Greatly Erdős Cardinals with some generalizations to the Chang and Ramsey properties

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Abstract

• We define a notion of *order* of indiscernibility type of a structure by analogy with Mitchell order on measures; we use this to define a hierarchy of strong axioms of infinity defined through normal filters, the *α-weakly Erdős hierarchy.* The filters in this hierarchy can be seen to be generated by sets of ordinals where these indiscernibility orders on structures dominate the *canonical functions*.
• The limit axiom of this is that of *greatly Erdős* and we use it to calibrate some strengthenings of the Chang property, one of which, CC^+, is equiconsistent with a Ramsey cardinal, and implies that $\omega_3 = (\omega_2^+)^K$ where K is the core model built with non-overlapping extenders - if it is rigid, and others which are a little weaker. As one corollary we have:

Theorem If $\mathrm{CC}^+ \wedge \neg\Box_{\omega_2}$ then there is an inner model with a strong cardinal.

• We define an α-Jónsson hierarchy to parallel the α-Ramsey hierarchy, and show that κ being α-Jónsson implies that it is α-Ramsey in the core model.

Key words: Ramsey cardinal, Chang Property, Erdős cardinal, core model
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1. Introduction

This paper is essentially an essay in set-theoretic indiscernibility: after one has left Gödel's constructible universe L, but before one reaches measurable cardinals in the hierarchy of large cardinals, there is the intermediate zone where the notions of Erdős and Ramsey cardinals come into focus.

We seek here to analyse this structure in detail. The notion of Erdős and Ramsey cardinals are far from new and have been extensively investigated (see, *e.g.* [4].) In particular for inner model theorists there is the fundamental absoluteness results of Mitchell and Jensen that relate such cardinals to the inner models that can support them: the *core models* ([8], [7], [19], [24]). L is too thin to allow any but the smallest amount of indiscernibility properties for first order structures. However it was an early result concerning the first core model (the so-called *Dodd-Jensen* core model K^{DJ}) that properties such as the Erdős and Ramsey properties of a cardinal κ would relativise to this model.

Jensen's proof of this fact made use of a principal lemma that became known as the "Indiscernibles Lemma". The thrust of the Lemma was that a first order structure inside the core model K^{DJ} would have

URL: `http://www.maths.bris.ac.uk/~mapdw` (P.D. Welch)

a set of *good indiscernibles* in K^{DJ} if it had one such in V. This lemma was first used as a component of the Rigidity Lemma for K^{DJ} (assuming no inner model of a measurable cardinal; see [7] Lemma 16.18). However it could also be equally well used to show that if κ were a Ramsey cardinal in V, for example, then it would be such in K (see [9] or [18]). Thus K^{DJ} could accommodate at least some of the large cardinals below that of a measurable cardinal. Subsequently in larger core models one saw also that, *e.g.* the Ramsey or smaller α-Erdős properties would relativise from V to them.

It is one of the main motives of this paper to free the Jensen Indiscernibles lemma from just these two purposes, and use it to analyse the detailed structure of indiscernible sets in V, and not just for structures in K^{DJ} or other core models. The methods employed are neither restricted to levels of a core model nor to structures within it. This process of freeing up the lemma was first done by applying the Lemma not just to K but (unsurprisingly) to levels of, or structures in, any universal weasel (see [26]) which are still "core models" of a sort, or models of the form $L[E]$. Then the same methods were used to look at *countably closed* structures in general in [28], these are essentially any transitive structure M with a countable closed filter U, so that $\langle M, \in, U\rangle$ was amenable, with U measuring the subsets of some $\kappa \in M$. From such a structure a good indiscernible set $I \subseteq \kappa$ could be obtained with $\mathrm{ot}(I) \geq \omega_1$ ([28] Thm 2.1). Dually, given $\langle M, \in\rangle$ and a good set of indiscernibles $I \subseteq \kappa \in M$ then given some $L_\kappa[A] \in M$, with $H^M_\kappa \subseteq L_\kappa[A]$, an amenable filter U could be found, with $\langle M', \in, U\rangle$ countably closed, and $L_\kappa[A] \in M'$ (Thm 1.1 *op.cit.)*

We bring out this duality fully in this paper and emphasise in Section 2 the back and forth interaction between structures, their countably closed measures, and indiscernible sets. In Section 2.2 we define an operation $I(\mathcal{A}, M)$ obtaining indiscernibles I for a structure $\mathcal{A}$ from a measure M; dually then in 2.3 there is an operation $M(\mathcal{A}, I)$ to go from a good indiscernible set to a measure M. There is not a lot of novelty to these constructions, but done in sufficient generality and in the right way these operations are inverse to one another as shown at Lemmata 2.19 and 2.20. Subsection 2.4 then establishes a general form of the Jensen Indiscernibles Lemma.

We are able to put a wellfounded relation on indiscernibles sets I for a given structure $\mathcal{A} \in V$ and thus define a *partial order* on such sets for a structure: $o_{\mathcal{A}}(I)$ (see Def.4.27). In $L[E]$ models this becomes a wellorder, and, naturally enough, one obtains the right correspondence between the indiscernible sets and ω-closed filters E_ν on the extender sequence E of the model, thus reinforcing the impression that $o_{\mathcal{A}}(I)$ is something of a Mitchell order on indiscernible sets.

Given the potential nature of the accumulating indiscernible sets I for structures on some cardinal κ, it is natural to develop notions to express the degree of indiscernibility available. (Of course having a normal measure U on the whole of $\mathcal{P}(\kappa)$ in V says that we have a maximal amount of indiscernibility available; we wish to look at situations where we do not necessarily have such a normal measure.) This "degree of indiscernibility" can be cashed out by analogy with the α-Mahlo hierarchy (Section 4):

Definition 4.1 *Let κ have uncountable cofinality, and let $\mathcal{A}$ be a κ-structure, $X \subseteq \kappa$. Let*

$$t_{\mathcal{A}}(X) = \{\alpha \in \kappa \cap \lim : \textit{ there exists a set } I \subseteq \alpha \cap X \textit{ of g.i's for } \mathcal{A} \textit{ cofinal in } \alpha\}.$$

Using this we then define a hierarchy of normal filters $\mathcal{F}_\alpha$ potentially for all $\alpha < \kappa^+$; these are generated by suprema of sets of nested indiscernibles for structures $\mathcal{A}$ on κ using the above basic $t_{\mathcal{A}}(X)$ operation. A cardinal κ will be *weakly α-Erdős* when $\mathcal{F}_\alpha$ is non-trivial. We define a cardinal as *greatly Erdős* as follows:

Definition 4.5 κ *is* greatly Erdős *iff there is a non-trivial normal filter $\mathcal{F}$ on κ such that $\mathcal{F}$ is closed under $t_{\mathcal{A}}(-)$ for every κ-structure $\mathcal{A}$.*

We may characterize this in two ways, firstly:

Corollary 4.10 *Let κ be a cardinal of uncountable cofinality. Then the following are equivalent:*
(i) κ is greatly Erdős;
(ii) $\mathcal{G} = \bigcup_{\alpha<\kappa^+} \mathcal{F}_\alpha \not\ni \varnothing$;
(iii) κ is α-weakly Erdős for all $\alpha < \kappa^+$.

Secondly we are able (Section 4.2) to give a restatement of the greatly Erdős property in terms of the bounding of any *canonical function* for $\alpha < \kappa^+$ by measure one sets in this filter hierarchy (Cor.4.25).

One influence on this work has been the paper of Donder and Lewinsky [11] where they analysed in detail properties weaker than that of ω_1-Erdős. They defined a number of games with players choosing ordinals and sets of indiscernibles, and defined a spectrum of large cardinal properties weaker than ω_1-Erdős by assuming Player I never had a winning strategy in such games. In one sense our paper is looking at what can be done in analogy with this idea, below the Ramsey property. Remarkably (to us at any rate) some of the same games emerge but with the assumption that Player *II* has a winning strategy. We prove an equivalence of the form:

Theorem 5.6 *If $\boldsymbol{V} = \boldsymbol{K}$ then II has a winning strategy in $\widetilde{G}_\lambda(\mathcal{A})$ for all λ-structures $\mathcal{A}$ iff λ is Ramsey.*

We relate these to strengthenings of the Chang property where we require of the property that many Chang substructures exist. One strengthening CC^+ is equiconsistent with a Ramsey cardinal in K (Sect. 6.2); another is *prima facie* weaker.

Definition 6.2 CC^+ *holds iff for any structure, $\mathcal{A} = \langle H_{\omega_2}, (A_n)_{n<\omega}\rangle$, there is a structure, $\mathcal{B}$, with the same domain, but possibly extra predicates, extending $\mathcal{A}$, such that for any countable $X \prec \mathcal{B}$ and $\gamma < \omega_2$ there is $Y \prec \mathcal{B}$ with $Y \supseteq X$, $X \cap \omega_1 = Y \cap \omega_1$, and* $\sup(Y \cap \omega_2) > \gamma$.

CC^+ is also interesting as it implies that $\boldsymbol{K}$ correctly computes the successor cardinal of ω_2 (assuming that $\boldsymbol{K}$ is the core model built from non-overlapping extenders and $\neg 0^{\P}$ ("not zero pistol ")). This is shown in Sect 6.1 as follows.

Theorem 6.7 ($\neg \mathbf{0}^{\P}$) *Suppose that* CC^+ *holds. Then $\omega_3 = (\lambda^+)^{\boldsymbol{K}}$, where $\lambda = \omega_2$.*

Our original motivations had been twofold: firstly to define (Def.3.29) a parallel notion of α-*Jónssonness* which would hopefully mirror the α-Ramsey hierarchy of Feng [12]. If this were to work well, then an α-Jónsson cardinal κ should be in K an α-Ramsey cardinal. This is achieved in Section 3 at Theorem 3.34. We also extend a result of Feng's on Π^1_n-indescribability, by defining a transfinite notion of Π^1_α-indescribability *via* finite length (and so determined) games (Cor. 3.24). We there also calculate the length of Feng's α-Ramsey hierarchy: Lemma 3.43 shows it to be the height of the least transitive admissible set with with $V_{\kappa+1}$ as an element. In Sect 3.3 we define another hierarchy notion on Ramseyness that of α-*very Ramsey* in terms of Player *II* having a winning strategy in an α-long game. This is stronger than Feng's hierarchy. We show (Lemma 3.50) that ω_1-very Ramseyness implies measurability in $\boldsymbol{K}$.

Our second motivation had been to look at finite versions of the *mutual stationarity property* of Foreman and Magidor [13].

Definition 6.18 *The* Intermediate Chang's Conjecture (ICC) *holds iff for any ω_2-structure $\mathcal{A}$, and stationary sets $S_0 \subseteq \omega_1$ and $S_1 \subseteq \omega_2 \cap \mathrm{Cof}_{\omega_1}$, $\mathcal{A}$ has a substructure $\mathcal{B} \prec \mathcal{A}$ such that* $\sup(\mathcal{B} \cap \omega_1) \in S_0$ *and* $\sup(\mathcal{B} \cap \omega_2) \in S_1$.

This of course is nothing other than a souped-up version of the Chang property. We wished to ascertain its strength in relation to the Chang and Ramsey properties. It indeed turns out to be strictly intermediate (Sect. 6.3). The notions of α-weakly Erdős and greatly-Erdős arose whilst trying to understand these relationships.

To summarise we shall have the following relationships. Only the first, the last and the penultimate implications are known to be irreversible, but at least one of the middle three must also be:

$\mathrm{Con}(\mathrm{ZFC}+\mathrm{SCC}) \leftrightarrow \mathrm{Con}(\mathrm{ZFC}+\exists\kappa(\kappa$ measurable$) \rightarrow$
$\mathrm{Con}(\mathrm{ZFC}+\mathrm{CC}^{+}) \leftrightarrow \mathrm{Con}(\mathrm{ZFC}+(\exists\kappa)(\kappa$ Ramsey $)) \rightarrow$
$\mathrm{Con}(\mathrm{ZFC}+(\exists\kappa)(\kappa$ virtually Ramsey $) \rightarrow$
$\mathrm{Con}(\mathrm{ZFC}+\mathrm{ICC}) \rightarrow$
$\mathrm{Con}(\mathrm{ZFC}+(\exists\kappa)(\kappa$ greatly Erdős$)) \rightarrow$
$\mathrm{Con}(\mathrm{ZFC}+(\exists\kappa)(\kappa$ almost Ramsey $) \rightarrow$
$\mathrm{Con}(\mathrm{ZFC}+(\exists\kappa)(\kappa\ \omega_1$ Erdős $)$.

1.1. Preliminaries

We fix several proper classes: $\mathrm{On}=\{x : x$ is an ordinal $\}$, $\mathrm{Card}=\{x : x$ is a cardinal $\}$, $\mathrm{Sing}=\{x : x$ is a singular ordinal $\}$, $\mathrm{Cof}_\alpha=\{\beta\in\mathrm{On} : \mathrm{cf}(\beta)=\alpha\}$, and $\mathrm{Lim}=\{\alpha\in\mathrm{On} : \alpha$ is a limit ordinal $\}$. Define $\mathrm{Cof}_{>\alpha}$ and $\mathrm{Cof}_{\geqslant\alpha}$ analogously to Cof_α. We also abbreviate "α is a limit ordinal" by $\lim(\alpha)$. For $X, Y\subseteq\mathrm{On}$ we write $\sup(X)$ and $\max(X)$ for the supremum and maximum of X, where they exist. $X\trianglelefteqslant Y$ if X is an initial segment of Y, *i.e.* if $Y\cap\sup(X)=X$. For κ a regular cardinal we let $\mathcal{C}_\kappa$ denote the club filter on κ, namely those substes of κ that contain closed and unbounded subsets of κ. Two sets $X, Y\subseteq\kappa$ are said to be "equal modulo $\mathcal{C}_\kappa$ if $X\cap Y\in\mathcal{C}_\kappa$; we write in this case $X =_{\mathcal{C}_\kappa} Y$ (or simply $X=Y$ if the subscript is understood).

The diagonal intersection of a sequence $\langle X_\alpha : \alpha<\kappa\rangle$ of subsets of κ is defined to be $\Delta_{\alpha<\kappa}X_\alpha =_{\mathrm{df}} \{\gamma<\kappa : (\forall\alpha<\gamma)\gamma\in X_\alpha\}$. If $\langle X_\alpha : \alpha<\kappa\rangle$ and $\langle X'_\alpha : \alpha<\kappa\rangle$ are sequences of subsets of κ such that $\{X_\alpha : \alpha<\kappa\}=\{X'_\alpha : \alpha<\kappa\}$ then $\Delta_{\alpha<\kappa}X_\alpha =_{\mathcal{C}_\kappa} \Delta_{\alpha<\kappa}X'_\alpha$.

For $Y\in[P(\kappa)]^{\leqslant\kappa}$, we give a definition modulo the club filter of ΔY. Set

$$\Delta Y=\Delta_{\alpha<\eta}X_\alpha=\{\alpha<\kappa : (\forall\beta<\alpha)(\text{if } f(\beta) \text{ is defined then } \alpha\in f(\beta))\},$$

where $\eta=|Y|$ and $f:\eta\to Y$ enumerates Y. Of course, this definition *only* makes sense modulo the club filter. Thus, if we write $\Delta Y=X$ it will implicitly be meant that the equality holds modulo the club filter. Note that if $\eta<\kappa$ then $\Delta Y=\bigcap Y$ (modulo $\mathcal{C}_\kappa$). We also define $\Delta_{x\in[A]^k}X_x =_{\mathrm{df}} \{\gamma<\lambda : (\forall x\in[A\cap\gamma]^k)\gamma\in X_x\}$ when $A\subseteq\lambda$.

For any regular uncountable cardinal κ, there is a sequence $\langle f_\alpha : \alpha<\kappa^+\rangle$ uniquely defined modulo the club filter of members of ${}^\kappa\kappa$ such that for all $\alpha<\beta<\kappa^+$, $f_\alpha <_{\mathcal{C}_\kappa} f_\beta$. These are the *canonical functions* for κ. If $\alpha<\kappa^+$ and $g:\kappa\to\alpha$ is a surjection then $f_\alpha =_{\mathcal{C}_\kappa} \overline{g}$ where $\overline{g}:\kappa\to\kappa$ is given by $\overline{g}(\gamma)=\mathrm{otp}((g^{-1})''\gamma)$.

1.1.1. J-hierarchies and core models

We shall assume familiarity with all the notions of the constructible J-hierarchies, as well as their finestructural properties. We refer the reader for all such concepts to [29] or [22].

For an ordinal α, an *α-structure* is structure of the form $\mathcal{A}=\langle J^{\vec{A}}_\alpha,\in,\vec{A},B_0,\ldots,B_m,\ldots\rangle$ where $\vec{A}=\langle A_0,\ldots,A_n\rangle$ (n finite) and $m\leq\omega$, which is *amenable*, *i.e.* for any $x\in J^{\vec{A}}_\alpha$, for any $A_i, B_j, x\cap A_i, x\cap B_j\in J^{\vec{A}}_\alpha$. If $\gamma\leqslant\beta$ then

$$\mathcal{A}\restriction\gamma =_{\mathrm{df}} \langle J^{\vec{A}}_\gamma,\in,A_0\cap J^{\vec{A}}_\gamma,\ldots,A_n\cap J^{\vec{A}}_\gamma,B_0\cap J^{\vec{A}}_\gamma,\ldots,B_m\cap J^{\vec{A}}_\gamma,\ldots\rangle.$$

We state some elementary facts as a Lemma.

Lemma 1.1. *For any sets or classes $\vec{A} = \langle A_0, \ldots, A_n \rangle$, with $n < \omega$,*

i. $\langle J_\alpha^{\vec{A}}, \in, \vec{A} \rangle$ *is amenable;*

ii. if $M = \langle J_\alpha^{\vec{A}}, \in, \vec{A}, B_0, \ldots, B_m, \ldots \rangle$ *is amenable then there is a* Σ_1*-Skolem function,* h_M*, for* M *that is uniformly* $\Sigma_1(M)$ *for all such* M*;*

iii. for M *as in ii., there is* $i < \omega$*,* $p \in M$*, such that* $h_M(i, \langle p, - \rangle) \restriction \mathrm{On} \cap M : \mathrm{On} \cap M \to M$ *is a bijection; if* α *is closed under Gödel pairing, then* p *can be omitted;*

iv. for all M *as in ii., there is a uniformly* $\Sigma_1(M)$ *well-ordering of* M*.*

Note that it follows from Lemma 1.1 that any α-structure, $\mathcal{A}$, has uniformly definable Skolem functions, so the *Skolem Hull* of a set X inside the structure $\mathcal{A}$, denoted $\mathrm{Hull}^{\mathcal{A}}(X)$ may be defined. On many occasions we shall pretend that $\vec{A}$ is simply a single predicate A as they can be coded up as such, and that the B-list is empty, as these latter predicates can usually be carried along in any proof.

We place here some facts concerning the core model built assuming there is no mouse with a measure of Mitchell order > 0, this is known as "$\neg 0^{\mathrm{sword}}$". Again the first part of [29] describes this core model model in detail, and we shall expect the reader to be familiar with it, as well as the core model built using extender hierarchies without overlapping extenders. These are described in [29] Ch. 7. We shall denote the core model by "$\boldsymbol{K}$", but which one may depend on the context: if $\neg 0^{\mathrm{sword}}$ is assumed, this is often called "the core built using measures of order 0" and written $\boldsymbol{K}^{MOZ}$. The original Dodd-Jensen Core Model was built assuming there was no inner model with a measurable cardinal, this is sometimes written $\boldsymbol{K}^{DJ}$. Most of our results are true assuming $\neg 0^{\P}$ ("not zero pistol ") and this assumption means that a core model may be built with a strong cardinal, but that there is no closed and unbounded in On class of indiscernibles for it. It is the largest model possible with non-overlapping extenders. Which model we are referring to will, we hope, always be clear from the context. If "$\boldsymbol{K}$" is not further specified, then it will mean the core model built from non-overlapping extenders.

There is a further small point to be made concerning the relativisations of properties such as Ramseyness to a core model. One should note that when $\boldsymbol{K}$ is the model built using non-overlapping extenders, if this model contains a (in fact the unique) strong cardinal δ then above δ there are no large cardinal properties holding that one cannot have in L. Hence if $\kappa > \delta$ is Ramsey it will not be so $\boldsymbol{K}$. This contrasts with $\boldsymbol{K}^{MOZ}$ where the Ramseyness of a κ relativises to it, irrespective of the existence of 0^{sword}. We thus find ourselves in the sequel only proving relativisation results assuming that the extenders of $\boldsymbol{K}$ do not overlap the κ under consideration. One way of doing this is to add the hypothesis of $\neg 0^{\P}$ (whereas we should not have needed to add "$\neg 0^{sword}$" had we been working all the time with measures of order zero. This is only a matter of convenience of proof-layout, rather than anything deep.

2. Good indiscernibles and the Indiscernibles lemma

The main object of study in this paper are sets of *good indiscernibles* for first order structures $\mathcal{A}$. These will be defined in the first subsection. In the next subsection 2.2 we define an operation $I(\mathcal{A}, M)$ obtaining Indiscernibles I for a structure $\mathcal{A}$ from a measure M; dually then in 2.3 there is an operation $M(\mathcal{A}, I)$ to go from a good indiscernible set to a measure M. That these operations are dual to one another is shown at Lemma 2.19 and Lemma 2.20.

Subsection 2.4 establishes a general form of the Jensen Indiscernibles Lemma.

2.1. Good indiscernibles

Definition 2.1. *Let $\mathcal{A}$ be a structure, and let $I \subseteq \mathcal{A} \cap \mathrm{On}$. Then I is a set of* indiscernibles for *$\mathcal{A}$ if whenever $\varphi \in \mathcal{L}_{\mathcal{A}}$ is an n-ary formula and $\alpha_0, \ldots, \alpha_{n-1}$, $\alpha'_0, \ldots, \alpha'_{n-1} \in I$ and $\alpha_0 < \ldots < \alpha_{n-1}$ and $\alpha'_0 < \ldots < \alpha'_{n-1}$, then $\mathcal{A} \vDash \varphi(\alpha_0, \ldots, \alpha_{n-1})$ iff $\mathcal{A} \vDash \varphi(\alpha'_0, \ldots, \alpha'_{n-1})$.*

Definition 2.2. *Let $\mathcal{A} = \langle J_\kappa^{\vec{A}}, \in, \vec{A}, \ldots \rangle$ be a κ-structure. A set $I \subseteq \kappa$ is a set of* good indiscernibles (g.i.'s) *for $\mathcal{A}$ iff for all $\gamma \in I$,*

i. $\mathcal{A} \restriction \gamma \prec \mathcal{A}$

ii. *$I \backslash \gamma$ is a set of indiscernibles for $\langle J_\kappa^{\vec{A}}, \in, \vec{A}, \ldots, \xi \rangle_{\xi < \gamma}$, i.e. whenever $\varphi \in \mathcal{L}_{\mathcal{A}}$ is an $(n+k)$-ary formula and $\beta_0 < \ldots < \beta_{k-1} < \gamma$, $\alpha_0, \ldots, \alpha_{n-1}$, $\alpha'_0, \ldots, \alpha'_{n-1} \in I \backslash \gamma$, and $\alpha_0 < \ldots < \alpha_{n-1}$ and $\alpha'_0 < \ldots < \alpha'_{n-1}$, then*

$$\mathcal{A} \vDash \varphi(\beta_0, \ldots, \beta_{k-1}, \alpha_0, \ldots, \alpha_{n-1}) \text{ iff } \mathcal{A} \vDash \varphi(\beta_0, \ldots, \beta_{k-1}, \alpha'_0, \ldots, \alpha'_{n-1})$$

Good indiscernibles were introduced by Jensen in the paper [9], and are necessary for the statement of the Jensen Indiscernibles Lemma (Theorem 2.3), which we shall often mention. Typically, well-known large cardinal properties that have formulations in terms of indiscernibles have a similar formulation in terms of good indiscernibles.

We say that κ is α-Erdős iff for any κ-structure $\mathcal{A}$, there is a set, I, of g.i.'s for $\mathcal{A}$ such that $\mathrm{otp}(I) = \alpha$. Also recall κ being Ramsey may be formulated so that it is equivalent to being κ-Erdős.

The Jensen Indiscernibles Lemma, or rather some general lemmas from [26], [28] distilled from Jensen's original proof [9] will be used repeatedly. For the record we state one version of this lemma formulated for universal weasels.

Theorem 2.3. (Jensen Indiscernibles Lemma) ($\neg 0^{\P}$) *Let $\boldsymbol{K}' = L[E']$ be a universal weasel and let $\mathcal{A} = \langle J_\kappa^{E'}, \in, E', A \rangle \in \boldsymbol{K}'$ be a κ-structure, and suppose I is a set of good indiscernibles for $\mathcal{A}$ with $\mathrm{cf}(\mathrm{otp}(I)) > \omega$.*

Then there is a set $I' \supseteq I$ of good indiscernibles for $\mathcal{A}$ with $I' \in \boldsymbol{K}'$.

