Games for Supervaluation and Dependency

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• It is possible to use 2 person perfect information games to give an epistemic variation on the Kripkean semantic approach to introducing a partially defined truth predicate to a formal language.
• We shall review the notion of such games, emphasising particularly open games.
• We then see how a Kripkean fixed point using Strong Kleene truth tables can be characterised using open games (due to Martin\(^1\)).
• We then consider such games for supervaluation fixed points.
• Then we sketch H. Leitgeb’s notion of dependency and see likewise how we can characterise grounded sentences in terms of strategies for particular games.

\(^1\)D.A. Martin *Revision and its rivals* Phil. Issues, 8, 1997
Typically a game between two players is played on integers, or 0’s and 1’s. Let $A$ be a set of infinite sequences of natural numbers $k \in \mathbb{N}$. 

The game $G_A$ is defined as follows:

I plays $k_0, k_2, k_4, \ldots$

II plays $k_1, k_3, k_5, \ldots$

We say I wins if $x = (k_0, k_1, \ldots) \in A$.

We say $G_A$, or just $A$, is determined if either player has a winning strategy in this game.

Thus a strategy for player I is a rule or function that takes the even length sequence $k_0, \ldots, k_{2n-1}$ played so far, and tells him/her what to play for $k_{2n}$. Analogously for II.
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Thus a strategy for player $I$ is a rule or function that takes the even length sequence $k_0, \ldots, k_{2n-1}$ played so far, and tells him/her what to play for $k_{2n}$. Analogously for $II$. 
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The game $G_A$ is defined as follows:

- \( I \) plays $k_0$
- \( II \) plays

Therefore, a strategy for player \( I \) is a rule or function that takes the even length sequence $k_0, \ldots, k_{n-1}$ played so far, and tells him/her what to play for $k_n$. Analogously for \( II \).
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The game $G_A$ is defined as follows:

- Player I plays $k_0, k_2$
- Player II plays $k_1$

We say I wins if $x = (k_0, k_1, \ldots) \in A$.

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Typically a game between two players is played on integers, or 0’s and 1’s. Let \( A \) be a set of infinite sequences of natural numbers \( k \in \mathbb{N} \).

The game \( G_A \) is defined as follows:

1. \( I \) plays \( k_0 \ k_2 \)
2. \( II \) plays \( k_1 \ k_3 \)

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We say \( G_A \), or just \( A \), is determined if either player has a winning strategy in this game.

Thus a strategy for player \( I \) is a rule or function that takes the even length sequence \( k_0, \ldots, k_{2n-1} \) played so far, and tells him/her what to play for \( k_{2n} \). Analogously for \( II \).
Infinite two person games

- Typically a game between two players is played on integers, or 0’s and 1’s. Let $A$ be a set of infinite sequences of natural numbers $k \in \mathbb{N}$.
- The game $G_A$ is defined as follows:

  I plays $k_0 \ k_2 \ \ldots \ k_{2n} \ldots$
  II plays $k_1 \ k_3$

- We say I wins if $x = (k_0, k_1, \ldots) \in A$.
- We say $G_A$, or just $A$, is determined if either player has a winning strategy in this game.
- Thus a strategy for player I is a rule or function that takes the even length sequence $k_0, \ldots, k_{2n-1}$ played so far, and tells him/her what to play for $k_{2n}$. Analogously for II.
Typically a game between two players is played on integers, or 0’s and 1’s. Let $A$ be a set of infinite sequences of natural numbers $k \in \mathbb{N}$.

The game $G_A$ is defined as follows:

- $I$ plays $k_0, k_2, \ldots, k_{2n}, \ldots$
- $II$ plays $k_1, k_3, \ldots, k_{2n+1}, \ldots$

We say $I$ wins if $x = (k_0, k_1, \ldots) \in A$.

We say $G_A$, or just $A$, is determined if either player has a winning strategy in this game.

Thus a strategy for player $I$ is a rule or function that takes the even length sequence $k_0, \ldots, k_{2n}-1$ played so far, and tells him/her what to play for $k_{2n}$. Analogously for $II$. 
Typically a game between two players is played on integers, or 0’s and 1’s. Let $A$ be a set of infinite sequences of natural numbers $k \in \mathbb{N}$.

The game $G_A$ is defined as follows:

- Player I plays $k_0 \ k_2 \ \ldots \ k_{2n} \ldots$
- Player II plays $k_1 \ k_3 \ \ldots \ k_{2n+1} \ldots$

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The game $G_A$ is defined as follows:

- $I$ plays $k_0 \; k_2 \; \ldots \; k_{2n} \ldots$
- $II$ plays $k_1 \; k_3 \; \ldots \; k_{2n+1} \ldots$

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\[
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I & \text{ plays } k_0 \ k_2 \ \ldots \ k_{2n} \ldots \\
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\end{align*}
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• We say $I$ wins if $x = (k_0, k_1, \ldots) \in A$.

