Games for Supervaluation and Dependency

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26.iii.2006

Introduction

- It is possible to use *2 person perfect information games* to give an epistemic variation on the Kripkean semantic approach to introducing a partially defined truth predicate to a formal language.
- We shall review the notion of such games, emphasising particularly *open* games.
- We then see how a Kripkean fixed point using Strong Kleene truth tables can be characterised using open games (due to Martin[1]).
- We then consider such games for supervaluation fixed points.
- Then we sketch H. Leitgeb's notion of *dependency* and see likewise how we can characterise grounded sentences in terms of strategies for particular games.

[1] D.A.Martin *Revision and its rivals* Phil. Issues, **8**, 1997

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• Thus a strategy for player I is a rule or function that takes the even length sequence $k_0, \ldots, k_{2n-1}$ played so far, and tells him/her what to play for k_{2n}. Analogously for II.

Definition (i) Let $s = (s_0, \ldots, s_m)$; then
$N_s = \{\vec{k} \in \mathbb{N}^{\mathbb{N}} \mid k_i = s_i (i \leq m)\}$ is a *basic open set*.
(ii) Let A be a union of basic open sets. Then A is called *open*.

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Proof: *Observation:* Suppose I has no winning strategy. Then whatever he plays as k_0 II has a reply k_1 so that I has no winning strategy τ in the "sub-game" from this point onwards. (Otherwise he *does* have a strategy in G_A: play such a k_0; when II has replied k_1 then he continues with τ.)

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• Note: Determinacy of A symmetrically implies the determinacy of its complement. Hence "Closed" Determinacy holds too.

Strong Kleene Logic

We consider a first order language $\mathcal{L}$ (in $\vee, \neg, \exists$ and extending that of arithmetic), and add a new monadic predicate symbol $\dot{T}$ to obtain $\mathcal{L}_{\dot{T}}$. (We assume a recursive gödelisation of the language so that, *e.g.* each sentence σ of $\mathcal{L}_{\dot{T}}$ obtains a code number $\ulcorner\sigma\urcorner$, plus ....)

If we are given an $\mathcal{L}_{\dot{T}}$-structure $\mathcal{M}$ and are in possession of a partial evaluation of $\dot{T}$ in $\mathcal{M}$ as $A = (A^+, A^-)$ as to what is in or out of $T_{\mathcal{M}}$, the Strong Kleene rules allow us to extend that partial valuation to $\Gamma(A) =_{df} (\Gamma(A)^+, \Gamma(A)^-)$:

Rules for Strong Kleene

- If $\varphi \in \mathcal{L}$ is atomic then
$\ulcorner\varphi\urcorner \in \Gamma(A)^+ \quad (\Gamma(A)^-) \Longleftrightarrow \mathcal{M} \models \varphi \quad (\mathcal{M} \models \neg\varphi)$.

- If $\varphi \in \mathcal{L}_{\dot{T}}$ is $\dot{T}(\ulcorner\sigma\urcorner)$ then
$\ulcorner\varphi\urcorner \in \Gamma(A)^+ \quad (\Gamma(A)^-)$ iff $\ulcorner\sigma\urcorner \in A^+ \quad (\ulcorner\sigma\urcorner \in A^-)$.

- $\ulcorner\varphi\urcorner \in \Gamma(A)^+$ if φ is:

(i)	$\psi \vee \chi$	&	either $\ulcorner\psi\urcorner$ or $\ulcorner\chi\urcorner \in \Gamma(A)^+$;
(ii)	$\neg\psi$	&	$\ulcorner\psi\urcorner \in \Gamma(A)^-$;
(iii)	$\exists v\psi(v)$	&	for some n $\ulcorner\psi[\underline{n}]\urcorner \in \Gamma(A)^+$.

- $\ulcorner\varphi\urcorner \in \Gamma(A)^-$ if φ is:

(i)	__________	&	both $\ulcorner\psi\urcorner$ and $\ulcorner\chi\urcorner \in \Gamma(A)^+$;
(ii)	__________	&	$\ulcorner\psi\urcorner \in \Gamma(A)^+$;
(iii)	__________	&	for all n $\ulcorner\psi[\underline{n}]\urcorner \in \Gamma(A)^+$.

Games for Strong Kleene Logic

- We take a first order structure $\mathcal{M}$ appropriate for a language $\mathcal{L}_{\mathcal{M}}$.
- We define a game G_{φ} where $\varphi \in \mathcal{L}_{\mathcal{M}}$ is a sentence.

Martin's games for Strong Kleene fixed points

Fix $\varphi \in \mathit{Sent} \cap \mathcal{L}_{\dot{T}}$. G_φ is as follows: Player I tries to verify φ, Player II to falsify φ; they list sentences φ_m.