A proof of this version (under the assumption $\neg 0^{\mathrm{sword}}$) can be found in [26]. As for Jensen's original arguments formulated for the Dodd-Jensn core model $\boldsymbol{K}^{DJ}$ built below a single measurable cardinal, it relies on establishing a correspondence between measures and sets of indiscernibles. However this relationship does not depend on the theory of inner models and will be presented in the next two subsections below, without any reference to $\boldsymbol{K}$.

2.2. $I(\mathcal{A}, M)$: Obtaining Indiscernibles from a Measure

In this subsection, we consider a reasonably general class of structures with measures ("pre-useful structures"), and describe a canonical way that indiscernibles for a given structure may be obtained from the measure. This canonical procedure will not always give rise to a set of indiscernibles, but we describe circumstances, here and in the following subsection, in which it does.

Definition 2.4. *A structure $M = \langle J_\alpha^{\vec{A}}, \in, \vec{A}, U \rangle$ is* pre-useful *iff*

i. M is an α-structure, i.e. *M is amenable;*

ii. $\langle J_\alpha^{\vec{A}}, \in, \vec{A} \rangle \vDash \mathrm{ZFC}^-$ *and M has a largest cardinal $\lambda = \lambda(M)$;*

iii. $U = U(M)$ *is an* M*-normal* M*-ultrafilter on* λ.

Lemma 2.5. *There is a sequence of* Σ_1*-formulae* $\langle \varphi_n(v_0, v_1) : 0 < n < \omega \rangle$ *with the following properties. If we suppose that* $M = \langle J_\alpha^{\vec{A}}, \in, \vec{A}, U \rangle$ *is any pre-useful structure,* $\lambda = \lambda(M)$*, and* $U = U(M)$*, then for any* $g : [X]^{<p} \to \lambda$ *(with* $p \leqslant \omega$*) a regressive function with* $g \in M$ *and* $X \in U$*, and for each* $n < p$*, there is a unique set* $Y_n^{g,M} \in M \cap U$*, with* $M \vDash \varphi_n(g, Y_n^{g,M})$*. Moreover* $Y_n^{g,M} \subseteq X$ *is homogeneous for* $g \restriction [X]^n$*.*

Proof. Let $h = h_M$ be the canonical $\Sigma_1(M)$ Σ_1-Skolem function for M.

We proceed by induction on n. In the case $n = 1$, it follows from normality of the measure that there exists some homogeneous set $Y \in U$ for $g \restriction [X]^1 \to \lambda$. It is easy to see that the property "Y is homogeneous for $g \restriction [X]^1$" is uniformly $\Sigma_1\{Y, g\}$ so an application of h suffices to give φ_1.

Suppose $n = k + 1$ and $\langle \varphi_m(v_0, v_1) : 0 < m < n \rangle$ has been constructed with the required properties. Given g, for each $x \in [X]^k$ define $g_x : [X \backslash x]^1 \to \lambda$ by $g_x(\gamma) = g(x \cup \{\gamma\})$. Since $M \vDash \mathrm{ZFC}^-$, the function $S_g = \{\langle \langle x, \gamma \rangle, g_x^{-1}(\gamma) \rangle : x \in [X]^k, \gamma < \min(x)\}$ is in M.

By amenability, the set $S' = \{\langle x, \gamma \rangle : x \in [X]^k, \gamma < \min(x), g_x^{-1}(\gamma) \in U\}$ is also in M. Similarly, the set $T = \{y \in U : (\exists x \in [X]^k)(\exists \gamma \in \lambda) y = g_x^{-1}(\gamma)\} \in M$.

Just as in the case $n = 1$, for each $x \in [X]^k$ there is $\gamma < \min(x)$ such that $g_x^{-1}(\gamma) \in U$. So the set $F_g = \{\langle x, \gamma \rangle : x \in [X]^k, \gamma$ is the least ordinal such that $g_x^{-1}(\gamma) \in U\} \in M$ is a function. Now,

$$
\begin{aligned}
v = F_g \quad \Leftrightarrow \quad & v \text{ is a function with } v : [X]^k \to \lambda \text{ and for all } \langle x, \gamma \rangle \in v(g_x^{-1}(\gamma) \in U \\
& \text{and}(\forall \beta < \gamma) g_x^{-1}(\gamma) \notin U) \\
\\
\Leftrightarrow \quad & (\exists Z, s, t)((s = \lambda) \wedge (t = [X]^k) \\
& \wedge Z = T \\
& \wedge (v \text{ is a function with } v : t \to s) \\
& \wedge (\forall u \in v)((\exists x \in t)(\exists \gamma \in s)(\exists z \in Z)(u = \langle x, \gamma \rangle \wedge (g_x^{-1}(\gamma) = z \\
& \wedge (\forall \beta < \gamma)(\forall z' \in Z)(z \neq g_x^{-1}(\gamma))) \\
\\
\Leftrightarrow \quad & (\exists Z, s, t)((s \text{ is an ordinal and } s \in U) \wedge (t = \{x \in \mathrm{dom}(g) : |x| = k\}) \\
& \wedge Z = \{y \in U : (\exists x \in t)(\exists \gamma \in s) y = g_x^{-1}(\gamma)\} \\
& \wedge (v \text{ is a function with } v : t \to s) \\
& \wedge (\forall u \in v)((\exists x \in t)(\exists \gamma \in s)(\exists z \in Z)(u = \langle x, \gamma \rangle \wedge (g_x^{-1}(\gamma) = z \\
& \wedge (\forall \beta < \gamma)(\forall z' \in Z)(z \neq g_x^{-1}(\gamma)))
\end{aligned}
$$

So F_g is Σ_1-definable in M from g. Note that F_g is also regressive since each $F_g(x) = g_x(\beta) < \min(x)$, for some $\beta \in g_x^{-1}(F_g(x))$, since $g_x^{-1}(F_g(x)) \in U$ is non-empty. By induction hypothesis, $Y = Y_k^{F_g,M}$ is homogeneous for F_g. Let γ be such that F_g"$[Y]^k = \{\gamma\}$ and let $Y' = Y \cap \Delta_{x \in [Y]^k} g_x^{-1}(\gamma)$. Recall that $\Delta_{x \in [A]^k} X_x =_{\mathrm{df}} \{\gamma < \lambda : (\forall x \in [A \cap \gamma]^k) \, \gamma \in X_x\}$.

Now,

$$\begin{aligned} y' = Y' \quad &\Leftrightarrow \quad (\exists w, y)(w = F_g \wedge \varphi_k(w, y) \wedge y' = y \cap \Delta_{x \in [\lambda]^k} g_x^{-1}(\gamma)) \\ &\Leftrightarrow \quad (\exists w, y, u)(w = F_g \wedge \varphi_k(w, y) \wedge y' = \{v \in y : (\forall x \in [v]^k) v \in g_x^{-1}(u)\} \wedge u = \gamma) \\ &\Leftrightarrow \quad (\exists w, y, u, s)(w = F_g \wedge \varphi_k(w, y) \wedge \ s = S_g \wedge \\ &\qquad y' = \{v \in y : (\forall x \in [v]^k) v \in s(\langle x, u \rangle)\} \wedge \ (\exists z \in [y]^k)(w(z) = u)). \end{aligned}$$

$s = S_g$ is uniformly Σ_1 in g, so Y' is uniformly Σ_1-definable from g. Let $\varphi_n(v_0, v_1)$ be such that

$$M \vDash \varphi_n(g, y) \text{ iff } y = Y'$$

It remains to show that Y_n has the required properties. That $Y' \in U$ follows from normality of U since the sequence $\langle g_x^{-1}(\gamma) : x \in [X]^k \rangle \in M$ and $g_x^{-1}(\gamma) \in U$ whenever $x \in [Y]^k$.

To see that Y' is homogeneous for $g \upharpoonright [X]^n$, let $x \in [Y']^n$. Then,

$$\begin{aligned} g(x) &= g_{x \setminus \{\max(x)\}}(\max(x)) \\ &= \gamma \end{aligned}$$

since $\max(x) \in g^{-1}_{x \setminus \{\max(x)\}}(\gamma)$, since $\max(x) \in Y' \subseteq \Delta_{x \in [Y]^k} g_x^{-1}(\gamma)$. □

Note 2.6. *We emphasise the uniformity in the above: if we fix a predicate symbol $\dot{U}$ then the sequence $\langle \varphi_n(v_0, v_1) : 0 < n < \omega \rangle$ of Lemma 2.5 may be assumed to be the same for all pre-useful structures M (for a language containing the symbol $\dot{U}$) such that the interpretation of $\dot{U}$ in M is $U(M)$.*

Definition 2.7. *A pre-useful structure M with $\lambda = \lambda(M)$ is* useful *iff $\bigcap_{0<n<\omega} Y_n^{g,M}$ is unbounded in λ for every $g : [X]^{<p} \to \lambda$ with $g \in M, X \in U$ and $p \leqslant \omega$.*

Lemma 2.8. *If M is pre-useful and $U(M)$ is ω-complete then M is useful.*

Proof. $(\lambda(M) \setminus \alpha) \cap \bigcap_{0<n<\omega} Y_n^{g,M} \neq \varnothing$ by definition of ω-completeness, for each $\alpha < \lambda(M)$, since $Y_n^{g,M} \in U$ for each g and n. □

Lemma 2.9. *There is a uniform sequence of Σ_1-formulae $\langle \psi_n(v_0, v_1) : 0 < n < \omega \rangle$ such that if M is a useful structure and $\mathcal{A} \in M$ is a λ_0-structure with $\lambda(M) \leqslant \lambda_0$ and $\mathcal{A} \upharpoonright \lambda(M) \prec \mathcal{A}$, then there is a unique set $X_n^{\mathcal{A},M} \in M$ for each $n < \omega$ with $M \vDash \psi_n(\mathcal{A}, X_n^{\mathcal{A},M})$. Moreover, each $X_n^{\mathcal{A},M} \in U$ and $\bigcap_{0<n<\omega} X_n^{\mathcal{A},M}$ is a set of good indiscernibles for $\mathcal{A}$, cofinal in $\lambda(M)$.*

Proof. Let $\lambda = \lambda(M), U = U(M)$, and $C = \{\alpha < \lambda : \mathcal{A} \upharpoonright \alpha \prec \mathcal{A}\}$. Define $g_{\mathcal{A}} : [C]^{<\omega} \to \lambda$ as follows for $x = \{x_1, \ldots, x_{2k}\} \in [\lambda]^{2k}$,

$$g_{\mathcal{A}}(x) = \begin{cases} 0 & \text{if } \{x_1, \ldots, x_k\} \text{ and } \{x_{k+1}, \ldots, x_{2k}\} \text{ have the same type over } \langle \mathcal{A}, (\xi) \rangle_{\xi < x_0} \\ \varphi & \text{where } \varphi \text{ is the least counterexample, otherwise} \end{cases}$$

and set $g_{\mathcal{A}}(x) = 0$ for $x \in [\lambda]^{2k+1}$. Since $\mathcal{A} \in M$, the satisfaction relation for $\mathcal{A}$ is also an element of M, hence $g_{\mathcal{A}} \in M$ (using the ZFC$^-$ property from ii of 2.4. Moreover, since the satisfaction relation for $\mathcal{A}$ is Σ_1-definable from $\mathcal{A}$, $g_{\mathcal{A}}$ is also Σ_1-definable from $\mathcal{A}$.

$g_{\mathcal{A}}$ is regressive and $C \in U$ (since U is normal and C contains a club) so we can apply Lemma 2.5. For each n let $\psi_n(v_0, v_1)$ be the Σ_1-formula $(\exists w)(w = g_{v_o} \wedge \varphi_n(w, v_1))$. So $X_n^{\mathcal{A},M} = Y_n^{g_{\mathcal{A}},M}$.

That $I = \bigcap_{0<n<\omega} X_n^{\mathcal{A},M}$ is a set of good indiscernibles for $\mathcal{A}$ is now immediate. □

Definition 2.10. *Let M be a useful structure, and $\mathcal{A} \in M$ be an α-structure with $\lambda(M) \leqslant \alpha$. Then, in the notation of the previous lemma, set $I(\mathcal{A}, M) = \bigcap_{0<n<\omega} X_n^{\mathcal{A},M}$.*

Definition 2.11. *If $\mathcal{A}$ is a κ-structure, M is useful, $\mathcal{A} \upharpoonright \lambda(M) \prec \mathcal{A}$ and $\mathcal{A} \upharpoonright \lambda(M) \in M$, then let $I(\mathcal{A}, M) = I(\mathcal{A} \upharpoonright \lambda(M), M)$.*

Remark 2.12. *The construction of $I(\mathcal{A}, M)$ is absolute between transitive ZF^--models containing $\mathcal{A}$ and M.*

2.3. $M(\mathcal{A}, I)$: Obtaining a Measure from Indiscernibles

Let κ be an uncountable cardinal (not necessarily regular) and let $\mathcal{A} = \langle J_\kappa^{\vec{A}}, \in, \vec{A}, \ldots \rangle$ be a κ-structure satisfying ZFC. Suppose that I is a set of good indiscernibles for I, $\gamma^* = \sup(I)$, and $\mathrm{otp}(I)$ is a limit ordinal. We keep $\kappa, \mathcal{A}, I$,and γ^* fixed throughout this subsection, and construct a useful structure $M = M(\mathcal{A}, I)$ with $\lambda(M) = \gamma^*$, $\mathcal{A} \upharpoonright \gamma^* \in M$, and $I \subseteq I(\mathcal{A}, M)$.

The details of this construction will be used later, so the following definitions also serve to fix notation. We use the hypothesis that $\mathcal{A} \vDash \mathrm{ZFC}$ to ensure that for any $\gamma < \kappa$ $(H_{(\gamma^+)^{\mathcal{A}}})^{\mathcal{A}} \in \mathcal{A}$. Denote this set by $H(\gamma)$. We could weaken this assumption somewhat, but this level of generality will suffice for our purposes.

By Lemma 1.1 there is $f_{\mathcal{A}} = h_{\mathcal{A}}(i, -)$ (some $i < \omega$), a $\Sigma_1(M)$ bijection $f_{\mathcal{A}} : \kappa \leftrightarrow J_\kappa^{\vec{A}}$. $\mathcal{A} \upharpoonright \gamma$ is closed under $f_{\mathcal{A}}$ for each $\gamma \in I$, by elementarity.

Definition 2.13. *For any $\gamma, \gamma' \in I, \gamma < \gamma'$, and $n < \omega$ define*

i. $\mathcal{A}_\gamma^{n,I} = \mathrm{Hull}^{\mathcal{A}}((\gamma + 1) \cup \vec{\gamma}) \cap H(\gamma)$ *where* $\vec{\gamma} \in [I \backslash (\gamma + 1)]^n$

ii. $\pi_{\gamma\gamma'}^{n,I,\mathcal{A}} : \mathcal{A}_\gamma^{n,I} \to \mathcal{A}_{\gamma'}^{n,I}$ *by* $\pi_{\gamma\gamma'}^{n,I,\mathcal{A}}(\tau(\vec{\nu}, \gamma, \vec{\gamma})) = \tau(\vec{\nu}, \gamma', \vec{\gamma})$ *for each $\mathcal{A}$-term τ,* $\vec{\nu} \in [\gamma]^{<\omega}$, *and* $\vec{\gamma} \in [I \backslash (\gamma' + 1)]^n$

iii. $U_\gamma^{n,I,\mathcal{A}} = \{X \in P(\gamma) \cap \mathcal{A}_\gamma^{n,I} : \gamma \in \pi_{\gamma\gamma'}^{n,I,\mathcal{A}}(X)\}$

iv. $\mathcal{B}_\gamma^{\mathcal{A},I} = \bigcup_{n<\omega} \mathcal{A}_\gamma^{n,I}$

v. $U_\gamma^{\mathcal{A},I} = \bigcup_{n<\omega} U_\gamma^{n,I,\mathcal{A}}$

vi. $\pi_{\gamma\gamma'}^{\mathcal{A},I} = \bigcup_{n<\omega} \pi_{\gamma\gamma'}^{n,I,\mathcal{A}}$

vii. $\overline{\mathcal{B}}_\gamma^{\mathcal{A},I} = \langle \mathcal{B}_\gamma^{\mathcal{A},I}, U_\gamma^{\mathcal{A},I} \rangle$

Lemma 2.14. *Let $\gamma, \gamma', \gamma'' \in I, \gamma < \gamma' < \gamma''$ and $n < \omega$.*

i. $\mathcal{A}_\gamma^{n,I}$ *is well-defined and transitive;* *ii.* $\mathcal{A}_\gamma^{n,I} \in \mathcal{A}_\gamma^{n+1,I}$;

iii. $\pi_{\gamma\gamma'}^n : \mathcal{A}_\gamma^{n,I} \to \mathcal{A}_{\gamma'}^{n,I}$ *is a well-defined elementary embedding with critical point γ;*

iv. $\pi_{\gamma\gamma'}^{n,I,\mathcal{A}} = \pi_{\gamma\gamma'}^{n+1,I,\mathcal{A}} \upharpoonright \mathcal{A}_\gamma^{n,I}$; *v.* $\pi_{\gamma\gamma''}^{n,I,\mathcal{A}} = \pi_{\gamma'\gamma''}^{n,I,\mathcal{A}} \circ \pi_{\gamma\gamma'}^{n,I,\mathcal{A}}$;

vi. $U_\gamma^{n,I,\mathcal{A}} \in \mathcal{A}_\gamma^{n+2,I}$; *vii.* $U_\gamma^{n,I,\mathcal{A}} = U_\gamma^{n+1,I,\mathcal{A}} \cap \mathcal{A}_\gamma^{n,I}$;

viii. $\overline{\mathcal{B}}_\gamma^{\mathcal{A},I}$ *is amenable;* *ix.* $U_\gamma^{I,\mathcal{A}}$ *is a normal $\mathcal{B}_\gamma^{\mathcal{A},I}$-ultrafilter;*

x. $\pi_{\gamma\gamma'}^{\mathcal{A},I}$ *is well-defined and is an elementary embedding in $\mathcal{L}_{\mathcal{A}}$;*

xi. $\pi^{\mathcal{A},I}_{\gamma\gamma''} = \pi^{\mathcal{A},I}_{\gamma'\gamma''} \circ \pi^{\mathcal{A},I}_{\gamma\gamma'}$; *xii* $\pi^{\mathcal{A},I}_{\gamma\gamma'} : \overline{\mathcal{B}}^{\mathcal{A},I}_{\gamma} \to \overline{\mathcal{B}}^{\mathcal{A},I}_{\gamma'}$ *is Q-preserving.*

Proof. i. Take $\vec{\gamma}, \vec{\delta} \in [I\backslash(\gamma+1)]^n$ and $\vec{\beta} \in [\gamma+1]^{<\omega}$. If $x \in H(\gamma) \subseteq \mathcal{A} \restriction \min(\vec{\gamma} \cup \vec{\delta})$, $\varepsilon = f^{-1}_{\mathcal{A}}(x)$, and φ is a formula of $\mathcal{L}_{\mathcal{A}}$ then, by good indiscernibility,

$$\mathcal{A} \vDash \varphi(\vec{\beta}, x, \vec{\gamma}) \Leftrightarrow \mathcal{A} \vDash \varphi(\vec{\beta}, f_{\mathcal{A}}(\varepsilon), \vec{\gamma}) \Leftrightarrow \mathcal{A} \vDash \varphi(\vec{\beta}, f_{\mathcal{A}}(\varepsilon), \vec{\delta}) \Leftrightarrow \mathcal{A} \vDash \varphi(\vec{\beta}, x, \vec{\delta})$$

Hence, $\mathrm{Hull}^{\mathcal{A}}((\gamma+1)\cup\vec{\gamma}) \restriction H(\gamma) = \mathrm{Hull}^{\mathcal{A}}((\gamma+1)\cup\vec{\delta}) \restriction H(\gamma)$ and $\mathcal{A}^{n,I}_{\gamma}$ is well-defined. Moreover, $\mathcal{A}^{n,I}_{\gamma}$ is transitive since if $x \in \mathcal{A}^{n,I}_{\gamma}$ then there exists $f \in \mathcal{A}^{n,I}_{\gamma}$ such that f maps γ onto x, since γ is the largest cardinal of $\mathcal{A}^{n,I}_{\gamma}$ and, hence, every element of x is definable in $\mathcal{A}^{n,I}_{\gamma}$ from f and some $\xi < \kappa$.

ii.

$$\begin{aligned} \mathcal{A}^{n,I}_{\gamma} &= \mathrm{Hull}^{\mathcal{A}}((\gamma+1)\cup\vec{\gamma}) \restriction H(\gamma) \\ &= \mathrm{Hull}^{\mathcal{A}\restriction\delta}((\gamma+1)\cup\vec{\gamma}) \restriction H(\gamma) \\ &\in \mathrm{Hull}^{\mathcal{A}}((\gamma+1)\cup\vec{\gamma}\cup\{\delta\}) \restriction H(\gamma) \\ &= \mathcal{A}^{n+1,I}_{\gamma} \end{aligned}$$

where $\delta \in I\backslash(\max(\vec{\gamma})+1)$

iii. By good indiscernibility of I. iv. and v. are immediate.

vi. Suppose $X \in \mathcal{A}^{n,I}_{\gamma}$. Then there is a formula φ of $\mathcal{L}_{\mathcal{A}}$ and $\vec{\alpha} \in [\gamma]^{<\omega}$ such that , for any $\vec{\gamma} \in [I\backslash(\gamma+1)]^n$, X is the unique set in $H(\gamma)$ such that $\mathcal{A} \vDash \varphi(\vec{\alpha}, X, \gamma, \vec{\gamma})$. Then

$$\begin{aligned} X \in U^{n,I,\mathcal{A}}_{\gamma} &\Leftrightarrow \gamma \in \pi^{n,I,\mathcal{A}}_{\gamma\gamma'}(X) \\ &\Leftrightarrow (\exists X')(\mathcal{A} \vDash \varphi(\vec{\alpha}, X', \gamma', \vec{\gamma}) \wedge \gamma \in X') \text{where } \vec{\gamma} \in [I\backslash(\gamma+1)]^n \\ &\Leftrightarrow (\exists X')(\mathcal{A} \restriction \delta \vDash \varphi(\vec{\alpha}, X', \gamma', \vec{\gamma}) \wedge \gamma \in X') \\ &\qquad \text{where } \vec{\gamma}\cup\{\delta\} \in [I\backslash(\gamma+1)]^{n+1}, \max(\vec{\gamma}) < \delta \end{aligned}$$

However $(\exists X')(\mathcal{A} \restriction \delta \vDash \varphi(\vec{\alpha}, X', \gamma', \vec{\gamma}) \wedge \gamma \in X')$ can be expressed by a formula with parameters

$$\vec{\alpha} \in [\gamma]^{<\omega}, \gamma, \gamma' \in I\backslash(\gamma+1), \vec{\gamma}\cup\{\delta\} \in [I\backslash(\gamma+1)]^{n+1}$$

so

$$U^{n,I,\mathcal{A}}_{\gamma} = \{X \in \mathcal{A}^{n,I}_{\gamma} : X \in U^{n,I,\mathcal{A}}_{\gamma}\} \in \mathcal{A}^{n+2,I}_{\gamma},$$

since $\mathcal{A}^{n,I}_{\gamma} \in \mathcal{A}^{n+2,I}_{\gamma}$.

vii. By iv.

viii. Let $X \in \mathcal{B}^{\mathcal{A},I}_{\gamma}$. Then $X \in \mathcal{A}^{n,I}_{\gamma}$, for some $n < \omega$.

$$\begin{aligned} U^{I,\mathcal{A}}_{\gamma} \cap X &= U^{I,\mathcal{A}}_{\gamma} \cap (\mathcal{A}^{n,I}_{\gamma} \cap X) \text{ since } X \subseteq \mathcal{A}^{n,I}_{\gamma} \\ &= U^{n,I,\mathcal{A}}_{\gamma} \cap X \text{ since } U^{n,I,\mathcal{A}}_{\gamma} = \mathcal{A}^{n,I}_{\gamma} \cap U^{I,\mathcal{A}}_{\gamma} \\ &\in \mathcal{A}^{n+2,I}_{\gamma} \text{by vi} \\ &\subseteq \mathcal{B}^{\mathcal{A},I}_{\gamma}. \end{aligned}$$

ix. We haved proved that the measure is amenable with respect to $\mathcal{B}^{\mathcal{A},I}_\gamma$. Let $X \in \mathcal{A}^{n,I}_\gamma$, some $n < \omega$, then:

$$\pi^n_{\gamma\gamma'}(\kappa\backslash X)) \cup \pi^n_{\gamma\gamma'}(X) = (\pi^n_{\gamma\gamma'}(\kappa)\backslash\pi^n_{\gamma\gamma'}(X)) \cup \pi^n_{\gamma\gamma'}(X) = \pi^n_{\gamma\gamma'}(\kappa) \ni \gamma$$

so $\kappa\backslash X \in U^{n,I,\mathcal{A}}_\gamma \subseteq U^{I,\mathcal{A}}_\gamma$ or $X \in U^{n,I,\mathcal{A}}_\gamma \subseteq U^{I,\mathcal{A}}_\gamma$. Suppose $f \in \mathcal{A}^{n,I}_\gamma$ and $f : \gamma \to U^{n,I,\mathcal{A}}_\gamma$. Then

$$\gamma \in \Delta_{\alpha<\gamma}\pi^n_{\gamma\gamma'}(f(\alpha')) \subseteq \Delta_{\alpha<\gamma'}\pi^n_{\gamma\gamma'}(f)(\alpha) = \pi^n_{\gamma\gamma'}(\Delta_{\alpha<\gamma}f(\alpha)),$$

as required.

x. By iii and iv; xi. Follows immediately from v.

xii. It is sufficient to show that $\pi^{I,\mathcal{A}}_{\gamma\gamma'}$ is Σ_0-cofinal. By i, if $X \in U^{I,\mathcal{A}}_\gamma$ then $\pi^{I,\mathcal{A}}_{\gamma\gamma'}(X) \in U^{I,\mathcal{A}}_{\gamma'}$. It follows that $\pi^{I,\mathcal{A}}_{\gamma\gamma'}$ is Σ_0-preserving, since $\pi^{I,\mathcal{A}}_{\gamma\gamma'}$ is also elementary for $\mathcal{L}_\mathcal{A}$. To see that $\pi^{I,\mathcal{A}}_{\gamma\gamma'}$ is cofinal, it is sufficient to note that $\pi^{I,\mathcal{A}}_{\gamma\gamma'}(\mathcal{A}^{n,I}_\gamma) = \mathcal{A}^{n,I}_{\gamma'}$ for each $n < \omega$ and $\mathcal{B}^{\mathcal{A},I}_{\gamma'} = \bigcup_{n<\omega}\mathcal{A}^{n,I}_{\gamma'}$.