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Open Sets are determined

**Definition** (i) Let \( s = (s_0, \ldots, s_m) \); then

\[ N_s = \{ \vec{k} \in \mathbb{N}^N | k_i = s_i (i \leq m) \} \] is a **basic open set**.

(ii) Let \( A \) be a union of basic open sets. Then \( A \) is called **open**.
Open Sets are determined

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**Theorem**

*(Gale-Stewart) Open Determinacy holds.*
Open Sets are determined

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(i) Let \( s = (s_0, \ldots, s_m); \) then
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is a basic open set.
(ii) Let \( A \) be a union of basic open sets. Then \( A \) is called open.

**Theorem**
(Gale-Stewart) Open Determinacy holds.

**Proof:** **Observation:** Suppose \( I \) has no winning strategy. Then whatever he plays as \( k_0 \) \( II \) has a reply \( k_1 \) so that \( I \) has no winning strategy \( \tau \) in the “sub-game” from this point onwards. (Otherwise he *does* have a strategy in \( G_A \): play such a \( k_0 \); when \( II \) has replied \( k_1 \) then he continues with \( \tau \).)
This means $II$ will have a winning strategy (w.s.):
This means II will have a winning strategy (w.s.):
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$I$ plays
$II$ plays
This means \( II \) will have a winning strategy (w.s.):

\[
\begin{align*}
& I \text{ plays } k_0 \\
& II \text{ plays }
\end{align*}
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This means \( II \) will have a winning strategy (w.s.):
She plays \( k_1 \) so that \( I \) has no w.s. \( \tau_{k_0,k_1} \) from this point on.

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\begin{align*}
I & \text{ plays } k_0 \\
II & \text{ plays } k_1
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This means II will have a winning strategy (w.s.):
She plays $k_1$ so that I has no w.s. $\tau_{k_0,k_1}$ from this point on.
She then waits for $k_2$.

$I$ plays $k_0$ $k_2$
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This means II will have a winning strategy (w.s.):
She plays $k_1$ so that I has no w.s. $\tau_{k_0,k_1}$ from this point on.
She then waits for $k_2$.
By repeating the Observation, she has a play $k_3$ so that I has no w.s. from this point on.

\[
\begin{align*}
\text{I plays } & k_0 \quad k_2 \\
\text{II plays } & k_1 \quad k_3
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&I \text{ plays } k_0 \quad k_2 \quad \ldots \quad k_{2n} \ldots \\
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$$
\begin{align*}
I \text{ plays } & k_0 \ k_2 \ \ldots \ k_{2n} \ \ldots \\
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\end{align*}
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This means \( II \) will have a winning strategy (w.s.):
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She then waits for \( k_2 \).
By repeating the Observation, she has a play \( k_3 \) so that \( I \) has no w.s. from this point on.

\[
\begin{align*}
I \text{ plays } & k_0 \kern1.5em k_2 \kern1.5em \cdots \kern1.5em k_{2n} \cdots \\
II \text{ plays } & k_1 \kern1.5em k_3 \kern1.5em \cdots \kern1.5em k_{2n+1} \cdots
\end{align*}
\]

\( II \) can thus play to the end of the game, and at no finite point in the game has \( I \) won (Interpretation: at no finite position \( s = k_0, \ldots, k_n \) does \( A \) contain \( N_s \). This means \( \vec{k} \notin A \). QED
This means \( II \) will have a winning strategy (w.s.):
She plays \( k_1 \) so that \( I \) has no w.s. \( \tau_{k_0,k_1} \) from this point on.
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- Note: Determinacy of \( A \) symmetrically implies the determinacy of its complement. Hence “Closed” Determinacy holds too.
We consider a first order language $\mathcal{L}$ (in $\lor, \neg, \exists$ and extending that of arithmetic), and add a new monadic predicate symbol $\dot{T}$ to obtain $\mathcal{L}_{\dot{T}}$. (We assume a recursive gödelisation of the language so that, e.g. each sentence $\sigma$ of $\mathcal{L}_{\dot{T}}$ obtains a code number $\lceil \sigma \rceil$, plus ....)