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$\psi \vee \chi$	then Player i lists as φ_{m+1} :	either ψ or χ;
$\neg\psi$	then the other player lists as φ_{m+1} :	ψ;
$\exists v \psi(v)$	then Player i must list as φ_{m+1} :	$\varphi[\underline{n}]$ for some n;
$\dot{T}(\ulcorner\sigma\urcorner)$	then Player i lists as φ_{m+1} :	σ.

Play <u>terminates</u> if φ is atomic:
if φ_m is $\dot{T}(\underline{n})$ and n does not code a sentence, then player i *loses*;

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Play <u>terminates</u> if φ is atomic:
if φ_m is $\dot{T}(\underline{n})$ and n does not code a sentence, then player i *loses*; otherwise player i *wins* iff $(\varphi_m)_{\mathcal{M}}$.

• If the game continues for infinitely many steps it is declared a draw.

• This does not conform to the strict format of an open game, because it allows for infinite plays to be so declared, but this is inessential.

• Note the compositionality of the Strong Kleene rules is reflected in the game's rules.

Take $\mathcal{L}, \ldots \mathcal{M}$ with a partial evaluation $A = (A^+, A^-)$ as before, we define:

$\Gamma(A) =_{df} (\Gamma(A)^+, \Gamma(A)^-)$:

$\ulcorner\varphi\urcorner \in \Gamma(A)^+ \quad (\Gamma(A)^-$ resp. $)$ iff
$\forall U \supseteq A^+[U \cap A^- = \varnothing, \Longrightarrow (\mathcal{M}, U) \models \varphi \ \ (\neg\varphi$ resp.$)]$.

- This is again monotonic:

$A^{+/-} \subseteq B^{+/-} \Longrightarrow \Gamma(A)^{+/-} \subseteq \Gamma(B)^{+/-}$, and starting from $(\varnothing, \varnothing)$ we can build up a *least fixed point:*

$$(\Gamma^+_\infty, \Gamma^-_\infty) = (\bigcup_\alpha \Gamma^+_\alpha, \bigcup_\alpha \Gamma^-_\alpha).$$

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Rules for II: Idea: Agathe is trying to play a sequence Σ of sentences $(m_0, m_1, \ldots)$ of the language $\mathcal{L}^+$ such, if Ψ is the extension of $\dot{T}$ so listed, (*i.e.* $\Psi = \{\ulcorner\sigma\urcorner \mid \exists i\ m_i =$ "$\ulcorner\sigma\urcorner \in \dot{T}$" $\}$) then Σ is the complete theory of $\langle \mathbb{N}, +, \times, 0,', \ldots \Psi\rangle$. Additionally she is trying to ensure $\varphi \notin \Sigma$.

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We may think of the game as a series of queries *Is* $\tau \in \Sigma$*?*" by Ulrich (including the sentences as to whether "$\ulcorner\sigma\urcorner \in \dot{T}$"); Agathe is giving "*yes*" or "*no*" answers to these queries, sometimes together with some additional information. During the course of the game Ulrich may issue a *challenge* to Agathe's earlier replies. We (semi-)formally define this via a description of the possible moves.

Round k: Player I states "i_k". This can either be

(i) a sentence σ, of $\mathcal{L}^+$, in which case Player II's reply is either "σ" or "$\neg\sigma$" (meaning yes, "$\sigma \in \Sigma$" or, no, "$\neg\sigma \in \Sigma$"); additionally, if σ is of the form "$\exists v_0 \psi(v_0)$" and she wishes to answer "yes", then she must also choose some $n \in \mathbb{N}$ and at the same time play "$\psi[\overline{n}]$" as well as σ (so in this case m_k codes this pair of sentences).

(ii) Player I may alternatively state a *challenge* to Player II. The challenges may come in one of two forms: (A) either he plays $i_k \neq 0$ that is a code of a proof of "$0 = 1$" from a (finite) list $m_{i_1}, \ldots, m_{i_l}$ of sentences that II has already played in earlier rounds - thus demonstrating that II's list is inconsistent. His other form, (B) is that he may challenge an earlier assertion by II of the form " $\ulcorner\sigma\urcorner \notin \dot{T}$ ", or " $\ulcorner\sigma\urcorner \in \dot{T}$ ".
Additionally to these Rules, are requirements that Player II attempt to falsify φ: if queried by Player I on φ she *must* reply $\neg\varphi$ (or lose).