□

Corollary 2.15. *$\overline{\mathcal{B}}^{\mathcal{A},I}_\gamma$ is pre-useful for every $\gamma \in I$.*

Definition 2.16. *For any $\eta \in \lim(I)$, define*

$$\overline{\mathcal{B}}^{\mathcal{A},I}_\eta = \langle\langle\mathcal{B}^{\mathcal{A},I}_\eta, U^{\mathcal{A},I}_\eta\rangle, \langle\pi^{\mathcal{A},I}_\gamma\rangle_{\gamma\in I\cap\eta}\rangle = \lim\langle\langle\mathcal{B}^{\mathcal{A},I}_\gamma, U^{\mathcal{A},I}_\gamma\rangle, \langle\pi^{\mathcal{A},I}_{\gamma\gamma'}\rangle_{\gamma<\gamma'\in I\cap\eta}\rangle$$

Set $M(\mathcal{A}, I) = \overline{\mathcal{B}}^{\mathcal{A},I} = \overline{\mathcal{B}}^{\mathcal{A},I}_{\gamma^*}$, and $U^{\mathcal{A},I} = U^{\mathcal{A},I}_{\gamma^*}$.

Remark 2.17. *If $\eta \in \lim(I) \cap I$ then 2.13 and 2.16 both give definitions for $\overline{\mathcal{B}}^{\mathcal{A},I}_\eta$, but the two definitions agree.*

Lemma 2.18. *$M = M(\mathcal{A}, I)$ is a useful structure with $\lambda(M) = \gamma^*$ and $\mathcal{A} \restriction \gamma^* \in M$. Moreover, for any $g : [X]^{<p} \to \lambda$ with $g \in M, X \in U(M)$, there is $\gamma < \gamma^*$ such that $\bigcap_{0<n<\omega} Y^{g,M}_n \supseteq I\backslash\gamma$.*

Proof. That M is pre-useful and $\lambda(M) = \gamma^*$ and $\mathcal{A} \restriction \gamma^* \in M$ follows from the direct limit construction (the maps $\pi^{\mathcal{A},I}_{\gamma\gamma'}$ are Q-preserving, for all $\gamma \in I$, and are the identity below γ).

To prove that M is useful, let $g : [X]^{<p} \to \lambda$ with $g \in M, X \in U(M) = U^{\mathcal{A},I}$ and $p \leqslant \omega$. Let $\gamma \in I$ and $\overline{g} \in \overline{\mathcal{B}}^{\mathcal{A},I}_\gamma$ such that $\pi^{\mathcal{A},I}_\gamma(\overline{g}) = g$.

Let $N = \overline{\mathcal{B}}^{\mathcal{A},I}_\gamma$. Then $\pi^{\mathcal{A},I}_\gamma(Y^{\overline{g},N}_n) = Y^{g,M}_n$ for each $n < \omega$ by Lemma 2.5 and Note 2.6, since $\pi^{\mathcal{A},I}_\gamma$ is Q-preserving. Let $\gamma', \gamma'' \in I$ with $\gamma < \gamma' < \gamma''$. Then, since $Y^{\overline{g},N}_n \in U^{I,\mathcal{A}}_\gamma$, $\gamma \in \pi^{\mathcal{A},I}_{\gamma\gamma''}(Y^{\overline{g},N}_n)$ so

$$\gamma = \pi^{\mathcal{A},I}_{\gamma''}(\gamma) \in \pi^{\mathcal{A},I}_{\gamma''}(\pi^{\mathcal{A},I}_{\gamma\gamma''}(Y^{\overline{g},N}_n)) = \pi^{\mathcal{A},I}_\gamma(Y^{\overline{g},N}_n) = Y^{g,M}_n.$$

Similarly, $\pi^{\mathcal{A},I}_{\gamma\gamma'}(Y^{\overline{g},N}_n) \in U^{I,\mathcal{A}}_{\gamma'}$ implies $\gamma' \in \pi^{\mathcal{A},I}_{\gamma'\gamma''}(\pi^{\mathcal{A},I}_{\gamma\gamma'}(Y^{\overline{g},N}_n)) = \pi^{\mathcal{A},I}_{\gamma\gamma''}(Y^{\overline{g},N}_n)$ so

$$\gamma' = \pi^{\mathcal{A},I}_{\gamma''}(\gamma') \in \pi^{\mathcal{A},I}_{\gamma''}(\pi^{\mathcal{A},I}_{\gamma\gamma''}(Y^{\overline{g},N}_n)) = \pi^{\mathcal{A},I}_\gamma(Y^{\overline{g},N}_n) = Y^{g,M}_n.$$

So $I\backslash\gamma \subseteq Y^{g,M}_n$ for all $n < \omega$. Hence $\bigcap_{0<n<\omega} Y^{g,M}_n \supseteq I\backslash\gamma$ is unbounded in $\gamma^* = \lambda(M)$. □

Lemma 2.19. *$I \subseteq I(\mathcal{A}, M)$, where $M = M(\mathcal{A}, I)$.*

Moreover, if $\mathcal{A}' \in M$ is a γ^*-structure and γ', $\mathcal{A}'_0$ are such that $\pi^{\mathcal{A},I}_{\gamma'}(\mathcal{A}'_0) = \mathcal{A}'$ then $I\backslash\gamma' \subseteq I(\mathcal{A}', M)$. In particular, if $\mathcal{A}' \in M$ is definable from some $\beta < \gamma^*$ then $I\backslash(\beta + 1) \subseteq I(\mathcal{A}', M)$.

Proof. Let $\gamma = \min(I)$. Then $\pi_\gamma^{\mathcal{A},I}(\mathcal{A} \restriction \gamma) = \mathcal{A} \restriction \gamma^*$ and, in the notation of the proof of Lemma 2.9, $g_{\mathcal{A}\restriction\gamma} \in \overline{\mathcal{B}}_\gamma^{\mathcal{A},I}$ and $\pi_\gamma^{\mathcal{A},I}(g_{\mathcal{A}\restriction\gamma}) = g_{\mathcal{A}\restriction\gamma^*}$. Just as in the proof of Lemma 2.18,

$$I = I \backslash \gamma \subseteq \bigcap_{0<n<\omega} X_n^{\mathcal{A}\restriction\gamma^*,M} = I(\mathcal{A}, M).$$

The proof of the final two statements is just the same. □

Lemma 2.20. *$M = M(\mathcal{A}, I(\mathcal{A}, M))$, where $M = M(\mathcal{A}, I)$.*

Proof. Since $I \subseteq I(\mathcal{A}, M)$ (by Lemma 2.19) is cofinal, and $\overline{\mathcal{B}}_\gamma^{\mathcal{A},I} = \overline{\mathcal{B}}_\gamma^{\mathcal{A},I(\mathcal{A},M)}$ for each $\gamma \in I$ (by indiscernibility), for each $\gamma \in I(\mathcal{A}, M)$ there is $\gamma' \in I$ such that $\pi_\gamma^{\mathcal{A},I(\mathcal{A},M)} : \overline{\mathcal{B}}_\gamma^{\mathcal{A},I(\mathcal{A},M)} \to \overline{\mathcal{B}}_{\gamma'}^{\mathcal{A},I}$.

Consequently,

$$\begin{aligned} M(\mathcal{A}, I) &= \lim\langle\langle \mathcal{B}_\gamma^{\mathcal{A},I}, U_\gamma^{\mathcal{A},I}\rangle, \langle \pi_{\gamma\gamma'}^{\mathcal{A},I}\rangle_{\gamma<\gamma'\in I}\rangle \\ &= \lim\langle\langle \mathcal{B}_\gamma^{\mathcal{A},I(\mathcal{A},M)}, U_\gamma^{\mathcal{A},I(\mathcal{A},M)}\rangle, \langle \pi_{\gamma\gamma'}^{\mathcal{A},I(\mathcal{A},M)}\rangle_{\gamma<\gamma'\in I(\mathcal{A},M)}\rangle \\ &= M(\mathcal{A}, I(\mathcal{A}, M)) \end{aligned}$$

□

Lemma 2.21. *$U^{\mathcal{A},I}$ is generated by I. Consequently, if $\mathrm{cf}(\gamma^*) > \omega$ then $U^{\mathcal{A},I}$ is ω-complete.*

Proof. The argument from the proof of Lemma 2.18 shows that $U^{\mathcal{A},I}$ is generated by I. ω-completeness is immediate. □

Remark 2.22. *The construction of $M(\mathcal{A}, I)$ is absolute between transitive ZF^--models containing $\mathcal{A}$ and I.*

2.4. The Jensen Indiscernibles Lemma

We assume $\neg 0^\P$ and thus $\boldsymbol{K}$ is built using non-overlapping extenders. We prove Theorem 2.3 just for the universal weasel $\boldsymbol{K}$ - the generalisations to other universal weasels being straightforward - we fix an uncountable cardinal κ, a κ-structure $\mathcal{A} \in \boldsymbol{K}$, and a set, I, of good indiscernibles for $\mathcal{A} = \langle J_\kappa^E, \in, E, A\rangle \in \boldsymbol{K}$, with $\sup(I) = \gamma^*$ and $\mathrm{cf}(\mathrm{otp}(I)) > \omega$.

Each $\gamma \in I$ is a regular cardinal of $\mathcal{A}$. If not, let $\gamma \in I$ such that $\alpha = \mathrm{cf}(\gamma) < \gamma$. If $\gamma' \in I \backslash (\gamma + 1)$ then $\mathrm{cf}(\gamma') = \alpha = \mathrm{cf}(\gamma)$ by good indiscernibility. Let $f_\gamma, f_{\gamma'}$ be the $\lhd$-least maps witnessing the cofinalities of γ and γ', respectively, where $\lhd$ is the canonical $\Sigma_1(\mathcal{A})$ well-ordering of $\mathcal{A}$. Then $f_{\gamma'}(\beta) = f_\gamma(\beta)$ for all $\beta < \alpha$, by good indiscernibility, so $f_{\gamma'}$ is bounded below γ - a contradiction. Since J_κ^E is acceptable, for each $\gamma, \gamma' \in I$ with $\gamma < \gamma'$, for all $\alpha < \gamma$, $(P(\alpha))^{\mathcal{A}} \subseteq J_\gamma^E$, hence $(P(\alpha))^{\mathcal{A}} \in J_{\gamma'}^E$, hence $(P(\alpha))^{\mathcal{A}} \in J_\gamma^E$, by good indiscernibility.

So each $\gamma \in I$ is an inaccessible cardinal of $\mathcal{A}$. It follows that $\mathcal{A} \restriction \gamma \models \mathrm{ZFC}$ and, since $\mathcal{A} \restriction \gamma \prec \mathcal{A}$, that $\mathcal{A} \models \mathrm{ZFC}$, and the results of subsection 2.3 apply.

For each $\gamma \in I$, $\overline{\mathcal{B}}_\gamma^{\mathcal{A},I} = \langle J_\delta^E, \in, E, , U_\gamma^{\mathcal{A},I}\rangle$, where $\delta = \delta_\gamma^{\mathcal{A},I} = \mathrm{On} \cap \overline{\mathcal{B}}_\gamma^{\mathcal{A},I}$ (by the argument showing $\overline{\mathcal{B}}_\gamma^{\mathcal{A},I}$ is transitive, 2.14*i*). Let $M_\gamma^{\mathcal{A},I} = \langle J_\delta^E, \in, E, U_\gamma^{\mathcal{A},I}\rangle$.

It follows that $\overline{\mathcal{B}}^{\mathcal{A},I} = \langle J_\delta^{\overline{E}}, \in, \overline{E}, \overline{A}_1, \ldots, \overline{A}_n, U_\gamma^{\mathcal{A},I}\rangle$, where $\delta = \delta^{\mathcal{A},I} = \mathrm{On} \cap \overline{\mathcal{B}}^{\mathcal{A},I}$ for some $\overline{E}, \overline{A}_1, \ldots, \overline{A}_n$ with

$$\overline{E} \restriction \gamma^* = E \restriction \gamma^*, \overline{A}_1 \cap \gamma^* = A_1 \cap \gamma^*, \ldots, \overline{A}_n \cap \gamma^* = A_n \cap \gamma^*.$$

Let $M^{\mathcal{A},I} = \langle J_{\delta^{\mathcal{A},I}}^{\overline{E}}, \in, \overline{E}, U^{\mathcal{A},I}\rangle$ (so $M^{\mathcal{A},I}$ is the structure $M(\mathcal{A}, I)$ with some predicates omitted). For convenience, we define $M_{\gamma^*}^{\mathcal{A},I} = M^{\mathcal{A},I}$.

Lemma 2.23. *i. $U^{\mathcal{A},I}$ is an ω-complete measure with critical point γ^*, and is generated by I;*

ii. $M_\gamma^{\mathcal{A},I}$ is an acceptable J-structure for each $\gamma \in I \cup \{\gamma^*\}$;

iii. The maps $\pi_{\gamma\gamma'}^{\mathcal{A},I}$ and $\pi_\gamma^{\mathcal{A},I}$ are Q-preserving for each $\gamma, \gamma' \in I$, and $\rho^1_{M_\gamma^{\mathcal{A},I}} \leq \gamma$ for each $\gamma \in I \cup \{\gamma^*\}$.

Proof. Set $M = M^{\mathcal{A},I}$.

i. By Lemma 2.21.

ii. Each $J^E_{\delta_\gamma^{\mathcal{A},I}}$, $\gamma \in I$, is a union of acceptable structures and so each $M_\gamma^{\mathcal{A},I}$ is acceptable. That M is acceptable follows since acceptability is a Q-property and each $\pi_\gamma^{\mathcal{A},I}$ is Q-preserving.

In fact $J^E_{\delta_\gamma^{\mathcal{A},I}}$ is an initial segment of $\boldsymbol{K}$ with largest cardinal γ; for arbitrarily large $\delta' < \delta_\gamma^{\mathcal{A},I}$ we have that $\rho^1_{\delta'} = \gamma$. Hence the same is true of arbitrarily large $\delta' < \delta = \delta^{\mathcal{A},I} = \mathrm{On} \cap \mathcal{M}$ at γ^*. Elementary iteration and comparison arguments then show that $(J^E_\delta)^{\mathcal{M}}$ is also an initial segment of $\boldsymbol{K}$.

iii. Let $\alpha = \sup(\mathrm{On} \cap h_M``(\gamma^* \cup \{\mathcal{A} \restriction \gamma^*\}))$. Then $h_M``(\gamma^* \cup \{\mathcal{A} \restriction \gamma^*\}) = J^E_\alpha$, since for any $\beta \in \mathrm{On} \cap h_M``(\gamma^* \cup \{\mathcal{A} \restriction \gamma^*\})$ there is in M as above, a $\Sigma_1^M(\beta)$-definable map from γ^* onto J^E_β. Suppose for a contradiction that $\alpha < \delta$.

Note that $M' =_{\mathrm{df}} \langle J^E_\alpha, \in, E \restriction \alpha, U^{\mathcal{A},I} \cap J^E_\alpha \rangle$ is an amenable structure, since if x is in the given Skolem hull, then so is $x \cap U^{\mathcal{A},I}$. Moreover $M' \prec_{\Sigma_1} M$ with the whole structure M' useful and an element of M for the same reason.

In the notation of Lemma 2.9, we have, by Σ_1-elementarity, $X_n^{\mathcal{A}\restriction\gamma^*,M'} = X_n^{\mathcal{A}\restriction\gamma^*,M} \in M$ for each $n < \omega$ with $M' \vDash \psi_n(\mathcal{A}\restriction\gamma^*, X_n^{\mathcal{A}\restriction\gamma^*,M'})$. Moreover, each $X_n^{\mathcal{A}\restriction\gamma^*,M'} \in U \cap M'$ and

$$\bigcap_{0<n<\omega} X_n^{\mathcal{A}\restriction\gamma^*,M'} = \bigcap_{0<n<\omega} X_n^{\mathcal{A}\restriction\gamma^*,M}$$

is definable over M', and thus is in $\mathcal{M}$, and is a set of good indiscernibles for $\mathcal{A} \restriction \gamma^*$. In other notation, we thus have $I(\mathcal{A}, M) = I(\mathcal{A}, M') \in M$.

However then, by Lemma 2.20, $M = M(\mathcal{A}, I) = M(\mathcal{A}, I(\mathcal{A}, M)) \in M$. Contradiction.

Thus $\alpha = \delta$ and so $\gamma^* \geqslant \rho^1_M$. The same argument as above, replacing δ with $\delta_\gamma^{\mathcal{A},I}$, and using $\bigcap_{0<n<\omega} X_n^{\mathcal{A}\restriction\gamma, M_\gamma^{\mathcal{A},I}}$, shows that $J^E_{\delta_\gamma^{\mathcal{A},I}} = h_{M_\gamma^{\mathcal{A},I}}``(\gamma \cup \{\mathcal{A} \restriction \gamma\})$ for any $\gamma \in I$. Hence, $\gamma \geqslant \rho^1_{M_\gamma^{\mathcal{A},I}}$ for each $\gamma \in I$. That each $\pi_{\gamma\gamma'}^{\mathcal{A},I}$ is Q-preserving, is Lemma 2.14 xii. It follows from the direct limit construction that each $\pi_\gamma^{\mathcal{A},I}$ is also Q-preserving.

□

Now we temporarily assume $\neg 0^{\mathrm{sword}}$, prove the result under this assumption, and then sketch how to relax this constraint.

Lemma 2.24. ($\neg 0^{\mathrm{sword}}$) *(i) $M_\gamma^{\mathcal{A},I}$ is a mouse for each $\gamma \in I \cup \{\gamma^*\}$;*
(ii) $\rho^\omega_{M_\gamma^{\mathcal{A},I}} = \gamma$ for each $\gamma \in I \cup \{\gamma^*\}$;
(iii) each $\pi_\gamma^{\mathcal{A},I}$ is Σ^*-preserving.

Proof. Since E is the measure sequence of $\boldsymbol{K}$, and every initial segment of $\boldsymbol{K}$ is a mouse, all clauses of the definition of 'premouse' are inherited by each $M_\gamma^{\mathcal{A},I}$ ($\gamma \in I$), except, perhaps, the coherency condition for the top measure. Since each clause of the definition is a Q-property, and the maps $\pi_\gamma^{\mathcal{A},I}$ are Q-preserving, the same holds for $M = M^{\mathcal{A},I}$.

Let $\pi : M \to N = \mathrm{Ult}(M, U) = \langle J_{\delta'}^{E'}, \in, E', U' \rangle$, where $U = U^{\mathcal{A},I}$, be the ultrapower map (this exists by 2.23i). Set again $\delta = \delta^{\mathcal{A},I}$. Since π is Σ_0-preserving,

$$N \vDash (\forall \gamma < \pi(\gamma^*))(\exists X \in \pi(J_{\gamma^*}^{E}) = J_{\pi(\gamma^*)}^{E'})(X = N\|\gamma \text{ is a mouse }).$$

Hence, $N \vDash N\|\delta$ is a mouse .

Since $\kappa \subseteq \gamma^* \subseteq N$ it follows by absoluteness that $N\|\delta$ is a mouse (*cf* [29] Thm 4.5.5). Hence the structure $M' = \langle J_\delta^{E'}, \in, E', E'_\delta, U \rangle$ is a premouse or s-premouse, depending on whether $E'_\delta = \varnothing$ or not. (Note that $J_\delta^{\overline{E}} = J_\delta^{E'}$, by amenability of U, with $((\gamma^*)^+)^N = \delta$.)

Claim: M' is 0-iterable.

Suppose first the claim holds and we show that this demonstrates that M is a mouse. Then $E'_\delta = \varnothing$ (otherwise M' is an s-mouse, contradicting our smallness hypothesis of $\neg 0^{\text{sword}}$), and the coherency condition for M follows. Moreover, $M = M'$ is simply normally iterable above γ^*, since $\rho_M^1 = \gamma^*$ (Lemma 2.23iii) implies that any simple normal iteration above γ^* is a 0-iteration, and so $\rho_{M_i}^1 = \gamma^* \leqslant \kappa_i$ for each iterate M_i and critical point κ_i.

It follows in a standard fashion that M is normally iterable above γ^*.

Since M is normally iterable, it is coiterable with $\boldsymbol{K}$. Let M' (N') be the final premouse on the M-side ($\boldsymbol{K}$-side). Then, M' is an initial segment of a mouse, and hence it, and so M, are mice.

It remains to prove the claim. We do not do this in detail as we shall not refer to it again, and it is almost identical to proofs available in the literature. Define a simple normal 0-iteration $\mathcal{I}' = \langle M'_i, \pi'_{\mathrm{ij}} : i \leqslant j < \theta' \rangle$ of M' with indices $\langle \nu'_i : i + 1 < \theta' \rangle$ by setting ν'_i to be the least $\nu < \mathrm{ht}(M_i)$ such that

i. $E_\nu^{M'_i}$ is a total measure;

ii. $\nu > \sup\{\nu'_{i'} : i' \leqslant i\}$;

iii. if i is a limit ordinal then $\mathrm{cf}(\mathrm{crit}(E_\nu^{M'_i})) = \omega$ or $\{i' < i : \pi'_{i'i}(\nu'_{i'}) = \nu\}$ is bounded in i.

The iteration terminates if there is no such ν. Note that since N is iterable and we do not use either of the top measures in the iteration, the iteration exists. Let $M^* = M'_{\theta'} = \langle J_{\delta^*}^{E^*}, \in, E^*, E^*_{\delta^*}, U^* \rangle$. Just as in Theorem 4.5.6. of [29], $\theta' < (\max\{\omega_1, |M'|\})^+$, U^* is ω-complete, and whenever $\gamma < \delta^*$ is such that E^*_γ is a total measure of M^*, E^*_γ is ω-complete.

We show that, in addition, $E^*_{\delta^*}$ is ω-complete. Let

$$\pi^* : M^* \to M^{**} = \mathrm{Ult}(M^*, U^*) = \langle J_{\delta^{**}}^{E^{**}}, \in, E^{**}, E^{**}_{\delta^{**}}, U^{**} \rangle.$$

Let $\{X_n : n < \omega\} \subseteq E^*_{\delta^*} = E^{**}_{\delta^*}$, and let $\kappa^* = \mathrm{crit}(E^*_{\delta^*})$. Then, by Łoś Theorem, for each $n < \omega$,

$$\{\gamma < \kappa^* : M^* \vDash X_n \cap \gamma \in E^*_{\gamma+}\} \in U^*.$$

Let

$$\gamma^* \in \bigcap_{n<\omega} \{\gamma < \kappa^* : M^* \vDash X_n \cap \gamma \in E^*_{\gamma+}\}$$

(this exists by ω-completeness of U^*). However $E^*_{(\gamma^+)^{M^*}}$ is a total measure of M^*, hence ω-complete, so

$$\varnothing \neq \bigcap_{n<\omega} X_n \cap \gamma^* \subseteq \bigcap_{n<\omega} X_n,$$

as required.

The proof of Theorem 4.5.1 of [29] shows that any s-premouse that has only ω-complete total measures and is such that every proper initial segment is a mouse, is 0-iterable. Thus, the claim follows.