If we are given an $\mathcal{L}_{\dot{T}}$-structure $\mathcal{M}$ and are in possession of a partial evaluation of $\dot{T}$ in $\mathcal{M}$ as $A = (A^+, A^-)$ as to what is in or out of $T_M$, the Strong Kleene rules allow us to extend that partial valuation to $\Gamma(A) =_{df} (\Gamma(A)^+, \Gamma(A)^-)$:
Rules for Strong Kleene

• If $\varphi \in \mathcal{L}$ is atomic then
  \[ \Box \varphi \in \Gamma(A)^+ \quad (\Gamma(A)^-) \iff M \models \varphi \quad (M \models \neg \varphi). \]

• If $\varphi \in \mathcal{L}_\ddagger$ is $\ddagger T(\Box \sigma)$ then
  \[ \Box \varphi \in \Gamma(A)^+ \quad (\Gamma(A)^-) \text{ iff } \Box \sigma \in A^+ \quad (\Box \sigma \in A^-). \]

• $\Box \varphi \in \Gamma(A)^+$ if $\varphi$ is:
  (i) $\psi \lor \chi$ & either $\Box \psi$ or $\Box \chi \in \Gamma(A)^+$;
  (ii) $\neg \psi$ & $\Box \psi \in \Gamma(A)^-$;
  (iii) $\exists v \psi(v)$ & for some $n$ $\Box \psi[n] \in \Gamma(A)^+$.

• $\Box \varphi \in \Gamma(A)^-$ if $\varphi$ is:
  (i) _______ & both $\Box \psi$ and $\Box \chi \in \Gamma(A)^+$;
  (ii) _______ & $\Box \psi \in \Gamma(A)^+$;
  (iii) _______ & for all $n$ $\Box \psi[n] \in \Gamma(A)^+$.
• We take a first order structure $\mathcal{M}$ appropriate for a language $\mathcal{L}_{\mathcal{M}}$.
• We define a game $G_\varphi$ where $\varphi \in \mathcal{L}_{\mathcal{M}}$ is a sentence.
Fix $\varphi \in \text{Sent} \cap \mathcal{L}_\ddot{T}$. $G_\varphi$ is as follows: Player I tries to verify $\varphi$, Player II to falsify $\varphi$; they list sentences $\varphi_m$. 

Play terminates if $\varphi$ is atomic: if $\varphi_m$ is $\dot{T}(n)$ and $n$ does not code a sentence, then player $i$ loses; otherwise player $i$ wins iff $(\varphi_m)_M$. 

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Fix $\varphi \in \text{Sent} \cap \mathcal{L}_{\bar{T}}$. $G_\varphi$ is as follows: Player I tries to verify $\varphi$, Player II to falsify $\varphi$; they list sentences $\varphi_m$.

I plays

$\varphi = \varphi_0 \varphi_1 \varphi_3 \ldots$

II plays $\varphi_2 \ldots \varphi_m$.

If $\varphi_m$ is the last sentence played, and it was played by Player $i$:

- If $\varphi_m$ is $\psi \lor \chi$ then Player $i$ lists as $\varphi_m + 1$: either $\psi$ or $\chi$;
- $\neg \psi$ then the other player lists as $\varphi_m + 1$: $\psi$;
- $\exists v \psi(v)$ then Player $i$ must list as $\varphi_m + 1$: $\varphi[n]$ for some $n$;
- $\bar{T}(\llbracket \sigma \rrbracket)$ then Player $i$ lists as $\varphi_m + 1$: $\sigma$.

Play terminates if $\varphi$ is atomic:

- if $\varphi_m$ is $\bar{T}(n)$ and $n$ does not code a sentence, then player $i$ loses;
- otherwise player $i$ wins iff $(\varphi_m)_M$. 

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Fix $\varphi \in \text{Sent} \cap \mathcal{L}_{\tilde{T}}$. $G_\varphi$ is as follows: Player $I$ tries to verify $\varphi$, Player $II$ to falsify $\varphi$; they list sentences $\varphi_m$.

$I$ plays $\varphi = \varphi_0$
Fix $\varphi \in \text{Sent} \cap \mathcal{L}_\mathcal{T}$. $G_\varphi$ is as follows: Player I tries to verify $\varphi$, Player II to falsify $\varphi$; they list sentences $\varphi_m$.

I plays $\varphi = \varphi_0 \, \, \varphi_1$
Fix \( \varphi \in \text{Sent} \cap \mathcal{L}_{\bar{T}} \). \( G_\varphi \) is as follows: Player I tries to verify \( \varphi \), Player II to falsify \( \varphi \); they list sentences \( \varphi_m \).

I plays \( \varphi = \varphi_0 \ \varphi_1 \)

II plays \( \varphi_2 \)
Fix $\varphi \in Sent \cap L_\vec{T}$. $G_\varphi$ is as follows: Player I tries to verify $\varphi$, Player II to falsify $\varphi$; they list sentences $\varphi_m$.

I plays $\varphi = \varphi_0 \varphi_1 \varphi_3$

II plays $\varphi_2$
Fix $\varphi \in \text{Sent} \cap \mathcal{L}_{\hat{T}}$. $G_{\varphi}$ is as follows: Player $I$ tries to verify $\varphi$, Player $II$ to falsify $\varphi$; they list sentences $\varphi_m$.