The last rule together with the description at *Round k* comprise the *basic rules* of the game G_φ. If anybody disobeys these rules, then the first to do so forfeits the game. If Player I issues a "consistency" challenge, then if he is correct, he wins outright and the game is over; however if he is incorrect Player II wins outright. If Player I issues a challenge of type (B) to some earlier assertion by II of the form " $\ulcorner\sigma\urcorner \notin \dot{T}$ ", or " $\ulcorner\sigma\urcorner \in \dot{T}$ ", then play in G_φ is halted, and play starts afresh in G_σ (or $G_{\neg\sigma}$ if " $\ulcorner\sigma\urcorner \in \dot{T}$ " was the challenge). We can thus think of G_σ as a "subgame" of G_φ, but in any case it is defined from σ (or $\neg\sigma$) just as G_φ was defined from φ.

Ulrich wins $\mathbf{G}_{\boldsymbol{\varphi}}$ if and only if (i) II makes a mistake on the basic rules of the game (or any I-initiated subgame) at some stage, **or** (ii) he makes a successful challenge to the consistency of the sentences II is playing in G_φ or in the current subgame. Thus, for I to win, the overall game must be finite in length.

Note: *II* thus wins precisely when: (a) *I* makes a false accusation that her list is inconsistent; or (b) she manages to make infinitely many moves in G_φ or in any initiated subgame; or else (c) through the whole course of play *I* initiates infinitely many subgames.

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Lemma

I has a winning strategy in $G_\varphi \iff \varphi \in \Gamma^+_\infty$ *where* $\Gamma_\infty = \langle \Gamma^+_\infty, \Gamma^-_\infty \rangle$ *is the least Kripkean supervaluation fixed point.*

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Proof: $(\Longleftarrow)$ Suppose $\varphi \in \Gamma^+_{\alpha_0+1} \backslash \Gamma^+_{\alpha_0}$ (we'll say that "$\mathrm{rk}(\varphi) = \alpha_0$"). *II* then plays out some Σ and Ψ as extension for $\dot{T}$. If she follows the basic rule of declaring "$\neg\varphi \in \Sigma$" when asked, then Ψ cannot be compatible with $(\Gamma^+_{\alpha_0}, \Gamma^-_{\alpha_0})$ as $\varphi \in \Gamma^+_{\alpha_0+1}$. Hence either:

(i) For some $\sigma_1 \in \Gamma^+_{\alpha_0}$, $\sigma_1 \notin \Psi$, or:
(ii) For some $\sigma_1 \in \Gamma^-_{\alpha_0}$, $\sigma_1 \in \Psi$

As part of his strategy, *I* makes sure that he asks every query of the form "$\tau \in \dot{T}$" during his course of play. (Indeed *I* can ensure the consistency and also the completeness of the pertinent Σ of any infinite run of any subgame, by querying every possible sentence.)

At some point then, suppose Case (i) (or Case (ii)) holds, and, when queried on $?\dot{T}(\sigma_1)?$, *II* states " $\sigma_1 \notin \dot{T}$" (or " $\sigma_1 \in \dot{T}$" respectively). Then *I* on his next move calls a challenge to this statement, and the subgame G_{σ_1} (or $G_{\neg\sigma_1}$ respectively) is initiated. The point is that $\sigma_1 \in \Gamma^+_{\alpha_0}$ (or $\sigma_1 \in \Gamma^-_{\alpha_0}$). In either case (with the obvious extension of notation) $\alpha_1 =_{\mathrm{df}} \mathrm{rk}(\sigma_1) < \alpha_0$, and the subgame now being initiated is being played on a formula of lower rank. The $\Psi(\sigma_1)$ (or $\Psi(\neg\sigma_1)$) that *II* now will now try to play out is incompatible with $(\Gamma^+_{\alpha_1}, \Gamma^-_{\alpha_1})$, (if she adheres to the basic rule of declaring that "$\sigma_1 \notin \Psi(\sigma_1)$" (or "$\sigma_1 \in \Psi(\sigma_1)$")), and thus at some stage in the new game *I* may issue a challenge on the assertion that some $\sigma_2 \notin \setminus \in \Psi$, and we have for the resultant $\alpha_2 =_{\mathrm{df}} \mathrm{rk}(\sigma_2) < \alpha_1$. Proceeding in this way, as long as *II* does not lose through inconsistency, *I* eventually challenges with some σ_k with $\mathrm{rk}(\sigma_k) = 0$. However if $\sigma_k \in \Gamma^+_1$ then *every* Ψ is compatible with $(\Gamma^+_0, \Gamma^-_0) = (\varnothing, \varnothing)$ and *II* in her play of G_{σ_k}, simply cannot produce an extension Ψ and a *consistent* set of sentences Σ so that $\langle \mathbb{N}, +, \times, 0,', \ldots \Psi \rangle \models \Sigma$, whilst $\langle \mathbb{N}, +, \times, 0,', \ldots \Psi \rangle \models \neg\sigma_k$.