To finish the lemma, note that each $M^{\mathcal{A},I}_\gamma$ ($\gamma \in I$) is a sound premouse, being Q-embeddable into the mouse $M = M^{\mathcal{A},I}_{\gamma^*}$. Further it is 0-iterable, and thus iterable above γ for the same reason. It can thus be compared with $\boldsymbol{K}$ by such an iteration (since, recall, each $M^{\mathcal{A},I}_\gamma \restriction \gamma = \boldsymbol{K} \restriction \gamma$). As above this ensures it is a mouse, and indeed an initial segment of $\boldsymbol{K}$. As each γ is a $\boldsymbol{K}$ cardinal (ii) is trivial and (iii) is immediate. □

Lemma 2.25. ($\neg 0^{\text{sword}}$)

i. $M^{\mathcal{A},I} \in \boldsymbol{K}$. *In fact,* $M^{\mathcal{A},I} = \boldsymbol{K}\|\delta^{\mathcal{A},I}$.

ii. Set $I' = I(\mathcal{A}, M(\mathcal{A}, I)) = I(\mathcal{A}, M^{\mathcal{A},I})$. *Then* $I' \in \boldsymbol{K}$, $I' \supseteq I$, *and* I' *is a set of good indiscernibles for* $\mathcal{A}$.

Proof. i. Again let $M = M^{\mathcal{A},I}$. Since $\rho^1_M = \gamma^*$ the Σ_1 mastercode $A = A^1_M$ is a $\Sigma_1(M)$ subset of γ^* from which M may be defined. Consider the coiteration M with $\boldsymbol{K}$ and let M' (N') be the final mouse on the M-side ($\boldsymbol{K}$-side). M' is an initial segment of N' and A is Σ_1-definable over M' (note that the iteration of M to M' is simple and involves no truncations). However $\mathcal{P}(\gamma^*) \cap \Sigma_\omega(N') \subseteq \mathcal{P}(\gamma^*) \cap \boldsymbol{K}$. Thus $M \in \boldsymbol{K}$. However if M is not an initial segment of $\boldsymbol{K}$, the $M \neq M'$, and as initial segments of N' are sound, $M' = N'$. Thus $M = core(M') = core(N') = \boldsymbol{K}\|\delta^{\mathcal{A},I}$.

ii By Lemma 2.19 and Lemma 2.9. □

We now argue that the above lemmata also go through assuming only $\neg 0^{\P}$. The last Lemma is unproblematic, so we consider the previous one. Assume that $\boldsymbol{K}$ now denotes the core model built with non-overlapping extenders, coded into the predicate $E = E^{\boldsymbol{K}}$, Lemma 2.23 still gives an ω-closed filter U_{γ^*} on γ^*. As the E hierarchy is defined using extenders attached at indices derived from successor cardinals of the images of critical points, we have to modify the above. We set M again to be $M^{\mathcal{A},I}_{\gamma^*}$; then M is of the form $\langle J_\delta, \in, E^M, U^* \rangle$ with $\delta = \delta^{\mathcal{A},I}_{\gamma^*}$. Now take $F = F^*_{\gamma^*}$ to be the extender derived from the ultrapower embedding $\sigma : M \longrightarrow N_0 = Ult(M, U^*)$. Let $\lambda = \sigma(\gamma^*)$ be the *length* of F and then F is an extender which we should naturally index by $\nu = On \cap N_0$, and where $N_0 \models$ " λ is the largest cardinal." Let $\pi : N_0 \longrightarrow N_1 = Ult(N_0, F)$ be the ultrapower map of N_0 by F. Let $N = \langle J^{E^N}_\nu, \in, E^N, E^N_\nu, F^N \rangle$ where $E^N = E^{N_1} \restriction \nu$, E^N_ν is set to be $E^{N_1}_\nu$, and $F^N = F$. If F is coherent with E^{N_0} then $E^N_\nu = \varnothing$. In this case we then argue as before that N is 0-iterable, hence normally iterable, by finding a simple iterate $\tilde{N}$ of N where all measures below the critical point of the top extender F^{top} of $\tilde{N}$ are ω-complete. As F^{top} is the image of the ω-complete F^N it also is ω-complete. The argument then finishes as before with both $\tilde{N}$ and N being mice. The latter is then an initial segment of $\boldsymbol{K}$. However, even if we suppose $E^N_\nu \neq \varnothing$, as we do not have iterable premice with overlapping extenders by our smallness assumption, we cannot have that $cp(E^N_\nu) \neq \gamma^*$. However if we have that $cp(E^N_\nu) = \gamma^*$, by the initial segment condition for this style of extender (*cf* [29] p.252), we have that for unboundedly many $\bar{\lambda} < \lambda$ that $E^N_{\bar{\lambda}}$ is a non-empty extender with critical point γ^*. Thus $N_0 \models$ " $\exists\delta(o(\delta) = \sigma(\gamma^*)$." However then $M \models$ " $\exists\delta(o(\delta) = \gamma^*)$." In turn this would imply that we could form the class length iteration of M, namely $\langle M_\iota, (\pi_{\iota\eta})_{\iota\leq\eta<\infty}, (\kappa_\iota)_{\iota<\infty}, (U_\iota)_{\iota<\infty} \rangle$, using repeatedly the ω-complete U^* and its images U_ι, throughout all the ordinals. This would leave behind a model $\boldsymbol{W} = \bigcup_{\iota<\infty} H^{M_\iota}_{\kappa_\iota}$ with a strong cardinal δ with $o^{\boldsymbol{W}}(\delta) = \infty$, and the critical points $(\kappa_\iota)_{\iota<\infty}$ of the iteration measure would be indiscernibles for $\boldsymbol{W}$. Hence we should have $0^\P$ - contradicting our smallness assumption.

So this finishes the proof of the Indiscernibles Lemma 2.3 in the case that the universal weasel $\boldsymbol{K}'$ there is $\boldsymbol{K}$ itself. However, to finish off the full statement for a general universal weasel $\boldsymbol{K}'$, note that we used no properties of $\boldsymbol{K}$ beyond universality. The structures $M^{\mathcal{A},I}$ arising are now in $\boldsymbol{K}'$ since they agree up to γ^* with $E^{\boldsymbol{K}'} \restriction \gamma^*$, and indeed are now mice in $\boldsymbol{K}'$ and in fact end up as initial segments of $\boldsymbol{K}'$, as all iterations and comparisons are performed above γ^* as before. Hence, as $M^{\mathcal{A},I} \in \boldsymbol{K}'$ so will some $I' \in \boldsymbol{K}'$ with $I' \supseteq I$.

This finishes the proof of the full Jensen Indiscernibles Lemma 2.3. □

2.5. A criterion for good indiscernibles

We establish a useful criterion for the existence of sets of g.i.'s in $\boldsymbol{K}$. First we prove an elementary result on failures of condensation of $\boldsymbol{K}$.

Theorem 2.26. **($\neg 0^\P$)** *Let κ be a regular uncountable cardinal and suppose $j : \hat{M} \to \langle \boldsymbol{K}|\nu, \in, \ldots\rangle$ is an elementary embedding with $\mu = \mathrm{crit}(j) < \kappa \leqslant \mathrm{ht}(\hat{M})$ and $\nu = (\lambda^+)^{\boldsymbol{K}}$, where $\kappa \leq \lambda = j(\kappa)$. Then there is a mouse $N_0 \in H_\kappa$ that iterates past $M =_{df} \boldsymbol{K}^{\hat{M}}$, that is $M <^* N_0$. Further, for no $\xi < \kappa$ is $(o(\xi) \geq \kappa)_M$.*

Proof. Note that if any $\mu' \leq \mu$ satisfies $o^{\boldsymbol{K}}(\mu') \geq \tau$, where $\tau \in \mathrm{ran}(j)$ then then $\mu' < \mu$ as μ' is definable in $\boldsymbol{K} \restriction \nu$ from τ. Let $M = K^{\hat{M}}$; now coiterate K with M.

We now show that there is a mouse N_0 of cardinality less than κ which iterates past M in their comparison; in other words that $M <^* N_0$. If $\mathcal{P}(\mu) \cap \boldsymbol{K} \neq \mathcal{P}(\mu) \cap M$ then standard arguments show $\mathcal{P}(\mu) \cap \boldsymbol{K} \supset \mathcal{P}(\mu) \cap M$ and thus K is immediately truncated in the comparison iteration with M to such an N_0. If this did not happen then $\mathcal{P}(\mu) \cap \boldsymbol{K} = \mathcal{P}(\mu) \cap M$. Using pseudo-ultrapower techniques ([29]3.6.3 for example) we may then lift-up the embedding $j \restriction (M \restriction (\mu^+)^M)$ to a $\bar{j} : \boldsymbol{K} \longrightarrow \boldsymbol{W}$ with the latter a universal weasel, if wellfounded. However the latter cannot be wellfounded: for if so we should have that $\boldsymbol{W}$ is a simple iterate of $\boldsymbol{K}$ with iteration map $\bar{j}$ (as we are below $0^\P$); further there is η which is the index of the first extender used in this iteration with $\mathrm{crit}(E^{\boldsymbol{K}}_\eta) = \mu = \mathrm{crit}(\bar{j})$. By coherency $E^{\boldsymbol{W}}_\eta = \varnothing$. This is a contradiction unless $\eta \geq \sup j$ " $(\mu^+)^M) > \mu$. This would imply that μ is strong beyond $j(\tau)$ for some $\tau \in [\mu, (\mu^+)^M)$: $o^{\boldsymbol{K}}(\mu) > j(\tau)$. However above we argued that any such μ' strong beyond any such $j(\tau)$ is less than μ (and there can be only one such, as we are in the region of non-overlapping extenders), so this is a contradiction.

We conclude that $\boldsymbol{W}$ is illfounded. Now we use a standard argument to a contradiction. Let $\langle f_n, \gamma_n\rangle \in \boldsymbol{K}$ $(n < \omega)$ witness that $\boldsymbol{W}$ is ill-founded. That is, for each $n < \omega$, $\mathrm{dom}(f_n) \in \mu'$, $\gamma_n \in \bar{j}(\mathrm{dom}(f_n))$, and

$$\langle \gamma_{n+1}, \gamma_n\rangle \in \bar{j}(\{\langle \alpha, \beta\rangle : \alpha \in \mathrm{dom}(f_{n+1})), \beta \in \mathrm{dom}(f_n), \text{and } f_{n+1}(\alpha) \in f_n(\beta)\}).$$

Since $\mu' = (\mu^+)^{\boldsymbol{K}}$ we may, without loss of generality, assume $\mathrm{dom}(f_n) = \mu$ for each n. Let $X = \mathrm{Hull}^{\boldsymbol{K}|\eta}((\mu+1) \cup \{f_n : n < \omega\})$, where η is any sufficiently large regular cardinal and let $\tau : N \cong X \prec \boldsymbol{K}|\eta$ be the inverse transitive collapse. Note that $|N| < \kappa$ and is iterable (as it is elementarily embedded into the iterable $\boldsymbol{K}|\eta$); to complete the proof, we require that $M <^* N$. Suppose not. Then N and M coiterate to some $N' \subseteq M'$ via iteration maps σ and π. Let $j' : M' \to K'$ and $\pi' : \boldsymbol{K}|\nu \to K'$ arise from the application of the copying construction (Lemma 4.3.1 of [29]) to j and the M-side of the coiteration. (See Figure 1, the reader may ignore $\widetilde{M}, \boldsymbol{K}, \boldsymbol{K}'', \pi''$.)

Set $f'_n = \sigma(\overline{f}_n)$, where $\tau(\overline{f}_n) = f_n$. Then, for each n, since the coiteration is above μ and $\sigma \restriction \mu = \mathrm{id} \restriction \mu$,

$$\begin{aligned}\{\langle\alpha,\beta\rangle : \alpha,\beta < \mu \,\text{and}\, f'_{n+1}(\alpha) \in f'_n(\beta)\} &= \{\langle\alpha,\beta\rangle : \alpha,\beta < \mu \text{ and } \overline{f}_{n+1}(\alpha) \in \overline{f}_n(\beta)\} \\ &= \{\langle\alpha,\beta\rangle : \alpha,\beta < \mu \text{ and } f_{n+1}(\alpha) \in f_n(\beta)\}\end{aligned}$$

so,

$$\begin{aligned}\langle\gamma_{n+1},\gamma_n\rangle &\in j(\{\langle\alpha,\beta\rangle : \alpha,\beta < \mu \text{ and } f_{n+1}(\alpha) \in f_n(\beta)\}) \\ &= j(\{\langle\alpha,\beta\rangle : \alpha,\beta < \mu \text{ and } f'_{n+1}(\alpha) \in f'_n(\beta)\}) \\ &= j'(\{\langle\alpha,\beta\rangle : \alpha,\beta < \mu \text{ and } f'_{n+1}(\alpha) \in f'_n(\beta)\}) \\ &= \{\langle\alpha,\beta\rangle : \alpha,\beta < j'(\mu) \text{ and } j'(f'_{n+1})(\alpha) \in j'(f'_n)(\beta)\},\end{aligned}$$

where the penultimate line is a consequence of the copy construction, which states that $j' \restriction \nu_0 = j \restriction \nu_0$, where $\nu_0 > \mu$ is the first index of the iteration of M.

So $j'(f'_{n+1})(\gamma_{n+1}) \in j'(f'_n)(\gamma_n)$ for all $n < \omega$, a contradiction. For the last assertion of the theorem, note that $(\exists\xi(o(\xi) \geq \kappa))_M$ would imply that $(\exists\xi(o(\xi) \geq \kappa))_{M_\kappa}$, and thus the same in N_κ too. However this would imply $0^\P$ since the iterable N_κ could be iterated through all the ordinals and leave behind a model with a strong cardinal, and indiscernibles for the same; it thus easily generates an "$0^\P$ mouse." This concludes the proof. □

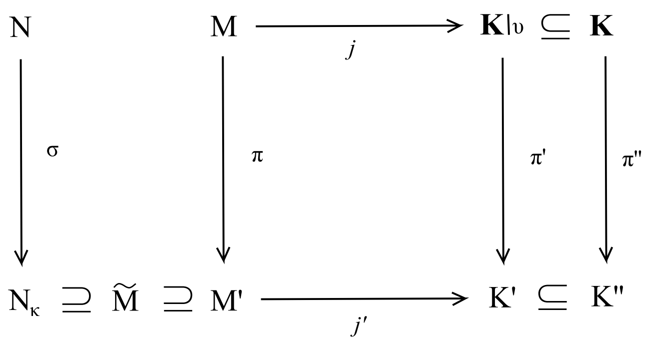

Figure 1: Comparison

Theorem 2.27. *(i) With the notation of the last theorem, suppose further that j"κ is cofinal in λ, and that $\mathcal{A} \in \mathrm{ran}(j)$ is a λ-structure. Then there is a set, $I \in \boldsymbol{K}$, of good indiscernibles for $\mathcal{A}$ of order type $\geq \kappa$.*
(ii) Lastly if in addition to the conditions of (i), that

$$\boldsymbol{K} \models \text{ ``} \{\alpha < \lambda : \boldsymbol{K} \models \alpha \text{ is measurable } \} \text{ is nonstationary in } \lambda\text{''}$$

holds then j"C is a set of good indiscernibles for $\mathcal{A}$ for some club $C \subseteq \kappa$.

Proof. For (i) consider an $\mathcal{A}$ as supposed. Let $j(\bar{\mathcal{A}}) = \mathcal{A}$. Then $\bar{\mathcal{A}}$ is an κ-structure. We consider the coiteration of $N = N_0$ with M. Let $\bar{N} = N_\kappa$, with iteration map $\sigma =_{df} \sigma^N_{0,\kappa} : N_0 \to \bar{N}$ then note that

$\mathcal{A}' \in \bar{N}$ where $\pi(\bar{\mathcal{A}}) = \mathcal{A}'$, with $\pi =_{df} \pi^M_{0,\kappa}$ the iteration map on the M side. Let π' be the copy-map iteration of π taking $\boldsymbol{K} \restriction \nu$ to K', copied up using j, and let $j' : M' \to K'$. As all extenders used in this latter copied iteration are total in $\boldsymbol{K}$ (ν is a $\boldsymbol{K}$-cardinal) we can consider also the iteration of $\boldsymbol{K}$ using the same extenders. This yields an iteration $\pi'' : \boldsymbol{K} \to \boldsymbol{K}''$ with the map π'' extending π'. Note that π " $\kappa \subseteq \kappa$. Again, as the copied iterated ultrapower map only takes ultrapowers using extenders in $\boldsymbol{K}$ which are below λ, without any critical point being sent up to λ, $\pi'(\lambda) = \lambda$. Moreover π' " $\lambda^{+\boldsymbol{K}} \subseteq \lambda^{+\boldsymbol{K}}$. Let $\tilde{\mathcal{A}} = \pi'(\mathcal{A})$. Then $\tilde{\mathcal{A}} = j'(\mathcal{A}')$ by commutativity.

First suppose that there is $\eta < On \cap \bar{N}$ with $\bar{N} \models$ "E_η is a full extender on $\mathcal{P}(\kappa)_{\bar{N}}$". If there is such then clearly $\mathcal{A}' \in J^{E^{\bar{N}}}_\eta$. From the extender we may clearly find an E_η-measure 1 set $\bar{I}$ of g.i's for $\mathcal{A}'$ in $J^{E^{\bar{N}}}_\eta$. Setting $I = j'$ " $\bar{I}$ then yields such a set of g.i's for $\tilde{\mathcal{A}}$. However the existence of such a set for $\tilde{\mathcal{A}} \in \boldsymbol{K}''$ itself, ensures, as $\boldsymbol{K}''$ is universal and thus we may apply the Indiscernibles Lemma, that there is such an $I_0 \supseteq I$ with $I_0 \in \boldsymbol{K}''$. By the elementarity of π'' there is such a set in $\boldsymbol{K}$ for $\mathcal{A}$ and we are done.

So otherwise there is no proper initial segment of $\bar{N}$, $J^{E^{\bar{N}}}_\eta$, with a full extender as above containing $\mathcal{A}'$. Note that for some $k < \omega$ $\rho^{k+1}_{\bar{N}} < \kappa \leq \rho^k_{\bar{N}}$ (because this property is ultimately inherited from the fact that for some $k < \omega$, $\rho^{k+1}_{N_0} \leq \mu \leq \rho^k_{N_0}$). However then $\bar{N}$ is an active mouse with a topmost extender $F^{\bar{N}}$ (measuring $\mathcal{P}(\kappa)_{\bar{N}}$) with the latter extender the image of the topmost extender F^{N_ι} applied unboundedly often below κ in the coiteration. Hence the same extender, so to speak, has been used unboundedly in κ. If $D = \{\kappa_\iota \mid \iota < \kappa\}$ are the critical points of the F^{N_ι} so used, then D forms a closed and unbounded set of g.i's for $\mathcal{A}'$, since D generates, as a final segment filter, $F^{\bar{N}}_\varnothing$ where the latter is the measure derived from the extender $F^{\bar{N}}$. (In other words we see that then the set of g.i.'s $\bigcap_{0<n<\omega} X^{\bar{\mathcal{A}},\bar{N}}_n$ (see Lemma 2.9) contains a final segment of D). However then j' " D is a sequence of g.i's unbounded in λ for $\tilde{\mathcal{A}}$. We now apply again the Jensen Indiscernibles Lemma (still in $\boldsymbol{V}[G]$) to deduce that $\tilde{\mathcal{A}}$ has a set of g.i's in $\boldsymbol{K}''$ unbounded in λ. Then the argument then finishes as above. This completes (i).

For (ii) note that the assumption implies that in the coiteration of N_0 with M we must have on a club subset of κ, C say, for $i < j \in C$ implies that $\sigma_{i,j}(\kappa_i) = j = \kappa_j$, and that $\sigma_{i,j}(E_{\nu_i}) = E_{\nu_j}$. In other words that the same extender and its images are used, and moreover these extenders and their images must be of Mitchell order 0: hence for such $i < j \in C$ we have $M \models$ "i is not measurable", and N_κ, $M_\kappa = M'$ model the same statements. Since the same order zero measure (extender) and its images is being used at the class C of its critical points, then C forms a class of g.i's for M_κ, and in particular for $\mathcal{A}'$. As each $i \in C$ is a fixed point for π, then pulling back C via the elementary map π just shows that $C = \pi^{-1}$"C forms a set of g.i.'s for $\bar{\mathcal{A}} = \pi^{-1}(\mathcal{A}')$, and hence j"C does so for $\mathcal{A}$, as required. □

3. Hierarchies of Ramsey and Jónsson Cardinals

In this section we describe and expand upon the theory of α-Ramsey cardinals due to Feng. We first in 3.1.1 extend the notion of Π^1_n-indescribability to a transfinite notion Π^1_α one, by *via* finite games. We use this to extend a result of Feng's then from n-Ramsey to α-Ramsey (Theorem 3.24). In Sect 3.1.2 we introduce an analogue, α-Jónsson cardinals, and relate them to α-Ramsey cardinals; answer in Sect. 3.2.2 a question about the the lengths of the hierarchy of filters associated with α-Ramsey cardinals; and, in the final subsection 3.2.3, consider a natural variant of the notion of an α-Ramsey cardinal which we call an α-very Ramsey cardinal.

3.1. The Cardinal Hierarchies

3.1.1. α-Ramsey Cardinals

The following definitions are due to Feng [12] after Baumgartner [2].

Definition 3.1. **(Feng)** *Let κ be a cardinal of uncountable cofinality. Let $\boldsymbol{I}$ be an ideal on κ containing all the bounded subsets of κ. For $X \in P(\kappa)$, say $X \in \mathcal{R}^+(\boldsymbol{I})$ iff for all club $C \subseteq \kappa$ and regressive $f : [X]^{<\omega} \to \kappa$ there exists $Y \in \boldsymbol{I}^+$ such that $Y \subseteq X \cap C$ and Y is homogeneous for f (where $\boldsymbol{I}^+ = P(\kappa)\backslash\boldsymbol{I}$).*

Let $\mathcal{R}(\boldsymbol{I}) = P(\kappa)\backslash\mathcal{R}^+(\boldsymbol{I})$.

[12] gives some useful equivalents:

Theorem 3.2. **(Feng)** *Let κ, $\boldsymbol{I}$ be as in Definition 3.1. Then $X \in \mathcal{R}^+(\boldsymbol{I})$ iff for any κ-structure $\mathcal{A} = \langle J_\kappa^{X,\vec{A}}, \in, X, \vec{A}\rangle$ there exists $Y \subseteq X$, $Y \in \boldsymbol{I}^+$ such that Y is a set of good indiscernibles for $\mathcal{A}$.*

If $\mathrm{NS}_\kappa \subseteq \boldsymbol{I}$ then $X \in \mathcal{R}^+(\boldsymbol{I})$ iff for all regressive $f : [X]^{<\omega} \to \kappa$ there exists $Y \in \boldsymbol{I}^+$ such that $Y \subseteq X$ and Y is homogeneous for f, i.e. one can omit the club C of Definition 3.1.

We prove a further equivalence:

Theorem 3.3. *(The case $X = \kappa, \boldsymbol{I} = [\kappa]^{<\kappa}$ is due to Mitchell, [18])*

Let κ, $\boldsymbol{I}$ be as in Definition 3.1. The following are equivalent:

a) $X \in \mathcal{R}^+(\boldsymbol{I})$

b) for any $A \subseteq \kappa$ there is a useful structure M and $Y \in \boldsymbol{I}^+$ such that $A \in M$, $Y \subseteq X$, $\lambda(M) = \kappa$, and $U(M)$ is generated by Y

Proof. ($a \Rightarrow b$) Let $Y \in \boldsymbol{I}^+$ be a set of good indiscernibles for $\mathcal{A} = \langle J_\kappa^{X,A}, \in, X, A\rangle$ such that $Y \subseteq X$. Let $M = M(\mathcal{A}, Y)$. Then $U(M)$ is generated by Y. The rest follows since $\kappa = \sup(Y) = \lambda(M)$.

($b \Rightarrow a$) Let $f : [X]^{<\omega} \to \kappa$ be regressive and $C \subseteq \kappa$ be club. Let M be a useful structure such that $f, C \in M$ ($\langle f, C\rangle$ maybe coded as a subset of κ) with $Y \in \boldsymbol{I}^+$ such that $Y \subseteq X$, $\lambda(M) = \kappa$, and $U(M)$ is generated by Y. $C \in U$ since $C \in M$ and U is normal. So $X \cap C \in U$ and $f' = f \restriction [X \cap C]^{<\omega} \in M$. f' is regressive so let $Y' = \bigcap_{0<n<\omega} Y_n^{f',M}$. Since each $Y_n^{f',M}$ contains a final segment of Y, and $\mathrm{cf}(\kappa) > \omega$, $Y' \supseteq Y\backslash\alpha$, some $\alpha < \kappa$. Hence, $Y' \in \boldsymbol{I}^+$, since $\alpha \in \boldsymbol{I}$ by assumption. □

For the sake of the exposition, the following two definitions differ slightly from those of [12].

