow \zeta^x$ then satisfies the Spector criterion:
$x \leq_{e\infty} y \longrightarrow (\tilde{x} \leq_{e\infty} y \longleftrightarrow \zeta^x < \zeta^y)$.

Lemma 2.13 *$\tilde{x}$ is (1-1) equivalent to the complete $\Sigma_2(L_{\zeta^x}[x])$ set of integers.*

Let $F = \{x \in {}^\omega 2 \mid x \in L_{\zeta^x}\}$. Then it is easy to see that F_0 is just F.

We may define a natural hierarchy of eventually infinite degrees through F in the spirit of [6] (where this is done for Q and hyperdegrees) by $\boldsymbol{e_0} = [0]_\sim$; $\boldsymbol{e_{\alpha+1}} = [\tilde{e}_\alpha]_\sim$; $\boldsymbol{e_\mu} \simeq l.u.b\{\boldsymbol{e_\alpha} \mid \alpha < \mu\}$ for $\mathrm{Lim}(\mu)$ if defined.

Question 1 What is the natural length of this hierarchy? That is, what is the least ρ so that $\boldsymbol{e_\rho}$ is undefined?

Theorem 2.14 *$\rho \geq Z$ where Z is the smallest ordinal δ so that L_δ is Σ_2-extendible, and is a limit of Σ_2-extendibles.*

Proof: Suppose $\mu < Z$ and $\mathrm{Lim}(\mu)$. Let $\overline{\mu} = \sup\{\zeta^{e_\alpha} \mid \alpha < \mu\} < Z$. Let $n \leq 2$ be least so that $\omega\rho^n_{\overline{\mu}} = \omega$, and let $A = A^n_{\overline{\mu}}$ be the Σ_n mastercode for $L_{\overline{\mu}}$. Clearly A is a $\leq_{e\infty}$ minimal upper bound for $\{\boldsymbol{e_\alpha} \mid \alpha < \mu\}$. By using Lemma 2.12 it is easy to see that if this not a least upper bound, but b is another minimal upper bound, $\leq_{e\infty}$-incomparable to it, then we must have $\zeta^b = \overline{\mu}$. By the obvious relativisation of 2.1 this would imply that $\overline{\mu}$ would be Σ_2-extendible, contrary to assumption. Hence $\boldsymbol{e_\mu}$ is defined. **Q.E.D.**

By way of contrast, the natural hierarchy of ∞-degrees has length ω - see [8].

Question 2 Are there minimal ∞-degrees or $e\infty$-degrees? When does a countable set of $e\infty$-degrees (or ∞-degrees) have minimal upper bounds?

(Minimal ∞-degrees are now known to exist, see [8] where also some very partial answers as to which countable sets of ∞-degrees have minimal upper bounds are given.)

We conclude with some amusing examples on the ∞-degrees which are exercises in the Barwise Compactness Theorem. The first says that any non-obviously not halting computation can be made to halt in some (perforce illfounded) model.

Lemma 2.15 *Suppose $p \notin 0^\nabla$. Suppose it is not provable in Σ_1-KP that $\varphi_p(0)\uparrow$. Then there is an end extension of L_λ which is a model of $KP +$ "$\varphi_p(0)\downarrow$."*

The second example is comparable with the fact that any countable model of ZFC can be extended to one of $ZFC + V = L$.

Lemma 2.16 *Suppose $x \notin F_0$,* i.e. $x \notin L_{\lambda^x}$. *Then there is a model of $KP +$ "$x \in F_0$" which is an end extension of L_{λ^x}.*

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