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- She thus will lose G_{σ_k} by breaking a basic rule.

Question: *Why did we require that she produce numeral evidence of the form $\psi[\underline{n}]$ when she gave an affirmative response to the query: "$\exists v_0 \psi(v_0)$"?*

Answer: Then the sets Σ being produced, if consistent, are true in the standard model $\langle \mathbb{N}, +, \times, 0,', \ldots \Psi \rangle$.

($\Longrightarrow$) Suppose $\varphi \notin \Gamma^+_\infty$. We describe how Agathe can win. By hypothesis there is $\Psi = \Psi(\varphi)$ with $\Psi \supseteq \Gamma^+_\infty, \Psi \cap \Gamma^-_\infty = \varnothing$, with $\langle \mathbb{N}, +, \times, 0,', \ldots \Psi \rangle \models \neg\varphi$. Whenever she is queried Agathe consults the above model and gives the appropriate reply. She thus will not lose G_φ on consistency grounds, nor by breaking any other basic rule. If she is challenged on her assertion "$\sigma_1 \notin \dot{T}$" then $\sigma_1 \notin \Gamma^+_\infty$. The subgame initiated is G_{σ_1}, but as $\sigma_1 \notin \Gamma^+_\infty$ she is no worse off than before, and can play just as well here continuing with Ψ since $\langle \mathbb{N}, +, \times, 0,', \ldots \Psi \rangle \models \neg\sigma_1$ If she is challenged on "$\sigma_1 \in \dot{T}$" then the subgame $G_{\neg\sigma_1}$ is initiated; but $\neg\sigma_1 \notin \Gamma^+_\infty$, (as $\sigma_1 \notin \Gamma^-_\infty$), and so she can still continue. Clearly she can keep this up no matter how many challenges that Ulrich issues, and she will ultimately win.

QED

Same $\mathcal{L}_{\mathcal{T}}$ etc as before.

Definition[2] φ *depends on* Ψ if and only if:

$\forall B, H \subseteq \mathcal{L}_{\mathcal{T}}$, $\varphi_{(\mathcal{M},H)} \Longleftrightarrow \varphi_{(\mathcal{M},\mathcal{B})}$ iff $\varphi_{(\mathcal{M},\Psi\cap H)} \Longleftrightarrow \varphi_{(\mathcal{M},\Psi\cap B)}$

- Equivalently: $\forall H \subseteq \mathcal{L}_{T}$, $\varphi_{(\mathcal{M},\Psi)} \Longleftrightarrow \varphi_{(\mathcal{M},\Psi\cap H)}$.

[2] H.Leitgeb *"What Truth Depends on"*, JPL, 34, 2005

Dependency

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Definition $D(\Psi) = \{\ulcorner\varphi\urcorner \mid \varphi \text{ depends on } \Psi\}$.

- D is monotonic, and Π^1_1; we set $\Phi_0 = \varnothing$; $\Phi_\infty = \bigcup_\alpha \Phi_\alpha$, where $\Phi_{\alpha+1} = D(\Phi_\alpha)$ etc.

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- Leitgeb calls the sentences in Φ_∞, the least fixed point, *grounded*. Of the *ungrounded*, we call those λ who depend on themselves, (*i.e.* $\{\lambda\}$), *self-referential*.

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A general result on positive monotone inductive fixed points

Theorem

(Moschovakis,Aczel) Suppose A is positive monotone inductive over a countable structure $(\mathbb{N}, +, \times, \ldots, \vec{R}, \vec{F})$*. Then there is an open set B in* $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ *so that* $n \in A$ *iff I has a winning strategy in the open game* G_{B_n} *where* $B_n =_{df} \{\vec{k} \mid (n, \vec{k}) \in B\}$

Characterising Dependency groundedness

We now describe the variant of the game which characterises the complete Π^1_1 set of *(dependency) grounded sentences* $\Phi_\infty = \bigcup_\alpha \Phi_\alpha$.
G^*_φ is played in a similar fashion, but now *II* must produce two extensions B_0, B_1 and sets of sentences Σ_0, Σ_1 (say she attends to queries about B_0, Σ_0 in even numbered rounds, and to B_1, Σ_1 on odd numbered ones). She is trying to ensure that $(\varphi)_{(\mathbb{N},B_0)} \Leftrightarrow (\neg\varphi)_{(\mathbb{N},B_1)}$ and so when queried about φ must answer accordingly. The basic rules are the same *mutatis mutandis* and *I* may still challenge on grounds of consistency, with the same outcomes. If however *II* has earlier asserted "$\sigma \in B_0$" and "$\sigma \notin B_1$" (or *vice versa*) then *I* may issue a challenge, and the subgame G^*_σ is initiated (again the subgame is the same as the game: she tries to produce two extensions $B_0(\sigma), B_1(\sigma)$ etc., etc.) Again, if neither player messes up their basic rules, then if the overall game lasts for infinitely many stages *II* wins.