Definition 3.4. **(Feng)** *Let κ be as above. Let*

$\boldsymbol{I}_{-1}^\kappa = [\kappa]^{<\kappa}$

$\boldsymbol{I}_{\alpha+1}^\kappa = \mathcal{R}(\boldsymbol{I}_\alpha^\kappa)$

$\boldsymbol{I}_\delta^\kappa = \bigcup_{\alpha<\delta} \boldsymbol{I}_\alpha^\kappa$ *for δ a limit ordinal.*

Say κ is α-Ramsey if $\kappa \notin \boldsymbol{I}_\alpha^\kappa$. More generally, say $X \subseteq \kappa$ is α-Ramsey if $X \notin \boldsymbol{I}_\alpha^\kappa$.

Definition 3.5. **(Feng)** *Let κ be as above. Let*

$\overline{\boldsymbol{I}}_{-1}^\kappa = \mathrm{NS}_\kappa$

$\overline{\boldsymbol{I}}_{\alpha+1}^\kappa = \mathcal{R}(\overline{\boldsymbol{I}}_\alpha^\kappa)$

$\overline{\boldsymbol{I}}_\delta^\kappa = \bigcup_{\alpha<\delta} \overline{\boldsymbol{I}}_\alpha^\kappa$ *for δ a limit ordinal.*

$X \subseteq \kappa$ is α-Ramsey^s if $X \notin \overline{\boldsymbol{I}}_\alpha^\kappa$. (The superscript "s" stands for "stationary".)

Remark 3.6. *Note that for all $n < \omega$, $\boldsymbol{I}_n^\kappa \subseteq \overline{\boldsymbol{I}}_n^\kappa \subseteq \boldsymbol{I}_{n+1}^\kappa$ so for all $\alpha \geqslant \omega$, κ is α-Ramsey iff κ is α-Ramsey^s. That is, the definitions differ only for $\alpha < \omega$. Note 0-Ramsey is the same as Ramsey.*

Definition 3.7. *Let κ be a cardinal of uncountable cofinality. For $X \subseteq \kappa$ and $\alpha \in \mathrm{On}$, let $G_R(X, \alpha)$ be the game with finitely many moves defined as follows:*

Set $D_{-1} = X, \alpha_{-1} = 1 + \alpha$

In the nth move
I chooses $\alpha_n < \alpha_{n-1}$ and some $\mathcal{A}_n = \langle J_\kappa^{D_{n-1},\vec{A}^n}, \in, D_{n-1}, \vec{A}^n\rangle$
II chooses $D_n \in [\kappa]^\kappa such that D_n \subseteq D_{n-1}$ and D_n is a set of good indiscernibles for $\mathcal{A}_n$
The first player to be unable to move loses.

Lemma 3.8. *The game $G_R(X, \alpha)$ is determined.*

Proof. We use the usual Gale-Stewart argument, [14]. Assume *I* does not have a winning strategy. Then for each move *I* makes, *II* can choose a move such that *I* does not have a winning strategy for the remaining game. This is a winning strategy for *II* since α_n must reach 0 in finitely many moves and then *I* cannot move. □

Lemma 3.9. *Let κ be a cardinal,* $\mathrm{cf}(\kappa) > \omega$, *and $X \subseteq \kappa, \alpha \in \mathrm{On}$.*
Then X is α-Ramsey iff (II) wins $G_R(X, \alpha)$.

Proof. By induction on α. The case $\alpha = 0$ is clear from the definitions and Theorem 3.2, and the limit case is trivial.

Suppose $\alpha = \beta + 1$. Suppose first that X is α-Ramsey. If *I* chooses $\alpha_0 < \beta$ then the result follows by induction hypothesis since X is β-Ramsey. Suppose $\alpha_0 = \beta$. Then, by Theorem 3.2, *II* can choose a β-Ramsey D_0 but then by induction hypothesis *II* wins $G_R(D_0, \beta)$. However this is exactly what remains of the game $G_R(X, \alpha)$.

Now suppose *II* wins $G_R(X, \alpha)$. Then let $\mathcal{A} = \left\langle J_\kappa^{X,\vec{A}}, \in, X, \vec{A}\right\rangle$. Let D_0 be *II*'s response according to its winning strategy to *I* playing $\alpha_o = \beta, \mathcal{A}_0 = \mathcal{A}$. Then *II* wins $G_R(D_0, \beta)$, hence, by induction hypothesis, D_0 is β-Ramsey. So $X \in \mathcal{R}^+(\boldsymbol{I}_\beta^\kappa) = (\boldsymbol{I}_\alpha^\kappa)^+$ as required. □

Definition 3.10. *Define the game $G_R^s(X, \alpha)$ just as $G_R(X, \alpha)$ except that II must play $D_n \in \mathrm{NS}_\kappa$ instead of $D_n \in [\kappa]^\kappa$.*

Remark 3.11. *By the same proof as Lemma 3.9, κ is α-Ramsey^s iff II wins $G_R^s(X, \alpha)$.*

Lemma 3.12. *Let κ be a cardinal,* $\mathrm{cf}(\kappa) > \omega$. *For $X \subseteq \kappa$ and $\alpha \in \mathrm{On}$, the game with finitely many moves defined below is equivalent to $G_R(X, \alpha)$ in the sense that the same player wins.*
Set $Y_{-1} = X, \alpha_{-1} = 1 + \alpha$
In the nth move
I chooses $\alpha_n < \alpha_{n-1}$ and some $A_n \subseteq \kappa$
II chooses a useful structure $M_n \ni M_{n-1}$ with $|M_n| = \kappa$ and a set $Y_n \subseteq Y_{n-1}$ such that $A_n \in M_n$, $\lambda(M) = \kappa$, and $U(M)$ is generated by Y_n.
The first player to be unable to move loses.
We denote this game by $G_R(X, \alpha)$ without fear of confusion.

Proof. Just as for Theorem 3.3. Note only that M_n can be chosen to contain M_{n-1} since M_{n-1} has cardinality κ and can be coded as a subset of κ. □

Definition 3.13. *Let β be an ordinal. Call a pair $\langle M, Y\rangle$ β-*Ramsey *(for κ, X) iff $H_\kappa \subseteq M$ and $p = \langle\langle\beta, \kappa\rangle, \langle M, Y\rangle\rangle$ is a valid (partial) play of $G_R(X, \alpha)$ (in the sense of Lemma 3.12) and is a winning position for II for some (equivalently, any) α with $1 + \alpha > \beta$.*

If $\langle M', Y'\rangle$ and $\langle M, Y\rangle$ are 0-Ramsey say that $\langle M', Y'\rangle$ extends $\langle M, Y\rangle$ iff $M \in M'$ and $Y \supseteq Y'$.

Remark 3.14. *Let κ be a cardinal, $\mathrm{cf}(\kappa) > \omega$. Then κ is α-Ramsey iff for every $A \subseteq \kappa$ and $\beta < \alpha$, there is a $(1 + \beta)$-Ramsey pair $\langle M, Y\rangle$ with $A \in M$ (here, we are allowing $\beta \in \{-1\} \cup \mathrm{On}$).*

The requirement that $H_\kappa \subseteq M$ is not necessary. However, since κ will always be inaccessible, $|H_\kappa| = \kappa$, and in the game $G_R(X, \alpha)$, H_κ could be coded in A_0, anyway. So there will not be a genuine loss of generality, but adopting this convention will help to streamline some of the following proofs.

We shall shortly show that Π^1_{2n}-formulae are absolute for n-Ramsey structures, and obtain an indescribability result using Łoś Theorem. However, to state this in the maximum generality it is necessary to first generalise the notion of absoluteness from formulae to games. As β increases, (ultrapowers of) β-Ramsey structures will correctly determine the winner of an increasing class of games.

Definition 3.15. *Let ζ be an ordinal and let $a_0, \dots, a_{k-1} \subseteq \zeta$ and $\Phi(v_0, \dots, v_{k+2})$ be a Σ_0-formula (without parameters). Let $G_\zeta(\Phi, \alpha, a_0, \dots, a_{k-1})$ be the game with two players, (Π) and (Σ), and finitely many moves such that , setting $\alpha_{-1} = \alpha$, in the nth move:*

(Σ) chooses some $\alpha_n < \alpha_{n-1}$ and $A_n \subseteq \kappa$

(Π) chooses some $D_n \subseteq \kappa$ such that $\Phi(\zeta, \overline{A}_n, \overline{D}_n, a_0, \dots, a_{k-1})$ where $\overline{D}_n = \langle D_k : k \leqslant n\rangle$ and $\overline{A}_n = \langle A_k : k \leqslant n\rangle$

The first player to be unable to move loses.

Definition 3.16. *Let $\alpha > 0$. Say that a property $Q(\zeta, a_0, \dots, a_{k-1})$ where ζ ranges over ordinals and $a_0, \dots, a_{k-1}$ range over subsets of ζ is*

- $\Pi^1_{2\cdot\alpha}$ *if there is Φ such that for all ζ and $a_0, \dots, a_{k-1} \subseteq \zeta$, $Q(\zeta, a_0, \dots, a_{k-1})$ holds iff (Π) wins $G_\zeta(\Phi, \alpha, a_0, \dots, a_{k-1})$*
- $\Sigma^1_{2\cdot\alpha}$ *if there is Φ such that for all ζ and $a_0, \dots, a_{k-1} \subseteq \zeta$, $Q(\zeta, a_0, \dots, a_{k-1})$ holds iff (Σ) wins $G_\zeta(\Phi, \alpha, a_0, \dots, a_{k-1})$*
- $\Sigma^1_{2\cdot\alpha+1}$ *if there is Φ such that for all ζ and $a_0, \dots, a_{k-1} \subseteq \zeta$, $Q(\zeta, a_0, \dots, a_{k-1})$ holds iff there exists $X \subseteq \zeta$ such that (Π) wins $G_\zeta(\Phi, \alpha, a_0, \dots, a_{k-1}, X)$*
- $\Pi^1_{2\cdot\alpha+1}$ *if there is Φ such that for all ζ and $a_0, \dots, a_{k-1} \subseteq \zeta$, $Q(\zeta, a_0, \dots, a_{k-1})$ holds iff for all $X \subseteq \zeta$, (Σ) wins $G_\zeta(\Phi, \alpha, a_0, \dots, a_{k-1}, X)$*

Remark 3.17. *If $0 < n < \omega$ then the notion of a Π^1_{2n} property given in Definition 3.16 is the same as the standard one. Note also that the game of Definition 3.15 is always determined and "X is α-Ramsey" is a $\Pi^1_{2\cdot(1+\alpha)}$ property, by Lemma 3.9.*

Definition 3.18. *If $G = G_\zeta(\Phi, \alpha, a_0, \dots, a_{k-1})$, and M is a transitive ZF^--model with $\zeta, a_0, \dots, a_{k-1} \in M$, define the relativisation, G^M, of G to M to be the game defined exactly as in Definition 3.15, but with the constraint that $A_n, D_n \in M$, for all n.*

Remark 3.19. *If $G_\kappa(\Phi, \alpha, a_0, \dots, a_{k-1}) = G_\kappa(\Phi', \alpha, a'_0, \dots, a'_{k-1})$ then $G_\kappa(\Phi, \alpha, a_0, \dots, a_{k-1})^M = G_\kappa(\Phi', \alpha, a'_0, \dots, a'_{k-1})^M$ by absoluteness of the Σ_0-formulae Φ and Φ', so G^M is well-defined.*

G^M is not necessarily a relativisation in the sense that all (partial) plays of G^M are in M: the ordinals α_n played by (Σ) may lie outside M. Intuitively, the ordinals should be understood as relating to the number of quantifiers. It is only the scope of these "quantifiers" that is restricted to M.

The following lemma shows that the "truth" of G^M, like that of Φ^M for a Π^1_n-formula Φ, depends only on G (resp. Φ) and M, not on the model in which it is evaluated.

Lemma 3.20. *Let* $G = G_\zeta(\Phi,\alpha,a_0,\dots,a_{k-1})$ *and* M *be as in Definition 3.18, and let* $N \ni \alpha, a_0,\dots,a_{k-1}, M$ *be a transitive model of* ZF^- *such that*

$$N \models (\exists v,w)(v = G_\zeta(\Phi,\alpha,a_0,\dots,a_{k-1})^M \wedge w =^v v)$$

Then, setting $\tilde{G} = G^M$,

$$N \models (\tilde{G} = G_\zeta(\Phi,\alpha,a_0,\dots,a_{k-1})^M \wedge \tilde{G} \text{ is determined })$$

and,

$$(\Pi) \text{ wins } G^M \quad \text{iff} \quad N \models (\Pi) \text{ wins } G^M$$

Proof. It is clear from Definitions 3.15 and 3.18 that the property "p is a play of G^M" is Σ_0 in parameters $\alpha, a_0,\dots,a_{k-1}$, and M, using the standard fact that "p is a finite sequence" is Σ_0 (see, e.g. [6]). Hence, $N \models \tilde{G} = G_\zeta(\Phi,\alpha,a_0,\dots,a_{k-1})^M$, since $G^M \subseteq N$. Since $N \models (\exists v,w)(v = \tilde{G} = G_\zeta(\Phi,\alpha,a_0,\dots,a_{k-1})^M \wedge w =^v v)$, the proof that G^M is determined (just as in 3.8) may be carried out in N.

If $N \models (\Sigma)$ wins G^M then let $\sigma \in N$ be a winning strategy for (Σ) in N. Then σ really is a winning strategy for (Σ), so (Σ)winsG^M. Similarly, $N \models (\Pi)$ wins G^M implies that (Π) wins G^M. However G^M is determined in N so this suffices to show that (Π)winsG^M iff $N \models (\Pi)$ wins G^M. □

Definition 3.21. *A transitive* ZF^-*-model* M *with* $\zeta \in M$ *is* $\Pi^1_{2\cdot\alpha}$correct (for ζ) *if for every* Σ_0 *formula* Φ, *and* $a_0,\dots,a_{k-1} \in P(\zeta) \cap M$,

$$(\Pi) \text{ wins } G_\zeta(\Phi,\alpha,a_0,\dots,a_{k-1}) \text{ iff } (\Pi) \text{ wins } G_\zeta(\Phi,\alpha,a_0,\dots,a_{k-1})^M.$$

Let $\langle f_\varepsilon : \varepsilon < \kappa^+\rangle$ *be a sequence of canonical functions for an uncountable cardinal* κ *and suppose* $\alpha < \kappa^+$. κ *is* $\Pi^1_{2\cdot\alpha}$-indescribable *if for every* Σ_0 *formula* Φ, *and* $a_0,\dots,a_{k-1} \subseteq \kappa$, *if*

(Π) *wins* $G_\kappa(\Phi,\alpha,a_0,\dots,a_{k-1})$ *then* (Π) *wins* $G_\zeta(\Phi,f_\alpha(\zeta),a_0\cap\zeta,\dots,a_{k-1}\cap\zeta)$ *for some* $\zeta < \kappa$.

Similarly for $\Sigma^1_{2\cdot\alpha}$, $\Pi^1_{2\cdot\alpha+1}$, *and* $\Sigma^1_{2\cdot\alpha+1}$.

Remark 3.22. *The preceding definition agrees with usual notion of* $\Pi^1_{2\cdot n}$*-indescribable, for* $n < \omega$. *Also note that the choice of the sequence* $\langle f_\varepsilon : \varepsilon < \kappa^+\rangle$ *is not important, as follows: Let* $\langle f'_\varepsilon : \varepsilon < \kappa^+\rangle$ *be some other sequence of canonical functions and suppose that for some* $\alpha < \kappa^+$, *for every* Σ_0 *formula* Φ *and* $a_0,\dots,a_{k-1} \subseteq \kappa$, *if* (Π) *wins* $G_\kappa(\Phi,\alpha,a_0,\dots,a_{k-1})$ *then* (Π) *wins* $G_\zeta(\Phi,f_\alpha(\zeta),a_0\cap\zeta,\dots,a_{k-1}\cap\zeta)$ *for some* $\zeta < \kappa$. *Fix* Φ *and* $a_0,\dots,a_{k-1} \subseteq \kappa$, *and let* C *be a club in* κ *such that for all* $\zeta \in C$, $f_\alpha(\zeta) = f'_\alpha(\zeta)$. *Then* (Π) *wins* $G_\kappa(\Phi \wedge (v_k \text{ is } unbounded\ ),\alpha,a_0,\dots,a_{k-1},C)$, *so there is* $\zeta < \kappa$ *such that:*

$$(\Pi) \text{ wins } G_\zeta(\Phi \wedge (v_k \text{ is unbounded }), f_\alpha(\zeta), a_0\cap\zeta,\dots,a_{k-1}\cap\zeta, C\cap\zeta,\zeta).$$

However then $\zeta \in \lim(C)\cap\kappa \subseteq C$, *so* $f'_\alpha(\zeta) = f_\alpha(\zeta)$, *so* (Π) *wins* $G_\zeta(\Phi,f'_\alpha(\zeta),a_0\cap\zeta,\dots,a_{k-1}\cap\zeta)$.

Lemma 3.23. *Suppose* $\langle M,Y\rangle$ *is* β*-Ramsey for* $X \subseteq \kappa$, *and* $\tilde{M} = \mathrm{Ult}(M,U), U = U(M)$. *If* $G = G_\kappa(\Phi,1+\beta,a_0,\dots,a_{k-1})$, $a_0,\dots,a_{k-1} \in M$, *and* $\beta \in M$ *then the following are equivalent:*

i. (Π) *wins* G*;*

ii. (Π) *wins* $G^{\tilde{M}}$*;*

iii. (Π) *wins* G^M*;*

i.e., M *and* $\tilde{M}$ *are* $\Pi^1_{2\cdot\beta}$ *(equivalently,* $\Sigma^1_{2\cdot\beta}$*) correct.*

Proof. By amenability of the measure U, the games G^M and $G^{\tilde{M}}$ are the same, so the equivalence of ii and iii is trivial.

We show that i is equivalent to ii by induction on β. The case $\beta = 0$ is immediate.

Suppose β is a limit ordinal. Then,

$$\begin{aligned}(\Pi) \text{ wins } G \quad &\Leftrightarrow \quad (\Pi) \text{ wins } G_\kappa(\Phi, \alpha, a_0, \ldots, a_{k-1}) \text{ for all } \alpha < \beta \\ &\Leftrightarrow \quad (\Pi) \text{ wins } (G_\kappa(\Phi, \alpha, a_0, \ldots, a_{k-1}))^{\tilde{M}} \text{ for all } \alpha < \beta \text{ (induction hypothesis)} \\ &\Leftrightarrow \quad (\Pi) \text{ wins } G^{\tilde{M}}\end{aligned}$$

Now suppose $\beta = \gamma + 1$ for some γ.

Suppose (Π) wins G. Let (γ, A) be any first move for (Σ) in $G^{\tilde{M}}$. Then (γ, A) is also a valid first move for (Σ) in G so let B be (Π)'s response according to its winning strategy for G.

Let $\vec{a} = a_0, \ldots, a_{k-1}$ and set $\underline{G}$ to be the $\Pi^1_{2\cdot\gamma}$ game derived from G, A, B in the obvious way, i.e. $\underline{G} = G_\kappa(\Psi, \gamma, \vec{a}, A, B)$ where $\Psi(\kappa, \overline{A}, \overline{D}, \vec{a}, A, B)$ holds iff $\Phi(\kappa, \langle A\rangle \frown \overline{A}, \langle B\rangle \frown \overline{D}, \vec{a})$ does. By choice of B, (Π) wins $\underline{G}$.

Let $\langle N, Z\rangle$ be γ-Ramsey such that $A, B \in N$ and $\langle N, Z\rangle$ extends $\langle M, Y\rangle$, and let $\tilde{N} = \mathrm{Ult}(N, W)$ where $W = U(N)$. By induction hypothesis, (Π) wins $\underline{G}^{\tilde{N}}$. Let $\tilde{G} = \underline{G}^{\tilde{N}}$. Then $\tilde{G} \in \tilde{N}$ and $\tilde{N} \models G_\kappa(\Psi, \gamma, \vec{a}, A, B) = \tilde{G}$. By Lemma 3.20, $\tilde{N} \models (\Pi)\,\mathrm{wins}\, G_\kappa(\Psi, \gamma, \vec{a}, A, B)$ so, by Łoś Theorem, using $H_\kappa \subseteq N$,

$$\{\alpha < \kappa : (\exists u)((\Pi) \text{ wins } G_\alpha(\Psi, f_\gamma(\alpha), a_0 \cap \alpha, \ldots, a_{k-1} \cap \alpha, A \cap \alpha, u))\} \in W \cap M \subseteq U,$$

where $[f_\gamma]_{\tilde{N}} = [f_\gamma]_{\tilde{M}} = \gamma$ (such a function exists since $\gamma \in M \subseteq N$). Note that $W \cap M \subseteq U$ because $Z \subseteq Y$ are the respective generating sets.

By Łoś Theorem again, using $H_\kappa \subseteq M$, $\tilde{M} \models (\exists u)((\Pi)\,\mathrm{wins}\, G_\kappa(\Psi, \gamma, \vec{a}, A, u))$. Take $B' \in \tilde{M}$ such that $\tilde{M} \models (\Pi)\,\mathrm{wins}\, G_\kappa(\Psi, \gamma, \vec{a}, A, B')$, and note that

$$G_\kappa(\Psi, \gamma, \vec{a}, A, B') \in \tilde{M} \text{ and } \tilde{M} \models G_\kappa(\Psi, \gamma, \vec{a}, A, B') = G', \text{ where } G' = G_\kappa(\Psi, \gamma, \vec{a}, A, B')^{\tilde{M}}.$$

By Lemma 3.20, (Π) wins $G_\kappa(\Psi, \gamma, \vec{a}, A, B')^{\tilde{M}}$. By definition of Ψ, this suffices to show that (Π) wins $G^{\tilde{M}}$ when it plays B' in response to (γ, A).

The converse is similar. Briefly: Suppose (Σ) wins G. Let $\langle N, Z\rangle$ extending $\langle M, Y\rangle$ be such that there is $A \in N$ such that for all B, (Σ) wins $\underline{G} = G_\kappa(\Psi, \gamma, \vec{a}, A, B)$, where $\Psi(\kappa, \overline{A}, \overline{D}, \vec{a}, A, B)$ holds iff $\Phi(\kappa, \langle A\rangle \frown \overline{A}, \langle B\rangle \frown \overline{D}, \vec{a})$ does. We may proceed as before to show that

$$\tilde{M} \models (\exists A)(\forall B)((\Sigma) \text{ wins } \underline{G} = G_\kappa(\Psi, \gamma, \vec{a}, A, B)),$$

and hence $\tilde{M} \models ((\Sigma) \text{ wins } G^{\tilde{M}})$. □

Corollary 3.24. *(The case* $\beta < \omega$ *is due to Feng.)*

If κ *is a* β*-Ramsey cardinal then* κ *is* $\Pi^1_{2\cdot(1+\gamma)+1}$*-indescribable for each* $\gamma < \min\{\beta, \kappa^+\}$*.*

Proof. Let κ be a β-Ramsey cardinal and let Φ and $\vec{a} \subseteq \kappa$ be such that (Σ) wins $G = G_\zeta(\Phi, \beta, \vec{a}, X)$ for all $X \subseteq \kappa$. Since κ is β-Ramsey, there is a $(1+\gamma)$-Ramsey $\langle M, Y\rangle$ such that $\beta, \vec{a} \in M$. Let $\tilde{M} = \mathrm{Ult}(M, U(M))$. By Lemma 3.23, $\tilde{M}$ is $\Sigma^1_{2\cdot(1+\gamma)}$ correct. So for each $X \in P(\kappa) \cap M$, (Σ) wins $G_\zeta(\Phi, \beta, \vec{a}, X)^M$. By Lemma 3.20,

$$\tilde{M} \models (\forall X \subseteq \kappa)((\Sigma) \text{ wins } G_\zeta(\Phi, \beta, \vec{a}, X)).$$

The result now follows by Łoś Theorem. □

Corollary 3.25. **(Feng)** *Suppose κ is β-Ramsey and $\gamma < \min\{\beta, \kappa^+\}$. Then there is a $f_\gamma(\zeta)$-Ramsey cardinal $\zeta < \kappa$, where $\langle f_\varepsilon : \varepsilon < \kappa^+\rangle$ is any sequence of canonical functions for κ.*

Proof. The property "λ is γ-Ramsey" is $\Pi^1_{2\cdot(1+\gamma)}$ over λ. □

3.1.2. α-Jónsson Cardinals

The following theorem establishes the equiconsistency of Ramsey and Jónsson cardinals.

Theorem 3.26. (Mitchell;[18]) *Suppose there is no inner model for a measurable cardinal, and κ is a Jónsson cardinal. Then $\boldsymbol{K} \models$ "κ is Ramsey."*

Mitchell also proved, [20], that, under the much weaker assumption that there is no inner model for a Woodin cardinal, any regular Jónsson cardinal is Ramsey in the Steel core model $\boldsymbol{K}$, if this exists.