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If $\varphi_m$ is the last sentence played, and it was played by Player $i$:

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- $\psi \lor \chi$ then Player $i$ lists as $\varphi_{m+1}$: either $\psi$ or $\chi$;
- $\neg \psi$ then the other player lists as $\varphi_{m+1}$: $\psi$;
- $\exists v \psi(v)$ then Player $i$ must list as $\varphi_{m+1}$: $\varphi[n]$ for some $n$;
- $\hat{T}(\neg \sigma)$ then Player $i$ lists as $\varphi_{m+1}$: $\sigma$.

Play terminates if $\varphi$ is atomic:

if $\varphi_m$ is $\hat{T}(n)$ and $n$ does not code a sentence, then player $i$ loses;
Fix $\varphi \in \text{Sent} \cap L_{\mathcal{T}}$. $G_\varphi$ is as follows: Player I tries to verify $\varphi$, Player II to falsify $\varphi$; they list sentences $\varphi_m$.

- I plays $\varphi = \varphi_0 \varphi_1 \varphi_3 \ldots$
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- $\neg \psi$ then the other player lists as $\varphi_{m+1}$: $\psi$;
- $\exists v \psi(v)$ then Player $i$ must list as $\varphi_{m+1}$: $\varphi[n]$ for some $n$;
- $\mathcal{T}(\overline{\sigma})$ then Player $i$ lists as $\varphi_{m+1}$: $\sigma$.

Play terminates if $\varphi$ is atomic:  
if $\varphi_m$ is $\mathcal{T}(n)$ and $n$ does not code a sentence, then player $i$ loses; 
otherwise player $i$ wins iff $(\varphi_m)_\mathcal{M}$.
• If the game continues for infinitely many steps it is declared a draw.

• This does not conform to the strict format of an open game, because it allows for infinite plays to be so declared, but this is inessential.

• Note the compositionality of the Strong Kleene rules is reflected in the game’s rules.
Take $\mathcal{L}, \ldots \mathcal{M}$ with a partial evaluation $A = (A^+, A^-)$ as before, we define:

$$\Gamma(A) = \text{df} \ (\Gamma(A)^+, \Gamma(A)^-):$$

$$\Gamma \varphi^- \in \Gamma(A)^+ \quad (\Gamma(A)^- \text{ resp. }) \text{ iff }$$

$$\forall U \supseteq A^+[U \cap A^- = \emptyset, \implies (\mathcal{M}, U) \models \varphi \ (\neg \varphi \text{ resp.})].$$

- This is again monotonic:
  $$A^+/^- \subseteq B^+/^- \implies \Gamma(A)^+/^- \subseteq \Gamma(B)^+/^-, \text{ and starting from } (\emptyset, \emptyset) \text{ we can build up a least fixed point:}$$
  $$\Gamma^{+/-}_\infty = (\bigcup \alpha \Gamma^+_{\alpha}, \bigcup \alpha \Gamma^-_{\alpha}).$$
We define a game $G_\varphi$, for $\varphi \in \mathcal{L}_T$ for which Player 1 has a winning strategy iff $\varphi \in \Gamma_\infty^\bot$. 

Rules for II: Idea: Agathe is trying to play a sequence $\Sigma$ of sentences $(m_0, m_1, \ldots)$ of the language $\mathcal{L}_T$ such, if $\Psi$ is the extension of $\dot{T}$ so listed, (i.e. $\Psi = \{\left\langle \sigma \right\rangle \mid \exists i \, m_i = \left\langle \sigma \right\rangle \in \dot{T}\}$) then $\Sigma$ is the complete theory of $\langle N, +, \times, 0, ', \ldots, \Psi \rangle$. Additionally she is trying to ensure $\varphi \not\in \Sigma$. 

We may think of the game as a series of queries Is $\tau \in \Sigma$? by Ulrich (including the sentences as to whether $\left\langle \sigma \right\rangle \in \dot{T}$); Agathe is giving "yes" or "no" answers to these queries, sometimes together with some additional information. During the course of the game Ulrich may issue a challenge to Agathe's earlier replies. We (semi-)formally define this via a description of the possible moves.
We define a game $G_\varphi$, for $\varphi \in \mathcal{L}_\mathcal{T}$ for which Player I has a winning strategy iff $\varphi \in \Gamma^+_{\infty}$.

*Ulrich* I $i_0 \ i_1 \ \cdots$

*Agathe* II $m_0 \ m_1 \ \cdots$
We define a game $G_\varphi$, for $\varphi \in \mathcal{L}_{\dot{T}}$ for which Player I has a winning strategy iff $\varphi \in \Gamma_\infty^+$.  