Lemma

I has a winning strategy in $G^*_\varphi \Longleftrightarrow \varphi \in \Phi_\infty$.

Proof: ($\Longleftarrow$) Suppose now $\varphi \in \Phi_{\alpha+1} \backslash \Phi_\alpha$; we set $\mathrm{rk}^D(\varphi) = \alpha$. By the definition of dependency, if $(\varphi)_{(\mathbb{N},B_0)} \Leftrightarrow (\neg\varphi)_{(\mathbb{N},B_1)}$, then $B_0 \cap \Phi_\alpha \neq B_1 \cap \Phi_\alpha$. Hence if *II* is trying to produce such B_0, B_1 for some $\sigma_1 \in \Phi_\alpha$ she must answer "$\sigma_1 \notin B_0$" and "$\sigma_1 \in B_1$" (or *vice versa*) when queried by Ulrich, and the latter may now issue a challenge and the game proceeds to $G^*_{\sigma_1}$. Now of course $\alpha_1 =_{\mathrm{df}} \mathrm{rk}^*(\sigma_1) < \alpha_0$, and if no one messes up their basic rules, we arrive as before at the situation of playing in $G^*_{\sigma_k}$ where $\mathrm{rk}^D(\sigma_k) = 0$. But here, as $\Phi_0 = \varnothing$, for every B, B', $(\sigma_k)_{(\mathbb{N},B)} \Leftrightarrow (\sigma_k)_{(\mathbb{N},B')}$.

$(\Longrightarrow)$ Suppose $\varphi \notin \Phi_\infty$. Then there exists B_0 with $(\varphi)_{(\mathbb{N},B_0)} \Leftrightarrow (\neg\varphi)_{(\mathbb{N},B_0\cap\Phi_\infty)}$ She may then play out the complete theories of the two models $\langle\mathbb{N},\ldots,B_0\rangle, \langle\mathbb{N},\ldots,B_0\cap\Phi_\infty)\rangle$. If she is challenged at some point on her assertions amounting to $\sigma \in B_0 \backslash B_0 \cap \Phi_\infty)$, then even when the subgame G^*_σ is initiated we have $\sigma \notin \Phi_\infty$ and she is no worse off than before, and can continue in the same fashion using some B_2 with $(\sigma)_{(\mathbb{N},B_2)} \Leftrightarrow (\neg\sigma)_{(\mathbb{N},B_2\cap\Phi_\infty)}$. QED

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(i) For Field's theory of truth[a] *over $\mathcal{M} = \mathbb{N}$, the set of 'ultimate truths', U, cannot be the set of sentences for which a player has a winning strategy in any open game. Moreover no game with even Σ^0_2-payoff set of infinite sequences can be used here.*

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(ii) The same is true for the set of stable truths under a Herzbergerian limit rule of a single *revision sequence starting with some (recursive) distribution of truth values.*

Theorem *There is a Σ^0_3 set, $B \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, so that $\sigma \in U$ iff I has a w.s. in G_{B_σ} where $B_\sigma = \{\vec{k} \in \mathbb{N}^{\mathbb{N}} \mid (\ulcorner\sigma\urcorner, \vec{k}) \in B\}$.*

[a] H. Field *A Revenge Immune solution of the semantic paradoxes*, JPL, 2003

Full Gupta-Belnap Revision Theoretic Truth

- The set S of *revision-theoretic stable truths*[3] (under any limit rule) forms a complete Π^1_2 set of integers.

- Again on general grounds, this set cannot be represented as the set of games for which I has a winning strategy. But it's complement can:

Theorem

There is a set Π^1_1 set, $B^S \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, so that $\sigma \notin S$ iff I has a w.s. in $G_{B^S_\sigma}$ where $B^s_\sigma = \{\vec{k} \in \mathbb{N}^{\mathbb{N}} \mid (\ulcorner\sigma\urcorner, \vec{k}) \in B^S\}$.

- However to prove this theorem requires going beyond Zermelo-Fraenkel set theory, and assuming the existence of large cardinals. We need *Determinacy*(Π^1_1).

[3] A.Gupta& N.Belnap *The Revision Theory of Truth*, MIT Press 1993