With the aim of proving similar theorems for α-Ramsey cardinals, we make some definitions.

Definition 3.27. *Let κ be a cardinal with uncountable cofinality. Let $\boldsymbol{I}$ be an ideal over κ containing all the bounded subsets of κ. For $X \in P(\kappa)$, say $X \in \mathcal{J}^+(\boldsymbol{I})$ iff for all club $C \subseteq \kappa$ and κ-structures $\mathcal{A} = \langle J_\kappa^{X,\vec{A}}, \in, X, \vec{A}\rangle$ there exists $Y \in \boldsymbol{I}^+$ such that $Y \subseteq X \cap C$ and $\mathrm{Hull}^{\mathcal{A}}(Y) \cap \mathrm{On} \neq \kappa$.*

Let $\mathcal{J}(\boldsymbol{I}) = P(\kappa) \backslash \mathcal{J}^+(\boldsymbol{I})$.

Lemma 3.28. *Let κ, $\boldsymbol{I}$ be as in Definition 3.27. For $X \in P(\kappa)$, $X \in \mathcal{J}^+(\boldsymbol{I})$ iff*

for all club $C \subseteq \kappa$ and α-structures $\mathcal{A} = \langle J_\alpha^{X,\vec{A}}, \in, X, \vec{A}\rangle$ with $\alpha \geqslant \kappa$ there exists $Y \in \boldsymbol{I}^+$ such that $Y \subseteq X \cap C$ and $\mathrm{Hull}^{\mathcal{A}}(Y) \cap \kappa \neq \kappa$.

Proof. To prove the nontrivial implication, let $X \in \mathcal{J}^+(\boldsymbol{I})$ and let $\mathcal{A} = \langle J_\alpha^{X,\vec{A}}, \in, X, \vec{A}\rangle$ with $\alpha \geqslant \kappa$.

The set $C = \{\gamma < \kappa : \mathrm{Hull}^{\mathcal{A}}(\gamma) \cap \kappa = \gamma\}$ is closed unbounded. For any $\gamma, \delta \in C$ with $\gamma \leqslant \delta$, let $\pi_\gamma : \mathcal{A}_\gamma \prec \mathcal{A}$ be the inverse transitive collapse of $\mathrm{Hull}^{\mathcal{A}}(\gamma)$ and $\pi_{\gamma\delta} = \pi_\delta^{-1} \circ \pi_\gamma : \mathcal{A}_\gamma \prec \mathcal{A}_\delta$. Let $\mathcal{A}' = \langle J_\kappa^{C,\Pi,A}, \in, C, \Pi, A\rangle$ where $\Pi = \langle \pi_{\gamma\delta} : \gamma, \delta \in C, \gamma \leqslant \delta\rangle$ and $A = \langle \mathcal{A}_\gamma : \gamma \in C\rangle$.

Since $X \in \mathcal{J}^+(\boldsymbol{I})$ there is $Y \subseteq X \cap C$ such that $\mathrm{Hull}^{\mathcal{A}'}(Y) \cap \kappa \neq \kappa$. Set $\mathcal{B} = \mathrm{Hull}^{\mathcal{A}}(Y)$. It is sufficient to show that $\mathcal{B} \cap \kappa = \mathrm{Hull}^{\mathcal{A}'}(Y) \cap \kappa$. Let $\varepsilon = f_\varphi^{\mathcal{A}}(a_0, \dots, a_{n-1}) \in \mathcal{B} \cap \kappa$, where $f_\varphi^{\mathcal{A}}$ is one of the (uniformly definable) Skolem functions of $\mathcal{A}$, and $a_0, \dots, a_{n-1} \in Y$. $C \cap \mathrm{Hull}^{\mathcal{A}'}(Y)$ is unbounded in κ by elementarity, so there is $\delta \in C \cap \mathrm{Hull}^{\mathcal{A}'}(Y)$ such that $\varepsilon, a_0, \dots, a_{n-1} < \delta$. However then

$$\varepsilon = \pi_\delta^{-1}(\varepsilon) = \pi_\delta^{-1}(f_\varphi^{\mathcal{A}}(a_0, \dots, a_{n-1})) = f_\varphi^{\mathcal{A}_\delta}(\pi_\delta^{-1}(a_0), \dots, \pi_\delta^{-1}(a_{n-1}))$$
$$= f_\varphi^{\mathcal{A}_\delta}(a_0, \dots, a_{n-1}) \in \mathrm{Hull}^{\mathcal{A}'}(Y) \qquad \text{since } \delta, a_0, \dots, a_{n-1} \in \mathrm{Hull}^{\mathcal{A}'}(Y).$$ □

Definition 3.29. *Let κ be as above. Let*

$\boldsymbol{J}^\kappa_{-1} = [\kappa]^{<\kappa}$

$\boldsymbol{J}^\kappa_{\alpha+1} = \mathcal{J}(\boldsymbol{J}^\kappa_\alpha)$

$\boldsymbol{J}^\kappa_\delta = \bigcup_{\alpha<\delta} \boldsymbol{J}^\kappa_\alpha$ *for δ a limit ordinal*

Say κ is α-Jónsson if $\kappa \notin \boldsymbol{J}^\kappa_\alpha$, and $X \subseteq \kappa$ is α-Jónsson if $X \notin \boldsymbol{I}^\kappa_\alpha$.

Definition 3.30. *Let κ be as above. For $X \subseteq \kappa$ and $\alpha \in \mathrm{On}$, let $G_J(X,\alpha)$ be the game with finitely many moves defined as follows:*

Set $D_{-1} = X, \alpha_{-1} = 1+\alpha$. In the n'th round:

I chooses $\alpha_n < \alpha_{n-1}$, a cub $C_n \subseteq \kappa$ and some κ-structure $\mathcal{M}_n = \langle J_\kappa^{D_{n-1},\vec{A}^{n-1}}, \in, D_{n-1}, \vec{A}^{n-1}\rangle$;
II chooses $D_n \in [\kappa]^\kappa$ with $D_n \subseteq D_{n-1} \cap C_n$ and $\mathrm{Hull}^{\mathcal{M}_n}(D_n) \neq \mathcal{M}_n$

A player who cannot move on his turn loses.

Lemma 3.31. *Let κ be a cardinal of uncountable cofinality and $X \subseteq \kappa, \alpha \in \mathrm{On}$. The game $G_J(X,\alpha)$ is determined, and κ is α-Jónsson iff II wins $G_J(X,\alpha)$.*

Proof. As for the α-Ramsey case. □

Definition 3.32. *Define 'α-Jónsson^s', $\overline{\boldsymbol{J}}^\kappa_\alpha$, and '$G^s_J(X,\alpha)$' analogously to their Ramsey counterparts.*

Remark 3.33. *'α-Jónsson^s', $\overline{\boldsymbol{J}}^\kappa_\alpha$, and '$G^s_J(X,\alpha)$' have equivalent properties to those stated in Remarks 3.6 and 3.11.*

3.2. More on α-Ramsey Cardinals

3.2.1. An equivalence of α-Ramsey^s and α-Jónsson^s in $\boldsymbol{K}$

Theorem 3.34. *Suppose $\neg 0^\P$, κ is regular and*
$\boldsymbol{K} \models \{\alpha < \kappa : \boldsymbol{K} \models \alpha$ is measurable$\}$ is non-stationary in κ.
If κ is an α-Jónsson^s cardinal then κ is an α-Ramsey^s cardinal in $\boldsymbol{K}$.

Proof. We induct on $\alpha \in \{-1\} \cup \mathrm{On}$ to show that $(\bar{\boldsymbol{J}}^\kappa_\alpha)^+ \cap \boldsymbol{K} \subseteq ((\bar{\boldsymbol{I}}^\kappa_\alpha)^{\boldsymbol{K}})^{+\boldsymbol{K}}$. It will then follow that if κ is α-Jónsson^s then $\kappa \in (\bar{\boldsymbol{J}}^\kappa_\alpha)^+ \cap \boldsymbol{K} \subseteq ((\bar{\boldsymbol{I}}^\kappa_\alpha)^{\boldsymbol{K}})^{+\boldsymbol{K}}$, so κ is an α-Ramsey^s cardinal in **K**.

The case $\alpha = -1$ is clear (the property of being a stationary set is Π_1), and the limit case follows immediately from the induction hypothesis.

Suppose $\alpha = \beta + 1$. Let $X \in (\bar{\boldsymbol{J}}^\kappa_\alpha)^+ \cap \boldsymbol{K} = \mathcal{J}^+(\bar{\boldsymbol{J}}^\kappa_\beta) \cap \boldsymbol{K}$.

Let $\mathcal{A} = \langle J_\kappa^{X,\vec{A}}, \in, X, \vec{A}\rangle \in \boldsymbol{K}$, and let $M' = \boldsymbol{K}|(\kappa^+)^{\boldsymbol{K}} \frown \langle \mathcal{A}\rangle$. By Lemma 3.28 for any cub C_0 there is $Y \in (\bar{\boldsymbol{J}}^\kappa_\beta)^+$ such that $Y \subseteq X \cap C_0$ and $\mathrm{Hull}^{M'}(Y) \cap \kappa \neq \kappa$. Fix such a Y. Let $j : M \cong \mathrm{Hull}^{M'}(Y) \prec \boldsymbol{K}|(\kappa^+)^{\boldsymbol{K}}$ be the inverse transitive collapse, and note $\mathcal{A} \in \mathrm{ran}(j)$.

By Theorem 2.27, there is a closed unbounded set $C \subseteq \kappa$ such that $C' =_{\mathrm{df}} j``C$ is a set of g.i.'s for $\mathcal{A}$. By Jensen Indiscernibles Lemma, there is $I \in \boldsymbol{K}, I \supseteq C'$ such that I is a set of g.i.'s for $\mathcal{A}$. Set $I' = I \cap X$. To conclude the proof it is sufficient to show that $I' \in (\bar{\boldsymbol{J}}^\kappa_\beta)^+ \cap \boldsymbol{K} \subseteq ((\bar{\boldsymbol{I}}^\kappa_\beta)^{\boldsymbol{K}})^{+\boldsymbol{K}}$ (where the inclusion is given by the induction hypothesis), for then I' witnesses that $X \in ((\bar{\boldsymbol{I}}^\kappa_\alpha)^{\boldsymbol{K}})^{+\boldsymbol{K}}$. Now:

$$I' = I \cap X \supseteq I \cap Y \supseteq C' \cap Y = (C' \cup \lim(C')) \cap \mathrm{ran}(j) \cap Y = (C' \cup \lim(C')) \cap Y.$$

The second equality holds since $C' = (C' \cup \lim(C')) \cap \mathrm{ran}(j)$, which in turn holds since C' is the image under j of the closed set C. However $(C' \cup \lim(C')) \cap Y \in (\bar{\boldsymbol{J}}^\kappa_\beta)^+$ since $Y \in (\bar{\boldsymbol{J}}^\kappa_\beta)^+$ and the club $(C' \cup \lim(C'))$ is in the dual of the normal ideal $(\bar{\boldsymbol{J}}^\kappa_\beta)$. So $I' \in (\bar{\boldsymbol{J}}^\kappa_\beta)^+ \cap \boldsymbol{K}$, as required.

The limit case is then trivial. □

Under a stronger anti-large cardinal assumption, it is possible to drop the assumption in Theorem 3.34 that κ is regular.

Corollary 3.35. *Suppose there in no inner model for a measurable cardinal. If κ is an α-Jónsson^s cardinal then κ is an α-Ramsey^s cardinal in **K**.*

Proof. In the paper [10], it is proved that under the assumption that there is no inner model for a measurable cardinal any Jónsson cardinal is weakly inaccessible. In particular, it is regular. The result now follows from Theorem 3.34. □

By showing that the Jónsson property is preserved under c.c.c.forcings, [5] proves that Jónsson cardinals are not necessarily Ramsey, since the latter are strongly inaccessible, whilst we can preserve the Jónsson property at κ whilst making $2^{\aleph_0} > \kappa$. We conclude this subsection by stating the same for α-Jónsson. The proof (which we omit) is essentially the same.

Lemma 3.36. *The statement "*κ is an α-Jónsson (α-Jónsson^s) cardinal*" is absolute for c.c.c. forcing. Hence if we suppose that κ is an α-Jónsson (α-Jónsson^s) cardinal, then there is a forcing extension in which κ remains α-Jónsson (α-Jónsson^s) but κ is not Ramsey (hence, not α-Ramsey or α-Ramsey^s).*

3.2.2. Completely Ramsey Cardinals

Definition 3.37. (Feng) *A regular uncountable cardinal is completely Ramsey iff it is α-Ramsey for all α. If κ is completely Ramsey then let θ_κ be the least ordinal α such that $\boldsymbol{I}^\kappa_{\alpha+1} = \boldsymbol{I}^\kappa_\alpha$.*

It is clear that θ_κ exists and is less than $(2^\kappa)^+$. In fact, an exact value may be found using the methods of [1]. That paper answers the same question for completely ineffable cardinals, and it will be useful to first introduce these cardinals and relate them to completely Ramsey cardinals.

Definition 3.38. ([2])
Let κ be a cardinal of uncountable cofinality. Let $\boldsymbol{I} \supseteq \mathrm{NS}_\kappa$ be an ideal. For $X \in P(\kappa)$, say $X \in \mathcal{Q}^+(\boldsymbol{I})$ iff for all $f : [X]^2 \to 2$ there exists $Y \in \boldsymbol{I}^+$ such that $Y \subseteq X$ and Y is homogeneous for f (where $\boldsymbol{I}^+ = P(\kappa)\backslash\boldsymbol{I}$).
Let $\mathcal{Q}(\boldsymbol{I}) = P(\kappa) \backslash \mathcal{Q}^+(\boldsymbol{I})$.

Definition 3.39. ([2])
Let κ be a cardinal of uncountable cofinality. Let
$Q^\kappa_{-1} = \mathrm{NS}_\kappa$
$Q^\kappa_{\alpha+1} = \mathcal{Q}(Q^\kappa_\alpha)$
$Q^\kappa_\delta = \bigcup_{\alpha<\delta} Q^\kappa_\alpha$ *for δ a limit ordinal*
Say κ is α-ineffable if $\kappa \notin Q^\kappa_\alpha$ and κ is completely ineffable if κ is α-ineffable for all α.

Remark 3.40. *It follows immediately from the definitions that any α-Ramsey^s cardinal is α-ineffable. Hence, any completely Ramsey cardinal is completely ineffable, by Remark 3.6.*

The key to the argument in [1] is a strong indescribability result, Lemma 3.42, which we shall be able to apply directly.

Notation 3.41. *Let α be an ordinal. Then $\mathcal{D}_\alpha$ is the smallest admissible transitive structure with $V_\alpha \in \mathcal{D}_\alpha$.*

Lemma 3.42. ([1]) *Suppose κ is completely ineffable, and $\mathcal{D}_{\kappa+1} \vDash \sigma(\kappa)$, where σ is a Σ_1-formula. Then there is $\alpha < \kappa$ such that $\mathcal{D}_{\alpha+1} \vDash \sigma(\alpha)$.*

Lemma 3.43. *Let κ be a completely Ramsey cardinal. Then $\theta_\kappa = \mathrm{On} \cap \mathcal{D}_{\kappa+1}$.*

Proof. We follow [1]. Let $\gamma = \gamma_\kappa = \mathrm{On} \cap \mathcal{D}_{\kappa+1}$.

Suppose first that $\gamma < \theta_\kappa$. Let $X \in \boldsymbol{I}^\kappa_{\gamma+1} \backslash \boldsymbol{I}^\kappa_\gamma$ and let $C \subseteq \kappa$ be a club and $f : [X]^{<\omega} \to \kappa$ a regressive function such that there is no $Y \in \boldsymbol{I}^{\kappa+}_\gamma$ with Y homogeneous for f. Define A and $m : A \to \gamma$ as follows:

$$A = \{Y \subseteq \kappa : Y \text{is homogeneous for } f\} \in \mathcal{D}_{\kappa+1}, \quad m(Y) = \min\{\delta < \gamma : Y \notin \boldsymbol{I}^{\kappa+}_\delta\}.$$

By admissibility of $\mathcal{D}_{\kappa+1}$, $\bigcup_{Y \in A} m(Y) < \gamma$. However this contradicts $X \in \boldsymbol{I}^\kappa_\gamma$, so $\theta_\kappa \leqslant \gamma$.

Now suppose $\theta_\kappa < \gamma_\kappa$. Without loss of generality we may assume that κ is the least completely Ramsey cardinal for which this occurs. Then $\mathcal{D}_{\kappa+1} \vDash (\exists\delta)(\boldsymbol{I}^\kappa_{\delta+1} = \boldsymbol{I}^\kappa_\delta \not\ni \kappa)$. By Lemma 3.42 (and Remark 3.40), there is $\alpha < \kappa$ such that $\mathcal{D}_{\alpha+1} \vDash (\exists\delta)(\boldsymbol{I}^\alpha_{\delta+1} = \boldsymbol{I}^\alpha_\delta \not\ni \alpha)$ but this implies that α is completely Ramsey and $\theta_\alpha < \gamma_\alpha$. Contradiction! □

3.2.3. Mahlo-Ramsey Cardinals

In imitation of the Mahlo hierarchy, we define another sense in which a cardinal could be "strongly" Ramsey, and show that it is much weaker than the property of being α-Ramsey.

Definition 3.44. *Let κ be a regular cardinal and let $\langle f_\alpha : \alpha < \kappa^+\rangle$ be a sequence of canonical functions.*
κ is 0-MR iff it is Ramsey.
κ is $(\alpha+1)$-MR iff for any $g : [\kappa]^{<\omega} \to 2$ there is $X \in \mathrm{NS}^+_\kappa$ such that X is homogeneous for g and
$(\forall\mu \in X)(\mu \text{ is } f_\alpha(\mu)\text{-MR})$.
κ is δ-MR for limit $\delta < \kappa^+$ iff it is α-MR for all $\alpha < \delta$.

Remark 3.45. *Since the sequence $\langle f_\alpha : \alpha < \kappa^+\rangle$ is unique modulo club, whether κ is α-MR does not depend on the choice of $\langle f_\alpha : \alpha < \kappa^+\rangle$.*

Lemma 3.46. *If κ is 1-Ramsey, it is α-MR for all $\alpha < \kappa^+$.*

Proof. We show by induction on $\alpha < \kappa^+$ that κ is α-MR.

That κ is 0-MR is immediate, as is the limit stage of the induction. Suppose $\alpha = \beta+1$, let $g : [\kappa]^{<\omega} \to 2$, and let $\langle M, Y\rangle$ be 1-Ramsey with $\langle f_\gamma : \gamma \leqslant \alpha\rangle, g \in M$ and set $U = U(M)$. By Lemma 3.23, $\langle M, Y\rangle$ is Π^1_2-correct, so $M' = \mathrm{Ult}_U(M) \vDash \kappa \text{ is } \beta\text{-MR}$, since "$\kappa$ is β-MR" is Π^1_2 over κ in parameter $a_0 = f_\beta$. By Łoś Theorem, $\{\mu < \kappa : \mu \text{ is } f_\beta(\mu)\text{-MR}\} \in U$. By ω-completeness of U,

$$X = \{\mu < \kappa : \mu \text{ is } f_\beta(\mu)\text{-MR}\} \cap \bigcap_{n<\omega} Y^{g,M}_n \neq \varnothing$$

and is homogeneous for g (see Lemma 2.5). Finally, we require $X \in \mathrm{NS}^+_\kappa$. Let $C \subseteq \kappa$ be closed unbounded and let $\langle N, Z\rangle$ be a 0-Ramsey pair extending $\langle M, Y\rangle$, with $C \in N$. Then $C \in U(N)$ since $U(N)$ is a normal N-ultrafilter, and

$$C \cap X = C \cap \{\mu < \kappa : \mu \text{ is } f_\beta(\mu)\text{-MR}\} \cap \bigcap_{n<\omega} Y^{g,M}_n \neq \varnothing$$

since

$$Y^{g,M}_n, \{\mu < \kappa : \mu \text{ is } f_\beta(\mu)\text{-MR}\} \in U \subseteq U(N),$$

and $U(N)$ is ω-complete. □

3.3. Longer Games

We briefly look at the consequences of modifying the definition of $G_R(X,\alpha)$ to allow longer games.

Definition 3.47. *Let κ be a regular uncountable cardinal. For $X \subseteq \kappa$ and $\alpha \leqslant \kappa$, let $G_r(X,\alpha)$ be the game with $(1+\alpha)$ moves defined as follows:*

Set $D_{-1} = X$ and in the ζth move

(I) *chooses some κ-structure $\mathcal{M}_\zeta = \langle J_\kappa^{\vec{A}}, \in, \vec{A}\rangle$;*

(II) *chooses $D_\zeta \in [\kappa]^\kappa$ such that $D_\zeta \subseteq D_\xi$ for all $\xi < \zeta$ and D_ζ is a set of good indiscernibles for $\mathcal{M}_\zeta$.*

II wins iff the game continues for $(1+\alpha)$ moves, i.e. iff the game does not terminate early due to II being unable to move.

If II has a winning strategy for the game $G_r(X,\alpha)$ then say X is α-very Ramsey.

For $n < \omega$, the games $G_r(X,n)$ and $G_R(X,n)$ coincide. However, the games of length greater than ω are distinct. If κ is measurable then *II* can win the longer games.

Lemma 3.48. *Suppose κ is a measurable cardinal. Then κ is κ-very Ramsey.*

Proof. Let U be a non-principal κ-complete ultrafilter on κ. Then for any structure $\mathcal{A} = \langle J_\kappa^{\vec{A}}, \in, \vec{A}\rangle$ there exists some $X_\mathcal{A} \in U$ such that $X_\mathcal{A}$ is a set of good indiscernibles for $\mathcal{A}$ (this is standard, see, e.g., [17]). It follows by κ-completeness of U that if *II* plays $D_\zeta = \bigcap\{X_{\mathcal{M}_\varepsilon} : \varepsilon \leqslant \zeta\}$ for each $\zeta < \kappa$ then *II* wins. □

Lemma 3.49. *Let κ be a regular uncountable cardinal. κ is completely Ramsey iff κ is ω-very Ramsey.*

Proof. ($\Leftarrow$) Suppose σ is a winning strategy for *II* in the game $G_r(\kappa,\omega)$ of Definition 3.47. Then σ provides a winning strategy for *II* in the game $G_R(\kappa,\alpha)$ of Definition 3.7 for every $\alpha \in \mathrm{On}$. (In the latter game, *II* ignores the ordinals α_n).

($\Rightarrow$) Suppose κ is completely Ramsey. Then $I =_{\mathrm{df}} \boldsymbol{I}^\kappa_{\theta_\kappa}$ is an ideal on κ such that $\mathcal{R}(I) = I \not\ni \kappa$. By Theorem 3.2, whenever $X \in I^+ = \mathcal{R}^+(I)$ and is a structure there is a set $Y \in I^+$ such that $Y \subseteq X$ and Y is a set of good indiscernibles for $\mathcal{A}$. Hence, since $\kappa \in I^+$, playing any valid move $D_n \in I^+$ in each turn is a winning strategy for *II*. □

The existence of a winning strategy for *II* can have strong consequences if the game is sufficiently long.

Lemma 3.50. *Suppose κ is ω_1-very Ramsey. Then κ is measurable in $\boldsymbol{K}$ or $0^\P$ exists.*

Proof. Assume that $0^\P$ does not exist.

Let $\mathbb{P} = \langle\{f : f \text{ is an injection } f : \alpha \to P({}^{<\omega}\kappa)^{\boldsymbol{K}}, \text{some } \alpha < \omega_1\}, \subseteq\rangle$. $\mathbb{P}$ is an ω-closed forcing, so adds no countable sets of ordinals, and does not collapse ω_1. Let G be generic for $\mathbb{P}$, and let $F = \bigcup G$. So $F : \omega_1 \to P({}^{<\omega}\kappa)^{\boldsymbol{K}}$ is a bijection.

Let σ be a winning strategy for *II* in $G_r(\kappa,\omega_1)$. Define a play, p, of $G_r(\kappa,\omega_1)$ (in the generic extension) by

$$p(\delta) = \langle J_\kappa^{F(\varepsilon)}, \in, F(\varepsilon)\rangle \text{ if } \delta = 2\cdot\varepsilon, \text{ some } \varepsilon$$
$$p(\delta) = \sigma(p \restriction \delta) \text{ if } \delta = 2\cdot\varepsilon{+}1, \text{ some } \varepsilon$$

and note that this is well-defined since each partial play $p \restriction \delta$ ($\delta < \omega_1$) is in $\boldsymbol{V}$.

By the Jensen Indiscernibles Lemma, for each $\varepsilon < \omega_1$ there is a set $p_\varepsilon \in \boldsymbol{K}$ with $p_\varepsilon \supseteq p(2\cdot\varepsilon{+}1)$ such that p_ε is a set of good indiscernibles for $\langle J_\kappa^{F(\varepsilon)}, \in, F(\varepsilon)\rangle$. Let $U = \{X \in P(\kappa)^{\boldsymbol{K}} : (\exists\varepsilon < \omega_1) p_\varepsilon \subseteq X\}$.