Ulrich I $i_0 \ i_1 \ \cdots$  

Agathe II $m_0 \ m_1 \ \cdots$  

Rules for II: Idea: Agathe is trying to play a sequence $\Sigma$ of sentences $(m_0, m_1, \ldots)$ of the language $\mathcal{L}^+$ such, if $\Psi$ is the extension of $\dot{T}$ so listed, (i.e. $\Psi = \{ \langle \sigma \rangle \mid \exists i \ m_i = \langle \sigma \rangle \in \dot{T} \}$ ) then $\Sigma$ is the complete theory of $\langle \mathbb{N}, +, \times, 0, ', \ldots \Psi \rangle$. Additionally she is trying to ensure $\varphi \notin \Sigma$. 
A game for supervaluation

We define a game $G_\varphi$, for $\varphi \in \mathcal{L}_\downarrow$ for which Player I has a winning strategy iff $\varphi \in \Gamma^+_{\infty}$.

*Ulrich* I $i_0$ $i_1$ ···

*Agathe* II $m_0$ $m_1$ ···

**Rules for II:** Idea: Agathe is trying to play a sequence $\Sigma$ of sentences $(m_0, m_1, \ldots)$ of the language $\mathcal{L}^+$ such, if $\Psi$ is the extension of $\dot{T}$ so listed, (i.e. $\Psi = \{ \lceil \sigma \rceil | \exists i \ m_i = "\lceil \sigma \rceil \in \dot{T}" \} \}$) then $\Sigma$ is the complete theory of $\langle \mathbb{N}, +, \times, 0, ', \ldots \Psi \rangle$. Additionally she is trying to ensure $\varphi \not\in \Sigma$.

We may think of the game as a series of queries *Is $\tau \in \Sigma$?* by Ulrich (including the sentences as to whether “$\lceil \sigma \rceil \in \dot{T}$”); Agathe is giving “yes” or “no” answers to these queries, sometimes together with some additional information. During the course of the game Ulrich may issue a *challenge* to Agathe’s earlier replies. We (semi-)formally define this via a description of the possible moves.
Round k: Player I states “$i_k$”. This can either be

(i) a sentence $\sigma$, of $\mathcal{L}^+$, in which case Player II’s reply is either “$\sigma$” or “$\neg\sigma$” (meaning yes, “$\sigma \in \Sigma$” or, no, “$\neg\sigma \in \Sigma$”); additionally, if $\sigma$ is of the form “$\exists v_0 \psi(v_0)$” and she wishes to answer “yes”, then she must also choose some $n \in \mathbb{N}$ and at the same time play “$\psi[\bar{n}]$” as well as $\sigma$ (so in this case $m_k$ codes this pair of sentences).

(ii) Player I may alternatively state a challenge to Player II. The challenges may come in one of two forms: (A) either he plays $i_k \neq 0$ that is a code of a proof of “$0 = 1$” from a (finite) list $m_{i_1}, \ldots, m_{i_l}$ of sentences that II has already played in earlier rounds - thus demonstrating that II’s list is inconsistent. His other form, (B) is that he may challenge an earlier assertion by II of the form “$\neg \sigma \not\in \hat{T}$”, or “$\neg \sigma \in \hat{T}$”.

Additionally to these Rules, are requirements that Player II attempt to falsify $\varphi$: if queried by Player I on $\varphi$ she must reply $\neg \varphi$ (or lose).
The last rule together with the description at Round $k$ comprise the basic rules of the game $G_\varphi$. If anybody disobeys these rules, then the first to do so forfeits the game. If Player $I$ issues a “consistency” challenge, then if he is correct, he wins outright and the game is over; however if he is incorrect Player $II$ wins outright. If Player $I$ issues a challenge of type (B) to some earlier assertion by $II$ of the form “$\neg \sigma \not\in \dot{T}$”, or “$\sigma \not\in \dot{T}$”, then play in $G_\varphi$ is halted, and play starts afresh in $G_\sigma$ (or $G_{\neg \sigma}$ if “$\sigma \not\in \dot{T}$” was the challenge). We can thus think of $G_\sigma$ as a “subgame” of $G_\varphi$, but in any case it is defined from $\sigma$ (or $\neg \sigma$) just as $G_\varphi$ was defined from $\varphi$.

**Ulrich wins $G_\varphi$** if and only if (i) $II$ makes a mistake on the basic rules of the game (or any $I$-initiated subgame) at some stage, or (ii) he makes a successful challenge to the consistency of the sentences $II$ is playing in $G_\varphi$ or in the current subgame. Thus, for $I$ to win, the overall game must be finite in length.
Note: II thus wins precisely when: (a) I makes a false accusation that her list is inconsistent; or (b) she manages to make infinitely many moves in $G_\varphi$ or in any initiated subgame; or else (c) through the whole course of play I initiates infinitely many subgames.
Note: II thus wins precisely when: (a) I makes a false accusation that her list is inconsistent; or (b) she manages to make infinitely many moves in $G_\varphi$ or in any initiated subgame; or else (c) through the whole course of play I initiates infinitely many subgames.