Claim: *U is an amenable ω-complete normal $\boldsymbol{K}$-ultrafilter.*

If the claim holds then U is "$\boldsymbol{K}$-correct" (*cf* [29] Sect.7.3) so, by Lemmata 7.3.8 and 7.3.9 (*op.cit.*), $U = E^{\boldsymbol{K}}_{\kappa^{+\boldsymbol{K}}}$ and the proof is complete. We prove the claim.

That U is an ultrafilter over $P(\kappa)^{\boldsymbol{K}}$ follows immediately from the construction. For ω-completeness, let $\{X_n : n < \omega\} \subseteq U$. Then there is a set $\{\varepsilon_n : n < \omega\}$ such that $p(2 \cdot \varepsilon_n + 1) \subseteq p_{\varepsilon_n} \subseteq X_n$. Since $\mathbb{P}$ is ω-closed, $\langle \varepsilon_n : n < \omega\rangle \in \boldsymbol{V}$ so $\varepsilon =_{\mathrm{df}} \sup_{n<\omega} \varepsilon_n < \omega_1$. So $\bigcap\{X_n : n < \omega\} \supseteq p(2 \cdot \varepsilon + 1) \neq \varnothing$.

To prove amenability of U, let $\langle X_\eta : \eta < \kappa\rangle \in \boldsymbol{K}$. Let $A = \{\langle \alpha, \beta\rangle : \alpha \in X_\beta\} \in \boldsymbol{K}$. Then there is $\varepsilon < \omega_1$ such that $p(2 \cdot \varepsilon) = \langle J^A_\kappa, \in, A\rangle$, so $p_\varepsilon \in \boldsymbol{K}$ is a set of good indiscernibles for $\langle J^A_\kappa, \in, A\rangle$. We show that for all $\eta < \kappa$, $X_\eta \in U$ iff $p_\varepsilon \backslash (\eta + 1) \subseteq X_\eta$.

Suppose first that $X_\eta \in U$. Let ε' such that $p_{\varepsilon'} \subseteq X_\eta$ and let

$$\gamma \in p(2 \cdot \varepsilon + 1) \backslash (\eta + 1) \subseteq p_\varepsilon \cap p_{\varepsilon'} \backslash (\eta + 1).$$

Then $\langle \gamma, \eta\rangle \in A$ so by good indiscernibility $\langle \gamma', \eta\rangle \in A$ for all $\gamma' \in p_\varepsilon \backslash (\eta + 1)$. So $p_\varepsilon \backslash (\eta + 1) \subseteq X_\eta$.

Now suppose $p_\varepsilon \backslash (\eta + 1) \subseteq X_\eta$. Then let ε' such that $p_{\varepsilon'}$ is a set of good indiscernibles for $\langle J^B_\kappa, \in, B\rangle$, where $B = p_\varepsilon \backslash (\eta + 1)$. Then $p_{\varepsilon'} \subseteq \kappa \backslash (\eta + 1)$ since η is definable in $\langle J^B_\kappa, \in, B\rangle$. So

$$p_{\varepsilon'} \cap B = p_{\varepsilon'} \cap p_\varepsilon \supseteq p(2 \cdot \max\{\varepsilon, \varepsilon'\} + 1) \neq \varnothing,$$

and $p_{\varepsilon'} \subseteq B = p_\varepsilon \backslash (\eta + 1) \subseteq X_\eta$ by indiscernibility. So $X_\eta \in U$. Hence,

$$\{\eta < \kappa : X_\eta \in U\} = \{\eta < \kappa : p_\varepsilon \backslash (\eta + 1) \subseteq X_\eta\} \in \boldsymbol{K}.$$

So U is amenable.

Finally, to show normality let $\langle X_\eta : \eta < \kappa\rangle \in \boldsymbol{K}$ such that $X_\eta \in U$. Let A and ε be as in the proof of amenability. Then $\Delta_{\eta<\kappa} X_\eta \supseteq \Delta_{\eta<\kappa} p_\varepsilon \backslash (\eta + 1) = p_\varepsilon \neq \varnothing$. □

Lemma 3.51 will show that the games $G_r(\kappa, \alpha)$ need not be determined in general. For $\alpha \geqslant \omega_1$, *I* not having a winning strategy is weaker in terms of consistency strength than *II* having a winning strategy (by Lemmas 3.48 and 3.50 the latter is equiconsistent with a measurable cardinal). However *I* not having a winning strategy for $G_r(\kappa, \omega + 1)$ is strictly stronger than than κ being completely Ramsey.

Lemma 3.51. *Suppose $\alpha < \beta \leqslant \kappa$ and I does not have a winning strategy for $G_r(\kappa, \beta)$. Then there is $\mu < \kappa$ such that I does not have a winning strategy for $G_r(\mu, \alpha)$.*

Proof. Assume for a contradiction that *I* has a winning strategy for $G_r(\mu, \alpha)$ for all $\mu < \kappa$. For each $\mu < \kappa$, let σ_μ be a winning strategy for *I* in $G_r(\mu, \alpha)$. Since κ is Ramsey, hence inaccessible, there is $A \subseteq \kappa$ such that $J^A_\kappa = V_\kappa$ and we may assume that each σ_μ is uniformly definable in $\mathcal{A}_0 = \langle V_\kappa, \in, A\rangle$ from μ and α.

We define a strategy τ for *I* in $G_r(\kappa, \alpha+1)$. Assume that p is a partial play of $G_r(\kappa, \alpha+1)$ that ends with *I* to play (possibly $p = \varnothing$). Let $\langle D_\xi : \xi < \zeta\rangle$ enumerate *II*'s moves in p, and let $B_p = \{\langle \varepsilon, \xi\rangle : \varepsilon \in D_\xi\}$. Let $\tau(p) = \langle V_\kappa, \in, A, B_p\rangle$.

Since *I* has no winning strategy for $G_r(\kappa, \beta)$, τ cannot be a winning strategy for $G_r(\kappa, \alpha + 1)$. Let p be a play of $G_r(\kappa, \alpha + 1)$ in which *II* wins, and let $\langle D_\xi : \xi \leqslant \alpha\rangle$ be *II*'s moves in p. W.l.o.g., assume $\alpha < \min(D_\xi)$ for all $\xi \leqslant \alpha$.

By construction of τ, $D = D_\alpha$ is a set of good indiscernibles for $\langle V_\kappa, \in, A, B_{p \restriction 2 \cdot \alpha}\rangle$. For $\gamma \in D$, we construct a play, q_γ, of $G_r(\gamma, \alpha)$ where *I* plays according to σ_γ and *II* plays $D_\zeta \cap \gamma$ in the ζth move, for each $\zeta < \alpha$. This gives a contradiction, since σ_γ is meant to be a winning strategy for *I* in $G_r(\gamma, \alpha)$, and so the game should terminate in (strictly) less than α turns.

q_γ is uniquely defined by the previous paragraph, but we must show that the definition makes sense. We induct on $\zeta < \alpha$ to show that $D_\zeta \cap \gamma$ is a valid move for *II* in the ζth turn.

Note first that $D_\zeta \cap \gamma$ is unbounded in γ whenever $\gamma \in D$ and $\zeta < \alpha$ since D_ζ is unbounded in κ and definable over $\langle V_\kappa, \in, A, B_{p\restriction 2\cdot\alpha}\rangle$ from $\zeta < \gamma$, and $\langle V_\kappa, \in, A, B_{p\restriction 2\cdot\alpha}\rangle \restriction \gamma \prec \langle V_\kappa, \in, A, B_{p\restriction 2\cdot\alpha}\rangle$.

Let $\xi < \alpha$, and assume the hypothesis holds for all $\zeta < \xi$. $q(2\cdot\varepsilon) = D_\varepsilon \cap \gamma$ is uniformly definable in $\mathcal{A}_\xi =_{\mathrm{df}} \langle V_\kappa, \in, A, B_{p\restriction 2\cdot\xi}\rangle$ from ε and γ, for each $\varepsilon < \xi$, and σ_γ is definable from γ and α, so $q_\gamma \restriction (2\cdot\xi+1)$ is definable from $\gamma \in D$ and parameters ξ and α. In particular, there is a formula $\varphi(v_0, v_1, v_2, v_3)$ of $\mathcal{L}_{\mathcal{A}_\xi}$ such that $\mathcal{A}_\xi \models (\forall x)(\varphi(\xi, \alpha, \gamma, x) \leftrightarrow x = q_\gamma(2\cdot\xi))$. By good indiscernibility, using $\gamma \in D \subseteq D_\xi$ and $\min(D_\xi) > \alpha > \xi$, for all $\gamma' \in D_\xi$ $\mathcal{A}_\xi \models (\forall x)(\varphi(\xi, \alpha, \gamma', x) \leftrightarrow x = q_\gamma(2\cdot\xi) \restriction \gamma')$.

To conclude the proof we must show that $D_\xi \cap \gamma$ is a set of g.i.'s for $q_\gamma(2\cdot\xi)$. For any formula ψ of $\mathcal{L}_{q_\gamma(2\cdot\xi)}$ and $a_0, \ldots, a_{n-1} \in q_\gamma(2\cdot\xi) \restriction \gamma'$, where $\gamma' \in D_\xi \cap \gamma$,

$$\begin{aligned} q_\gamma(2\cdot\xi) \restriction \gamma' \models \psi(a_0, \ldots, a_{n-1}) \quad &\text{iff} \quad \mathcal{A}_\xi \models (q_\gamma(2\cdot\xi) \restriction \gamma' \models \psi(a_0, \ldots, a_{n-1})) \\ &\text{iff} \quad \mathcal{A}_\xi \models (\exists x)(\varphi(\xi, \alpha, \gamma', x) \wedge x \models \psi(a_0, \ldots, a_{n-1})) \\ &\text{iff} \quad \mathcal{A}_\xi \models (\exists x)(\varphi(\xi, \alpha, \gamma, x) \wedge x \models \psi(a_0, \ldots, a_{n-1})) \\ &\text{iff} \quad \mathcal{A}_\xi \models (q_\gamma(2\cdot\xi) \models \psi(a_0, \ldots, a_{n-1})) \\ &\text{iff} \quad q_\gamma(2\cdot\xi) \models \psi(a_0, \ldots, a_{n-1}) \end{aligned}$$

so $q_\gamma(2\cdot\xi) \restriction \gamma' \prec q_\gamma(2\cdot\xi)$ for each $\gamma' \in D_\xi$.

Now, for any formula ψ of $\mathcal{L}_{q_\gamma(2\cdot\xi)}$, any $\gamma_1, \ldots, \gamma_n, \gamma'_1, \ldots, \gamma'_n \in D_\xi \cap \gamma$, and any $\delta_1, \ldots, \delta_m < \min\{\gamma_1, \ldots, \gamma_n, \gamma'_1, \ldots, \gamma'_n\}$,

$$\begin{aligned} q_\gamma(2\cdot\xi) \models \psi(\delta_1, \ldots, \delta_m, \gamma_1, \ldots, \gamma_n) \quad &\text{iff} \quad \mathcal{A}_\xi \models (q_\gamma(2\cdot\xi) \models \psi(\delta_1, \ldots, \delta_m, \gamma_1, \ldots, \gamma_n)) \\ &\text{iff} \quad \mathcal{A}_\xi \models (\exists x)(\varphi(\xi, \alpha, \gamma, x) \wedge x \models \psi(\delta_1, \ldots, \delta_m, \gamma_1, \ldots, \gamma_n)) \\ &\text{iff} \quad \mathcal{A}_\xi \models (\exists x)(\varphi(\xi, \alpha, \gamma, x) \wedge x \models \psi(\delta_1, \ldots, \delta_m, \gamma'_1, \ldots, \gamma'_n)) \\ &\text{iff} \quad \mathcal{A}_\xi \models (q_\gamma(2\cdot\xi) \models \psi(\delta_1, \ldots, \delta_m, \gamma'_1, \ldots, \gamma'_n)) \\ &\text{iff} \quad q_\gamma(2\cdot\xi) \models \psi(\delta_1, \ldots, \delta_m, \gamma'_1, \ldots, \gamma'_n) \end{aligned}$$

so $D_\xi \cap \gamma$ is a set of g.i.'s for $q_\gamma(2\cdot\xi)$, as required. □

Remark 3.52. *An obvious question about α-very Ramsey cardinals remains unanswered. Namely, how strong (in terms of consistency strength) is α-very Ramsey for countable α? One might also ask whether games of countable length are determined.*

4. Greatly Erdős Cardinals

We first give an overview of this section. Greatly Erdős cardinals arise naturally in Section 7 as a lower bound on the consistency strength of a variant of Chang's Conjecture. They also have some interesting properties in their own right which will be given here. They can be defined, and thought of, by way of analogy with greatly Mahlo cardinals. The definitions of these, together with the α-weakly Erdős cardinals, of which they a limit, and their basic properties are given in Sect. 4.1. The notion is that of building a hierarchy of normal filters $\mathcal{F}_\alpha$ *via* operators $t_{\mathcal{A}}$ for as long as these remain non-trivial.

In Sect. 4.2 we define a partial ordering on indiscernibility sets for a given structure $\mathcal{A}$ by analogy with the Mitchell order on measures. (In $\boldsymbol{K}$ this can be construed a wellorder for structures $\mathcal{A}$ that are initial segments of $\boldsymbol{K}$.) We then see at Cor. 4.25, how the filter hierarchies, and so the notions of α-weakly Erdős, and greatly Erdős, can be generated by by sets of ordinals where orders of indiscernible sets for a structure dominate canonical functions. Section 4.3 then rounds off by making some comments on obvious relativisations and orderings between the cardinals concerned.

4.1. Definitions

4.1.1. Greatly Erdős Cardinals

Greatly Erdős cardinals will be defined analogously to the widely studied greatly (weakly) Mahlo cardinals. We now consider an operator, $t_{\mathcal{A}}(-)$, on subsets of a cardinal κ and use this to define this notion.

Definition 4.1. *Let κ have uncountable cofinality, and let $\mathcal{A}$ be a κ-structure, $X \subseteq \kappa$. Let*
$t_{\mathcal{A}}(X) = \{\alpha \in \kappa \cap \lim$:*there exists a set $I \subseteq \alpha \cap X$ of g.i's for $\mathcal{A}$ cofinal in α*$\}$.

Lemma 4.2. (Basic Properties Lemma) *Let κ have uncountable cofinality. Let $\mathcal{A}$ and $\mathcal{B}$ be κ-structures and let $X, Y \subseteq \kappa$.*

Then

i. $X \subseteq Y \Rightarrow t_{\mathcal{A}}(X) \subseteq t_{\mathcal{A}}(Y)$

ii. if $\mathcal{A}$ extends $\mathcal{B}$ then $t_{\mathcal{A}}(X) \subseteq t_{\mathcal{B}}(X)$

iii. $t_{\mathcal{A}}(X \cup (\gamma + 1)) \subseteq t_{\mathcal{A}}(X) \cup (\gamma + 1)$ *for all $\gamma < \kappa$*

iv. $t_{\mathcal{A}}(X) \subseteq t_{\mathcal{B}}(X) \cup \gamma$ *if $\mathcal{B}$ is definable over $\mathcal{A}$ from parameters in γ*

Proof. i, ii and iii are immediate from the definition.

For iv, note that we may as well assume that $Lim(\gamma)$ and thus the finitely many predicates of $\mathcal{B}$ are definable over $\mathcal{A}$ from parameters in some $\gamma_0 < \gamma$, suppose that $\delta \in t_{\mathcal{A}}(X)\backslash\gamma$. Let $I \subseteq X$ witness $\delta \in t_{\mathcal{A}}(X)\backslash \gamma$. Then $I\backslash\gamma_0 \subseteq X$ is a set of good indiscernibles for $\mathcal{B}$, since, by the definition of good indiscernibles, the members of $I\backslash(\gamma_0)$ are indiscernible for formulae over $\mathcal{A}$ with parameters in γ_0. So $I\backslash\gamma_0$ witnesses $\delta \in t_{\mathcal{B}}(X)$. □

Example 4.3. *Let κ have uncountable cofinality. Suppose $\mathcal{A} = \langle J_\kappa^{\vec{A}}, \in, \vec{A}, \ldots \rangle$ is a κ-structure and $C \subseteq \kappa$ is closed unbounded.*

- *If κ is ω-Erdős then $t_{\mathcal{A}}(C)$ is stationary.*
- *If κ is Ramsey then $t_{\mathcal{A}}(C)$ contains a closed unbounded set.*

Proof. Let $C^* \subseteq \kappa$ be club. Let $\mathcal{A}^* = \langle J_\kappa^{\vec{A},C,C^*}, \in, \vec{A}, \ldots, C, C^* \rangle$. Then any set I of good indiscernibles for $\mathcal{A}^*$ must be a subset of $C \cap C^*$ since, by elementarity, $C \cap C^*$ is cofinal below each member of I.

If κ is Ramsey and I is a set of g.i.'s for $\mathcal{A}^*$ such that I is cofinal in κ then $\lim(I) \subseteq t_{\mathcal{A}}(C)$ is club in κ.

If κ is ω-Erdős and I is a set of g.i.'s for $\mathcal{A}^*$ of limit order type then $\sup(I) \in t_{\mathcal{A}}(C \cap C^*) \subseteq C^* \cap t_{\mathcal{A}}(C)$. □

In fact, we can strengthen the first part of Example 4.3: Lemma 4.4 also provides the motivation for the terminology 'greatly Erdős'.

Lemma 4.4. *Let κ have uncountable cofinality. The following are equivalent:*

- *κ is ω-Erdős*
- *$t_{\mathcal{A}}(\kappa)$ is stationary for every κ-structure $\mathcal{A}$*
- *$t_{\mathcal{A}}(C)$ is stationary for every κ-structure $\mathcal{A}$ and club $C \subseteq \kappa$*

Proof. Follows immediately from example 4.3 and the definitions. □

Definition 4.5. *κ is* greatly Erdős *iff there is a normal filter $\mathcal{F}$ on κ such that $\mathcal{F}$ is closed under $t_{\mathcal{A}}(-)$ for every κ-structure $\mathcal{A}$.*

4.1.2. α-weakly Erdős Cardinals

Definition 4.6 provides the appropriate analogue of α-Mahlo cardinals.

Definition 4.6. *Let κ be a cardinal of uncountable cofinality.*
Set

$\mathcal{F}_0 = \mathcal{C}_\kappa$, *the club filter at* κ

$\mathcal{F}_{\alpha+1} = \{X \subseteq \kappa : X \supseteq t_{\mathcal{A}}(Y)$, *for some* $Y \in \mathcal{F}_\alpha$ *and* κ*-structure*$\mathcal{A}\}$

$\mathcal{F}_\alpha = \{X \subseteq \kappa : X \supseteq \Delta Y$ *for some* $Y \subseteq \bigcup_{\beta<\alpha} \mathcal{F}_\beta, |Y| \leqslant \kappa\}$ *for limit* $\alpha < \kappa^+$

κ *is* α-weakly Erdős *iff* $\mathcal{F}_\alpha \not\ni \varnothing$.

In the remainder of this section, we show that the notions of greatly Erdős and α-weakly Erdős relate in a natural way, and that the $\mathcal{F}_\alpha$'s have the properties we would expect: they are filters, are increasing in α, and generate a witness to κ being greatly Erdős, when one exists. Note that by elementary reasoning using the definition of good indiscernibles, if κ is 1-weakly Erdős then it is strongly inaccessible.

First, we prove a useful characterisation of the sets X in $\mathcal{F}_\alpha$'s, by providing them with 'histories.'

Lemma 4.7. *Let κ be a cardinal of uncountable cofinality. Let $X \in \mathcal{F}_\alpha$, $\alpha < \kappa^+$. Then there is a sequence $\langle (X)_\beta : \beta \leqslant \alpha\rangle$ (defined modulo the club filter) and a κ-structure $\mathcal{A} = \mathcal{A}(X)$ such that:*

i.
$$\begin{aligned} (X)_0 &= \kappa \\ (X)_{\beta+1} &= t_{\mathcal{A}}((X)_\beta) \\ (X)_\beta &= \Delta_{\gamma<\beta}(X)_\gamma \text{ for limit } \beta \end{aligned}$$

ii. $(X)_\beta \in \mathcal{F}_\beta$ *for each* β

iii. $(X)_\alpha \subseteq X$

Moreover, any sequence $\langle (X)_\beta : \beta \leqslant \alpha\rangle$ satisfying i for some structure $\mathcal{A}$ is decreasing under $\supseteq$ (modulo the club filter, of course, since the sequence is only defined modulo the club filter).

Proof. ii follows immediately from i and Definition 4.6. We prove i and iii by induction on α. The case $\alpha = 0$ is immediate.

Suppose first that $\alpha = \beta + 1$. Let $X \in \mathcal{F}_\alpha$, i.e. $X \supseteq t_{\mathcal{B}}(Y)$, some $Y \in \mathcal{F}_\beta$. Apply the induction hypothesis to obtain $\langle (Y)_\gamma : \gamma < \alpha\rangle$ and a structure $\mathcal{A}(Y)$. Let $\mathcal{A}(X)$ be a structure with all predicates from $\mathcal{A}(Y)$ and $\mathcal{B}$. $\langle (X)_\gamma : \gamma \leqslant \alpha\rangle$ is then determined by i. Then it is clear that $(X)_\gamma \subseteq (Y)_\gamma$ for all $\gamma < \alpha$. Finally:

$$(X)_\alpha = t_{\mathcal{A}(X)}((X)_\beta) \subseteq t_{\mathcal{B}}((X)_\beta) \subseteq t_{\mathcal{B}}((Y)_\beta) \subseteq t_{\mathcal{B}}(Y) \subseteq X.$$

Suppose that α is a limit ordinal. Let $X \in \mathcal{F}_\alpha$, so $X \supseteq \Delta Y$ for some $Y \subseteq \bigcup_{\beta<\alpha} \mathcal{F}_\beta, |Y| \leqslant \kappa$. Write Y as $\{Y_\gamma : \gamma < \eta\}$ and apply the induction hypothesis to obtain $\langle (Y_\gamma)_\beta : \beta \leqslant \alpha_\gamma\rangle$ and κ-structures $\mathcal{A}(Y_\gamma)$ for each $\gamma < \eta$. Let $\mathcal{A}(X)$ be a κ-structure with a predicate coding $\langle \mathcal{A}(Y_\gamma) : \gamma < \eta\rangle$ in such a way that $\mathcal{A}(Y_\gamma)$ is definable over $\mathcal{A}(X)$ from γ for each $\gamma < \eta$. Let $\langle (X)_\beta : \beta \leqslant \alpha\rangle$ be the sequence defined from $\mathcal{A}(X)$ by clause i.

We show by a subsidiary induction on β that $(X)_\beta \subseteq \Delta_{\gamma<\eta}(Y_\gamma)_\beta$ (modulo the club filter) for all $\beta \leqslant \alpha$. It is immediate if $\beta = 0$.

Suppose $\beta = \delta + 1$. Then

$$\begin{array}{rcl} (X)_\beta & = & t_{\mathcal{A}(X)}((X)_\delta) \\ & \subseteq & t_{\mathcal{A}(X)}(\Delta_{\gamma<\eta}(Y_\gamma)_\delta) \text{ by induction hypothesis} \\ & \subseteq & t_{\mathcal{A}(X)}((Y_\gamma)_\delta \cup (\gamma+1)) \text{ for all } \gamma < \eta \\ & \subseteq & t_{\mathcal{A}(X)}((Y_\gamma)_\delta) \cup (\gamma+1) \text{ by Lemma } 4.2iii \\ & \subseteq & t_{\mathcal{A}(Y_\gamma)}((Y_\gamma)_\delta) \cup (\gamma+1) \text{ by Lemma } 4.2iv \\ & = & (Y_\gamma)_\beta \cup (\gamma+1) \end{array}$$

Hence $(X)_\beta \subseteq \Delta_{\gamma<\eta}((Y_\gamma)_\beta \cup (\gamma+1)) = \Delta_{\gamma<\eta}((Y_\gamma)_\beta$, as required.

Now, suppose β is a limit ordinal. Then

$$\begin{array}{rcl} (X)_\beta & = & \Delta_{\varepsilon<\beta}(X)_\varepsilon \\ & \subseteq & \Delta_{\varepsilon<\beta}\Delta_{\gamma<\eta}(Y_\gamma)_\varepsilon \\ & = & \Delta_{\gamma<\eta}\Delta_{\varepsilon<\beta}(Y_\gamma)_\varepsilon \\ & = & \Delta_{\gamma<\eta}(Y_\gamma)_\beta \end{array}$$

We now have that $(X)_\beta \subseteq \Delta_{\gamma<\eta}(Y_\gamma)_\beta$ (modulo the club filter) for all $\beta \leqslant \alpha$ and, in particular, $(X)_\alpha \subseteq \Delta_{\gamma<\eta}(Y_\gamma)_\alpha \subseteq X$.