**Lemma**

$I$ has a winning strategy in $G_\varphi \iff \varphi \in \Gamma^+_\infty$ where $\Gamma_\infty = \langle \Gamma^+_\infty, \Gamma^-\infty \rangle$ is the least Kripkean supervaluation fixed point.
Note: II thus wins precisely when: (a) I makes a false accusation that her list is inconsistent; or (b) she manages to make infinitely many moves in $G_\varphi$ or in any initiated subgame; or else (c) through the whole course of play I initiates infinitely many subgames.

**Lemma**

I has a winning strategy in $G_\varphi \iff \varphi \in \Gamma^+_{\infty}$ where $\Gamma_{\infty} = \langle \Gamma^+_{\infty}, \Gamma^-_{\infty} \rangle$ is the least Kripkean supervaluation fixed point.

**Proof:** ($\Leftarrow$) Suppose $\varphi \in \Gamma^+_{\alpha_0+1} \setminus \Gamma^+_{\alpha_0}$ (we’ll say that “$\text{rk}(\varphi) = \alpha_0$”). II then plays out some $\Sigma$ and $\Psi$ as extension for $\dot{T}$. If she follows the basic rule of declaring “$\neg \varphi \in \Sigma$” when asked, then $\Psi$ cannot be compatible with $(\Gamma^+_{\alpha_0}, \Gamma^-_{\alpha_0})$ as $\varphi \in \Gamma^+_{\alpha_0+1}$. Hence either:

(i) For some $\sigma_1 \in \Gamma^+_{\alpha_0}$, $\sigma_1 \notin \Psi$, or:

(ii) For some $\sigma_1 \in \Gamma^-_{\alpha_0}$, $\sigma_1 \in \Psi$

As part of his strategy, I makes sure that he asks every query of the form “$\tau \in \dot{T}$” during his course of play. (Indeed I can ensure the consistency and also the completeness of the pertinent $\Sigma$ of any infinite run of any subgame, by querying every possible sentence.)
At some point then, suppose Case (i) (or Case (ii)) holds, and, when queried on \( T(\sigma_1) \), II states “\( \sigma_1 \notin T \)” (or “\( \sigma_1 \in T \)” respectively). Then I on his next move calls a challenge to this statement, and the subgame \( G_{\sigma_1} \) (or \( G_{\neg \sigma_1} \) respectively) is initiated. The point is that \( \sigma_1 \in \Gamma^+_{\alpha_0} \) (or \( \sigma_1 \in \Gamma^-_{\alpha_0} \)). In either case (with the obvious extension of notation) \( \alpha_1 = \text{rk}(\sigma_1) < \alpha_0 \), and the subgame now being initiated is being played on a formula of lower rank. The \( \Psi(\sigma_1) \) (or \( \Psi(\neg \sigma_1) \)) that II now will now try to play out is incompatible with \( (\Gamma^+_{\alpha_1}, \Gamma^-_{\alpha_1}) \), (if she adheres to the basic rule of declaring that “\( \sigma_1 \notin \Psi(\sigma_1) \)” (or “\( \sigma_1 \in \Psi(\sigma_1) \)”), and thus at some stage in the new game I may issue a challenge on the assertion that some \( \sigma_2 \notin \Psi \), and we have for the resultant \( \alpha_2 = \text{rk}(\sigma_2) < \alpha_1 \). Proceeding in this way, as long as II does not lose through inconsistency, I eventually challenges with some \( \sigma_k \) with \( \text{rk}(\sigma_k) = 0 \). However if \( \sigma_k \in \Gamma^+_1 \) then every \( \Psi \) is compatible with \( (\Gamma^+_0, \Gamma^-_0) = (\emptyset, \emptyset) \) and II in her play of \( G_{\sigma_k} \), simply cannot produce an extension \( \Psi \) and a consistent set of sentences \( \Sigma \) so that \( \langle \mathbb{N}, +, \times, 0', \ldots, \psi \rangle \models \Sigma \), whilst \( \langle \mathbb{N}, +, \times, 0', \ldots, \psi \rangle \models \neg \sigma_k \).
• She thus will lose \( G_{\sigma_k} \) by breaking a basic rule.
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**Question:** Why did we require that she produce numeral evidence of the form $\psi[n]$ when she gave an affirmative response to the query: $\exists v_0 \psi(v_0)$?
She thus will lose $G_{\sigma_k}$ by breaking a basic rule.

**Question:** Why did we require that she produce numeral evidence of the form $\psi[n]$ when she gave an affirmative response to the query: “$\exists v_0 \psi(v_0)$”?

**Answer:** Then the sets $\Sigma$ being produced, if consistent, are true in the standard model $\langle \mathbb{N}, +, \times, 0', \ldots, \Psi \rangle$. 
(⇒) Suppose \( \varphi \notin \Gamma^+_\infty \). We describe how Agathe can win. By hypothesis there is \( \Psi = \Psi(\varphi) \) with \( \Psi \supseteq \Gamma^+_\infty \), \( \Psi \cap \Gamma^-\infty = \emptyset \), with \( \langle \mathbb{N}, +, \times, 0, ', \ldots ; \Psi \rangle \models \neg \varphi \). Whenever she is queried Agathe consults the above model and gives the appropriate reply. She thus will not lose \( G_\varphi \) on consistency grounds, nor by breaking any other basic rule. If she is challenged on her assertion “\( \sigma_1 \notin T \)” then \( \sigma_1 \notin \Gamma^+_\infty \). The subgame initiated is \( G_{\sigma_1} \), but as \( \sigma_1 \notin \Gamma^+_\infty \) she is no worse off than before, and can play just as well here continuing with \( \Psi \) since \( \langle \mathbb{N}, +, \times, 0, ', \ldots ; \Psi \rangle \models \neg \sigma_1 \). If she is challenged on “\( \sigma_1 \in \dot{T} \)” then the subgame \( G_{\neg \sigma_1} \) is initiated; but \( \neg \sigma_1 \notin \Gamma^+_\infty \), (as \( \sigma_1 \notin \Gamma^-\infty \)), and so she can still continue. Clearly she can keep this up no matter how many challenges that Ulrich issues, and she will ultimately win.

QED
Same $\mathcal{L}_T$ etc as before.

**Definition**\(^2\) $\varphi$ *depends on* $\Psi$ if and only if:

$$\forall B, H \subseteq \mathcal{L}_T, \varphi(M,H) \iff \varphi(M,B) \quad \text{iff} \quad \varphi(M,\Psi \cap H) \iff \varphi(M,\Psi \cap B)$$

• Equivalently: $\forall H \subseteq \mathcal{L}_T, \varphi(M,\Psi) \iff \varphi(M,\Psi \cap H)$.  

\(^2\)H. Leitgeb “What Truth Depends on”, JPL, 34, 2005
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**Definition** $D(\Psi) = \{ \text{⌜\varphi\⌝} | \varphi \text{ depends on } \Psi \}$.

- $D$ is monotonic, and $\Pi^1_1$; we set $\Phi_0 = \emptyset$; $\Phi_\infty = \bigcup_\alpha \Phi_\alpha$, where $\Phi_{\alpha+1} = D(\Phi_\alpha)$ etc.

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\(^2\)H. Leitgeb “What Truth Depends on”, JPL, 34, 2005
Dependency

Same $\mathcal{L}_T$ etc as before.

**Definition** $^2\varphi$ depends on $\Psi$ if and only if:

$$\forall B, H \subseteq \mathcal{L}_T, \varphi(M,H) \iff \varphi(M,B) \iff \varphi(M,\Psi \cap H) \iff \varphi(M,\Psi \cap B)$$

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**Definition** $D(\Psi) = \{ \upharpoonright \varphi \downharpoonleft | \varphi \text{ depends on } \Psi \}$.

- $D$ is monotonic, and $\Pi^1_1$; we set $\Phi_0 = \emptyset$; $\Phi_\infty = \bigcup_\alpha \Phi_\alpha$, where $\Phi_{\alpha+1} = D(\Phi_\alpha)$ etc.

- Leitgeb calls the sentences in $\Phi_\infty$, the least fixed point, *grounded*. Of the *ungrounded*, we call those $\lambda$ who depend on themselves, (i.e. $\{\lambda\}$), *self-referential*.

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$^2$H.Leitgeb “What Truth Depends on”, JPL, 34, 2005
A general result on positive monotone inductive fixed points

Theorem

(Moschovakis, Aczel) Suppose $A$ is positive monotone inductive over a countable structure $(\mathbb{N}, +, \times, \ldots, \vec{R}, \vec{F})$. Then there is an open set $B$ in $\mathbb{N} \times \mathbb{N}^\mathbb{N}$ so that $n \in A$ iff $I$ has a winning strategy in the open game $G_{B_n}$ where $B_n = \{\vec{k} \mid (n, \vec{k}) \in B\}$.
We now describe the variant of the game which characterises the complete $\Pi^1_1$ set of (dependency) grounded sentences $\Phi_\infty = \bigcup \Phi_\alpha$.

$G^*_\varphi$ is played in a similar fashion, but now $II$ must produce two extensions $B_0, B_1$ and sets of sentences $\Sigma_0, \Sigma_1$ (say she attends to queries about $B_0, \Sigma_0$ in even numbered rounds, and to $B_1, \Sigma_1$ on odd numbered ones). She is trying to ensure that $(\varphi)_{(\mathbb{N}, B_0)} \iff (\neg \varphi)_{(\mathbb{N}, B_1)}$ and so when queried about $\varphi$ must answer accordingly. The basic rules are the same *mutatis mutandis* and $I$ may still challenge on grounds of consistency, with the same outcomes. If however $II$ has earlier asserted “$\sigma \in B_0$” and “$\sigma \notin B_1$” (or *vice versa*) then $I$ may issue a challenge, and the subgame $G^*_\sigma$ is initiated (again the subgame is the same as the game: she tries to produce two extensions $B_0(\sigma), B_1(\sigma)$ etc., etc.)

Again, if neither player messes up their basic rules, then if the overall game lasts for infinitely many stages $II$ wins.
Lemma

\[ I \text{ has a winning strategy in } G^*_\varphi \iff \varphi \in \Phi_\infty. \]

**Proof:** (\(\iff\)) Suppose now \(\varphi \in \Phi_{\alpha+1} \setminus \Phi_\alpha\); we set \(\text{rk}^D(\varphi) = \alpha\). By the definition of dependency, if \((\varphi)(N,B_0) \iff (\neg \varphi)(N,B_1)\), then \(B_0 \cap \Phi_\alpha \neq B_1 \cap \Phi_\alpha\). Hence if II is trying to produce such \(B_0, B_1\) for some \(\sigma_1 \in \Phi_\alpha\) she must answer “\(\sigma_1 \notin B_0\)” and “\(\sigma_1 \in B_1\)” (or vice versa) when queried by Ulrich, and the latter may now issue a challenge and the game proceeds to \(G^*_{\sigma_1}\). Now of course \(\alpha_1 =_{\text{df}} \text{rk}^*(\sigma_1) < \alpha_0\), and if no one messes up their basic rules, we arrive as before at the situation of playing in \(G^*_{\sigma_k}\) where \(\text{rk}^D(\sigma_k) = 0\). But here, as \(\Phi_0 = \emptyset\), for every \(B, B'\), 
\[(\sigma_k)(N,B) \iff (\sigma_k)(N,B').\]
(⇒) Suppose $\varphi \not\in \Phi_\infty$. Then there exists $B_0$ with 
$(\varphi)_{(\mathbb{N}, B_0)} \iff (\neg \varphi)_{(\mathbb{N}, B_0 \cap \Phi_\infty)}$ 
She may then play out the complete theories of the two models $\langle \mathbb{N}, \ldots, B_0 \rangle, \langle \mathbb{N}, \ldots, B_0 \cap \Phi_\infty \rangle$. If she is challenged at some point on her assertions amounting to 
$\sigma \in B_0 \setminus B_0 \cap \Phi_\infty)$, then even when the subgame $G_{\sigma}^*$ is initiated we have $\sigma \not\in \Phi_\infty$ and she is no worse off than before, and can continue in the same fashion using some $B_2$ with 
$(\sigma)_{(\mathbb{N}, B_2)} \iff (\neg \sigma)_{(\mathbb{N}, B_2 \cap \Phi_\infty)}$. 
QED
**Question** Can any of this be done for other theories of truth?
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Theorem

(i) For Field’s theory of truth\(^a\) over \(\mathcal{M} = \mathbb{N}\), the set of ‘ultimate truths’, \(U\), cannot be the set of sentences for which a player has a winning strategy in any open game. Moreover no game with even \(\Sigma^0_2\)-payoff set of infinite sequences can be used here.

(ii) The same is true for the set of stable truths under a Herzbergerian limit rule of a single revision sequence starting with some (recursive) distribution of truth values.
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**Theorem**

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(ii) The same is true for the set of stable truths under a Herzbergerian limit rule of a single revision sequence starting with some (recursive) distribution of truth values.

**Theorem** There is a \(\Sigma^0_3\) set, \(B \subseteq \mathbb{N} \times \mathbb{N}^\mathbb{N}\), so that \(\sigma \in U\) iff \(I\) has a w.s. in \(G_{B_\sigma}\) where \(B_\sigma = \{ \vec{k} \in \mathbb{N}^\mathbb{N} | (\sigma, \vec{k}) \in B \}\).

\(^a\)H. Field *A Revenge Immune solution of the semantic paradoxes*, JPL, 2003
• The set $S$ of revision-theoretic stable truths$^3$ (under any limit rule) forms a complete $\Pi^1_2$ set of integers.

• Again on general grounds, this set cannot be represented as the set of games for which $I$ has a winning strategy. But it’s complement can:

**Theorem**

*There is a set $\Pi^1_1$ set, $B^S \subseteq \mathbb{N} \times \mathbb{N}^\mathbb{N}$, so that $\sigma \notin S$ iff $I$ has a w.s. in $G_{B^S}$ where $B^S_\sigma = \{ \vec{k} \in \mathbb{N}^\mathbb{N} | (\vec{\sigma}, \vec{k}) \in B^S \}$.*

• However to prove this theorem requires going beyond Zermelo-Fraenkel set theory, and assuming the existence of large cardinals. We need *Determinacy*($\Pi^1_1$).

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$^3$A.Gupta& N.Belnap *The Revision Theory of Truth*, MIT Press 1993