It now remains to show the final sentence of the lemma. It suffices to induct on β to show that $t_{\mathcal{A}}((X)_\beta) \subseteq (X)_\beta$. If $\beta = 0$, this is immediate since $(X)_0 = \kappa$.

Suppose $\beta = \delta + 1$. So $(X)_\beta = t_{\mathcal{A}}((X)_\delta)$. Let $\gamma \in t_{\mathcal{A}}((X)_\beta)$ and let I be a set of g.i.'s witnessing this. Then $I \subseteq (X)_\beta = t_{\mathcal{A}}((X)_\delta) \subseteq (X)_\delta$ by induction hypothesis. Hence, I witnesses $\gamma \in t_{\mathcal{A}}((X)_\delta) = (X)_\beta$.

Let β be a limit ordinal. Then

$$\begin{array}{rcl} t_{\mathcal{A}}((X)_\beta) & = & t_{\mathcal{A}}(\Delta_{\varepsilon<\beta}(X)_\varepsilon) \\ & \subseteq & \Delta_{\varepsilon<\beta}(t_{\mathcal{A}}((X)_\varepsilon)) \\ & \subseteq & \Delta_{\varepsilon<\beta}(X)_\varepsilon \\ & = & (X)_\beta \end{array}$$

where the third line uses the induction hypothesis. The induction is complete. □

Corollary 4.8. *Let κ be a cardinal of uncountable cofinality. For any κ-structure $\mathcal{A}$ and $\alpha < \kappa^+$, define a sequence $\langle (X_{\mathcal{A}})_\beta : \beta \leqslant \alpha \rangle$ by*

$$\begin{array}{rcl} (X_{\mathcal{A}})_0 & = & \kappa \\ (X_{\mathcal{A}})_{\beta+1} & = & t_{\mathcal{A}}((X_{\mathcal{A}})_\beta) \\ (X_{\mathcal{A}})_\beta & = & \Delta_{\gamma<\beta}(X_{\mathcal{A}})_\gamma \text{ for limit } \beta \end{array}$$

Then κ is α-weakly Erdős iff for every κ-structure $\mathcal{A}$, $(X_{\mathcal{A}})_\alpha \neq \varnothing$.

Proof. Follows from 4.7 and the definition. □

Lemma 4.9. *Let κ be a cardinal of uncountable cofinality. Then*

i. for all $\beta < \kappa^+$, $\mathcal{F}_\beta$ is a normal filter

ii. $\alpha < \beta \Rightarrow \mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$

iii. $\mathcal{F} = \bigcup_{\alpha<\kappa^+} \mathcal{F}_\alpha$ *is a normal filter closed under* $t_\mathcal{A}(-)$ *for all* κ*-structures* $\mathcal{A}$

Proof. To show i, we induct on β. The case $\beta = 0$ is immediate.

Suppose $\beta = \alpha + 1$. Then $\kappa \supseteq t_\mathcal{A}(\kappa)$ and, by induction hypothesis, $\kappa \in \mathcal{F}_\alpha$ so $\kappa \in \mathcal{F}_\beta$. That any superset of a member of $\mathcal{F}_\beta$ is itself a member of $\mathcal{F}_\beta$ is immediate from the definition.

Let $X_j \in \mathcal{F}_\beta$, $j < \kappa$. Then there exist $Y_j \in \mathcal{F}_\alpha$ and structures $\mathcal{A}_j$ s.t $X_j \supseteq t_{\mathcal{A}_j}(Y_j)$ for each $j < \kappa$. Let $\mathcal{B}$ be a κ-structure having a predicate coding $\langle \mathcal{A}_j : j < \kappa\rangle$ so that $\mathcal{A}_j$ is definable over $\mathcal{B}$ from j for each $j < \kappa$, and let $Z = \Delta_{j<\kappa} Y_j \in \mathcal{F}_\alpha$. Then

$$\begin{aligned}
\Delta_{j<\kappa} X_j &\supseteq \Delta_{j<\kappa}(t_{\mathcal{A}_j}(Y_j) \cup (j+1)) \\
&\supseteq \Delta_{j<\kappa} t_\mathcal{B}(Y_j) \text{ by Lemma 4.2 iv} \\
&= \Delta_{j<\kappa} t_\mathcal{B}(Y_j) \\
&\supseteq \Delta_{j<\kappa} t_\mathcal{B}((\Delta_{j<\kappa} Y_j \backslash (j+1))) \text{ by Lemma 4.2 i} \\
&= \Delta_{j<\kappa}(t_\mathcal{B}(\Delta_{j<\kappa} Y_j) \backslash (j+1)) \\
&= t_\mathcal{B}(\Delta_{j<\kappa} Y_j) \\
&\in \mathcal{F}_\beta
\end{aligned}$$

Now suppose that β is a limit ordinal. Then $\kappa = \Delta\{\kappa\} \in \mathcal{F}_\beta$ and, again, the superset clause is immediate from the definition. Let $X_j \in \mathcal{F}_\beta$, $j < \kappa$. Then there exist:

$$\eta_j \leqslant \kappa,\ Y_{j,i} \in \bigcup_{\alpha<\beta} \mathcal{F}_\alpha \text{ such that} X_j \supseteq \Delta_{\alpha<\eta_j} Y_{j,\alpha} \text{ for each } j < \kappa.$$

. By setting $Y_{j,i} = \kappa$ whenever it is undefined, we can w.l.o.g. assume that $\eta_j = \kappa$ for all $j < \kappa$. Then

$$\begin{aligned}
\Delta_{j<\kappa} X_j &\supseteq \Delta_{j<\kappa} \Delta_{\alpha<\eta_j} Y_{j,\alpha} \\
&= \Delta_{j<\kappa} \Delta_{\alpha<\kappa} Y_{j,\alpha} \\
&=_{\mathcal{C}_\kappa} \Delta_{j<\kappa} Y_{f(j)}
\end{aligned}$$

where f is any mapping of κ onto κ^2. So there is $C \in \mathcal{C}_\kappa = \mathcal{F}_0$ such that $C \cap \Delta_{j<\kappa} Y_{f(j)} \subseteq \Delta_{j<\kappa} X_j$ and this witnesses that $\Delta_{j<\kappa} X_j \in \mathcal{F}_\beta$.

To show ii, let $X \in \mathcal{F}_\alpha$. Let $\langle (X)_\varepsilon : \varepsilon \leqslant \alpha\rangle$ and $\mathcal{A} = \mathcal{A}(X)$ be as in Lemma 4.7. Then extend the sequence as far as β according to the clauses of 4.7i. Then, since the sequence is subset decreasing, we have $X \supseteq (X)_\alpha \supseteq (X)_\beta \in \mathcal{F}_\beta$ so $X \in \mathcal{F}_\beta$.

iii follows from i and ii, and the observation that if $Y \subseteq \mathcal{F}$, $|Y| \leqslant \kappa$ then $Y \subseteq \mathcal{F}_\alpha$, some $\alpha < \kappa^+$. □

Corollary 4.10. *Let* κ *be a cardinal of uncountable cofinality. Then the following are equivalent:*

i. κ *is greatly Erdős;*

ii. $\mathcal{G} = \bigcup_{\alpha<\kappa^+} \mathcal{F}_\alpha \not\ni \varnothing$*;*

iii. κ *is* α*-weakly Erdős for all* $\alpha < \kappa^+$.

Moreover, if $\mathcal{F}$ *is a filter witnessing that* κ *is greatly Erdős then* $\mathcal{F} \supseteq \mathcal{G} = \bigcup_{\alpha<\kappa^+} \mathcal{F}_\alpha$ *so* $\mathcal{G}$ *is a minimal witness that* κ *is greatly Erdős.*

Proof. iii.⇒ii. is immediate. ii.⇒i. follows from Lemma 4.9iii.

To see i.⇒iii. and the final sentence, let $\mathcal{F}$ be a filter witnessing that κ is greatly Erdős. Then by induction on α using definition 4.6 it follows that $\mathcal{F} \supseteq \mathcal{F}_\alpha$ for all $\alpha < \kappa^+$, so $\varnothing \notin \mathcal{F}_\alpha$ since $\varnothing \notin \mathcal{F}$. □

Corollary 4.11. *Let κ be a cardinal of uncountable cofinality. For any κ-structure $\mathcal{A}$, we may define the sequence $\langle (X_\mathcal{A})_\beta : \beta < \kappa^+ \rangle$ as in Corollary 4.8. Then κ is greatly Erdős iff for every κ-structure $\mathcal{A}$ and $\beta < \kappa^+$, $(X_\mathcal{A})_\beta \neq \varnothing$.*

Proof. Immediate from 4.8 and 4.10. □

4.2. Canonical Functions and Indiscernibility Orders

The Mitchell order is a well-founded partial ordering of total measures on a cardinal, where $U_1 < U_2$ iff $U_1 \in \mathrm{Ult}_{U_2}(V)$. $o(\lambda)$ is then usually defined to be the height of the Mitchell order on λ. We define a partial ordering on sets of good indiscernibles with the aim of understanding their relative "strength" in a similar fashion. In the sequel recall that we require that $\mathcal{A} \vDash \mathrm{ZFC}$ to construct $M(\mathcal{A}, J)$.

Definition 4.12. *Let $\mathcal{A}$ be a γ-structure. Suppose I, J are cofinal sets of good indiscernibles for $\mathcal{A}$.*

Say $I <_\mathcal{A} J$ iff $I \in M(\mathcal{A}, J)$.

Lemma 4.13. *For any γ-structure $\mathcal{A}$ the relation $<_\mathcal{A}$ is well-founded.*

Proof. Suppose $\langle I_n : n < \omega \rangle$ is such that $I_{n+1} <_\mathcal{A} I_n$ for each $n < \omega$. Then $I_{n+1} \in M(\mathcal{A}, I_n)$ for each n, so $M(\mathcal{A}, I_{n+1}) \in M(\mathcal{A}, I_n)$, since $M(\mathcal{A}, I_n) \vDash \mathrm{ZF}^-$ and $\mathcal{A} \in M(\mathcal{A}, I_n)$. However this gives an infinite descending $\in$-chain - contradiction. □

Lemma 4.14. *For any κ-structure $\mathcal{A}$ and $I, J \subseteq \kappa$ cofinal sets of g.i.'s for $\mathcal{A}$, if $I <_\mathcal{A} J$ then there is $\gamma < \kappa$ such that $J \backslash \gamma$ is a set of good indiscernibles for $\langle \mathcal{A}, I \rangle$.*

Proof. $\langle \mathcal{A}, I \rangle \in M(\mathcal{A}, J)$ so the result follows from Lemma 2.18. □

Definition 4.15. *Let $\mathcal{A}$ be a κ-structure.*

- *If I is a cofinal set of good indiscernibles for $\mathcal{A}$ then let $o_\mathcal{A}(I) = \mathrm{height}(I, <_\mathcal{A})$, i.e. if I is $<_\mathcal{A}$-minimal then set $o_\mathcal{A}(I) = 0$, and set $o_\mathcal{A}(I) = \sup(o_\mathcal{A}(J) + 1 : J <_\mathcal{A} I)$, otherwise.*
- *If I is a set of good indiscernibles for $\mathcal{A}$, $\lambda = \sup(I)$, then let $o_\mathcal{A}(I) = o_{\mathcal{A} \restriction \lambda}(I)$.*
- *Define $o_\mathcal{A} : \kappa \to \kappa$ by $o_\mathcal{A}(\lambda) = \sup(\{o_{\mathcal{A}\restriction\lambda}(I) + 1 : I \text{ is a cofinal set of g.i.'s for } \mathcal{A} \restriction \lambda\})$.*
- *If I is a set of good indiscernibles for $\mathcal{A}$ then let $o_\mathcal{A}(M(\mathcal{A}, I)) = o_\mathcal{A}(I)$.*

Remark 4.16. *The final clause of the definition makes sense since if $M(\mathcal{A}, I) = M(\mathcal{A}, J)$ then it follows immediately from Definition 4.12 and the earlier clauses of Definition 4.15 that $o_\mathcal{A}(I) = o_\mathcal{A}(J)$ ($o_\mathcal{A}(I)$ depends only on $M(\mathcal{A}, I)$, not on I itself).*

Definition 4.17. *Suppose $\boldsymbol{V} = \boldsymbol{K}$, $\mathrm{cf}(\gamma) > \omega$, and $\mathcal{A} = \langle J^E_\gamma, \in, E, \vec{A} \rangle$ is a γ-structure that satisfies* ZFC. *Suppose I, J are cofinal sets of good indiscernibles for $\mathcal{A}$.*

Say $I \sim_\mathcal{A} J$ iff $M^{\mathcal{A}, I} = M^{\mathcal{A}, J}$.

The next two lemmas show that in $\boldsymbol{K}$ the orderings $<_\mathcal{A}$ are, in effect, linear. In general, we cannot assume this.

Lemma 4.18. *Suppose* $\boldsymbol{V} = \boldsymbol{K}$, $\mathrm{cf}(\gamma) > \omega$, *and* $\mathcal{A} = \langle J^E_\gamma, \in, E, \vec{A}\rangle \vDash \mathrm{ZFC}$ *is a* γ*-structure.*

Then $<_\mathcal{A}$ *gives rise to a well-order of* $G = \{I \subseteq \gamma : I \text{ is a cofinal set of g.i.' s for } \mathcal{A}\}$ *modulo* $\sim_\mathcal{A}$ *in the sense that the relation* $<'_\mathcal{A}$ *on* $G/\sim_\mathcal{A}$ *given by*

$$[I]_{\sim_\mathcal{A}} <'_\mathcal{A} [I']_{\sim_\mathcal{A}} \text{ iff } (\exists J, J' \in G)(J \sim_\mathcal{A} I, J' \sim_\mathcal{A} I', \text{ and } J <_\mathcal{A} J')$$

is a well-defined strict well-ordering, where $[I]_{\sim_\mathcal{A}}$ *denotes the equivalence class of* I *w.r.t.* $\sim_\mathcal{A}$*. Moreover:*

$$[I]_{\sim_\mathcal{A}} <'_\mathcal{A} [I']_{\sim_\mathcal{A}} \text{ iff } M^{\mathcal{A},I} \in M^{\mathcal{A},I'} \text{ iff } o_\mathcal{A}(I) < o_\mathcal{A}(I').$$

Proof. We first prove that $[I]_{\sim_\mathcal{A}} <'_\mathcal{A} [I']_{\sim_\mathcal{A}}$ iff $M^{\mathcal{A},I} \in M^{\mathcal{A},I'}$. Suppose $[I]_{\sim_\mathcal{A}} <'_\mathcal{A} [I']_{\sim_\mathcal{A}}$ and $J \sim_\mathcal{A} I$, $J' \sim_\mathcal{A} I'$, and $J <_\mathcal{A} J'$. Then $J \in \mathrm{dom}(M(\mathcal{A}, J')) = \mathrm{dom}(M^{\mathcal{A},J'})$. However then $M^{\mathcal{A},I} = M^{\mathcal{A},J} \in M^{\mathcal{A},J'} = M^{\mathcal{A},I'}$ since $M^{\mathcal{A},J}$ may be constructed in the ZF^--model $M^{\mathcal{A},J'}$ from $\mathcal{A}, J \in M^{\mathcal{A},J'}$. For the converse, suppose $M^{\mathcal{A},I} \in M^{\mathcal{A},J}$. However then $I' = I(\mathcal{A}, M^{\mathcal{A},I}) \in M^{\mathcal{A},J}$ and $M^{\mathcal{A},I'} = M^{\mathcal{A},I}$ by Lemma 2.20, i.e. $I \sim_\mathcal{A} I'$ and $I' <_\mathcal{A} J$, as required.

The result now follows since Lemma 2.25 implies that $\{M^{\mathcal{A},I} : I \in G\}$ is well-ordered by $\in$. □

Lemma 4.19. *Suppose* $\boldsymbol{V} = \boldsymbol{K}$, $\kappa \geq \lambda$, $\mathrm{cf}(\lambda) > \omega$, *and* $\mathcal{A}$ *is a* κ*-structure satisfying* ZFC. *If* $\gamma = o_\mathcal{A}(\lambda)$ *then there is a sequence* $\langle I_\varepsilon : \varepsilon < \gamma\rangle$ *of sets of g.i.'s for* $\mathcal{A}$ *cofinal in* λ *such that for any* $\varepsilon < \varepsilon' < \gamma$, $I_\varepsilon <_\mathcal{A} I_{\varepsilon'}$.

Moreover, $\langle I_\varepsilon : \varepsilon < \gamma\rangle$ *may chosen such that whenever* I *is set of g.i.'s for* $\mathcal{A}$ *cofinal in* λ, $\langle I_\varepsilon : \varepsilon < \eta\rangle \in M^{\mathcal{A},I}$, *where* $\eta = o_\mathcal{A}(I)$.

Proof. Let $G = \{I \subseteq \lambda : I \text{ is a cofinal set of g.i.' s for } \mathcal{A}\}$. Using Lemma 4.18, for each $\varepsilon < \gamma$ let $I_\varepsilon = I(\mathcal{A}, M_\varepsilon)$, where M_ε is the unique member of $G' = \{M^{\mathcal{A},I} : I \in G\}$ such that $o_\mathcal{A}(M_\varepsilon) = \varepsilon$. If $\varepsilon < \varepsilon' < \gamma$ then $o_\mathcal{A}(I_\varepsilon) = o_\mathcal{A}(M_\varepsilon) = \varepsilon < \varepsilon' = o_\mathcal{A}(M_{\varepsilon'}) = o_\mathcal{A}(I_{\varepsilon'})$ so, by Lemma 4.18 again, $I_\varepsilon <_\mathcal{A} I_{\varepsilon'}$.

Moreover, if I is set of g.i.'s for $\mathcal{A}$ cofinal in λ and $\eta = o_\mathcal{A}(I)$ then $M^{\mathcal{A},I} = M^{\mathcal{A},I_\eta} = M_\eta$, since $\eta = o_\mathcal{A}(I_\eta)$. However $M_\varepsilon \in M_\eta$ for all $\varepsilon < \eta$, so the construction of $\langle I_\varepsilon : \varepsilon < \gamma\rangle \upharpoonright \eta$ may be performed absolutely in M_η, hence $\langle I_\varepsilon : \varepsilon < \gamma\rangle \upharpoonright \eta \in M_\eta = M^{\mathcal{A},I}$. □

One way this can be interpreted, for example when $\lambda = \kappa$, is as an exact calculation of the order type of ω-closed filters on the $E^{\boldsymbol{K}}$ sequence that contain $\mathcal{A}$. If κ is measurable, or even just Ramsey, of course, then this order type is κ^+. However if less than κ^+ we get an indication of how many such E_ν 'measure' $\mathcal{A}$: it is $o_\mathcal{A}(\lambda)$. With $\mathcal{A}$ a sufficiently rich structure, in the notation of the last part of the lemma, M_η will be some $\boldsymbol{K}_{\tilde{\eta}}$.

We now return to the general case and aim to characterise the members of the filters $\mathcal{F}_\alpha$ in terms of canonical functions and $o_\mathcal{A}$. To do this, it will be necessary to define a particularly uniform variant of the sequence $\langle (X_\mathcal{A})_\beta : \beta \leqslant \alpha\rangle$ from Corollary 4.8. Since any 1-weakly Erdős cardinal is ω-Erdős, hence necessarily inaccessible, we shall fix an inaccessible cardinal κ. This allows us to simplify the following by assuming that κ-structures have domain $V_\kappa = H_\kappa$, and satisfy ZFC.

Fix an uncountable cardinal κ and suppose $g : \kappa \to \alpha$ is a bijection, and $\alpha < \kappa^+$. It will be convenient to assume $g(0) = 0$. For any function $f : \lambda \to \mu$ set $G(f) = \{\langle \eta, \xi\rangle \in \lambda^2 : f(\eta) < f(\xi)\}$.

Suppose $\mathcal{A} = \langle V_\kappa, \in, A_1, \ldots, A_n, G(g)\rangle$ is a κ-structure. For any $\lambda \leqslant \kappa$, $G(g) \cap \lambda^2$ is a well-ordering of λ so there is a map $g^\lambda : \lambda \to \alpha_\lambda$ where $\alpha_\lambda = \mathrm{otp}(G(g) \cap \lambda^2)$ defined by $g^\lambda(\eta) = \mathrm{rk}_{G(g)\cap\lambda^2}(\eta)$.

If $\lambda = \kappa$ then $g^\lambda = g$, and if $\lambda < \kappa$ then g^λ is uniformly definable from λ in $\mathcal{A}$. Let $\pi_\lambda : (\alpha_\lambda + 1) \to (g``\lambda) \cup \{\alpha\}$ be the inverse of the transitive collapse, so $g \upharpoonright \lambda = \pi_\lambda \circ g^\lambda$.

Let $0 < \gamma \leqslant \alpha_\lambda$. Define a surjection $g^\lambda_\gamma : \lambda \to \gamma$ by $g^\lambda_\gamma(\eta) = \begin{cases} g^\lambda(\eta) & \text{if } g^\lambda(\eta) < \gamma \\ 0 & \text{otherwise} \end{cases}$ and note the five equivalent ways of defining the term on the left:

$$\begin{aligned} \pi_\lambda \circ g^\lambda_\gamma(\eta) &= \begin{cases} \pi_\lambda \circ g^\lambda(\eta) & \text{if } g^\lambda(\eta) < \gamma \\ \pi_\lambda(0) & \text{otherwise} \end{cases} \\ &= \begin{cases} g(\eta) & \text{if } g^\lambda(\eta) < \gamma \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} g(\eta) & \text{if } \pi_\lambda \circ g^\lambda(\eta) < \pi_\lambda(\gamma) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} g(\eta) & \text{if } g(\eta) < \pi_\lambda(\gamma) \\ 0 & \text{otherwise} \end{cases} \\ &= g^\kappa_{\pi_\lambda(\gamma)}(\eta) \end{aligned}$$

We make the following definition explicitly, rather than modulo the club filter.
For $\lambda \leqslant \kappa$, define sequences $\langle (X_{\mathcal{A}\restriction\lambda})'_\beta : \beta \leqslant \alpha_\lambda \rangle$, $\langle (\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\beta : \beta \leqslant \alpha_\lambda \rangle$, $\langle (B_{\mathcal{A}\restriction\lambda})'_\beta : \beta \leqslant \alpha_\lambda \rangle$ by

$$\begin{aligned} (B_{\mathcal{A}\restriction\lambda})'_0 &= 0 \\ (\mathcal{B}_{\mathcal{A}\restriction\lambda})'_0 &= \mathcal{A} \restriction \lambda \\ (X_{\mathcal{A}\restriction\lambda})'_0 &= \lambda \end{aligned}$$

For $\beta > 0$, set

$$\begin{aligned} (B_{\mathcal{A}\restriction\lambda})'_\beta &= \{\langle \xi, \eta \rangle \in \lambda^2 : \eta \in (X_{\mathcal{A}\restriction\lambda})'_{g^\lambda_\beta(\xi)}\} \\ (\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\beta &= \langle \mathcal{A} \restriction \lambda, (B_{\mathcal{A}\restriction\lambda})'_\beta, G(g^\lambda_\beta) \rangle \end{aligned}$$

$$(X_{\mathcal{A}\restriction\lambda})'_\beta = \begin{cases} t_{(\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\beta}((X_{\mathcal{A}\restriction\lambda})'_\delta) & \text{for } \beta = \delta + 1 \\ \Delta_{\xi<\lambda}(X_{\mathcal{A}\restriction\lambda})'_{g^\lambda_\beta(\xi)} & \text{for limit } \beta \end{cases}$$

Finally, we define a sequence, $\langle D_\beta : \beta \leqslant \alpha \rangle$ of club subsets of κ by setting $D_\beta = \{\eta < \kappa : (\mathcal{B}_\mathcal{A})'_\beta \restriction \eta \prec (\mathcal{B}_\mathcal{A})'_\beta\} \backslash (g^{-1}(\beta) + 1)$ for each $\beta < \alpha$, and $D_\alpha = \{\eta < \kappa : (\mathcal{B}_\mathcal{A})'_\alpha \restriction \eta \prec (\mathcal{B}_\mathcal{A})'_\alpha\}$.

The definitions of $(B_{\mathcal{A}\restriction\lambda})'_\beta$, $(\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\beta$, $(X_{\mathcal{A}\restriction\lambda})'_\beta$, and D_β are all dependent on g (and $\mathcal{A}$). When we need to emphasize this we shall write $(B_{\mathcal{A}\restriction\lambda})'_\beta(g)$, $(\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\beta(g)$, $(X_{\mathcal{A}\restriction\lambda})'_\beta(g)$, and $D_\beta(g, \mathcal{A})$, respectively.

Lemma 4.20. *For any $\beta \leqslant \alpha$, $D_\beta \backslash (\eta + 1) \subseteq D_{g^\kappa_\beta(\eta)}$.*

Proof. Let $\lambda \in D_\beta$ and $\eta < \lambda$. We require that $\lambda \in D_{g^\kappa_\beta(\eta)}$. However

$$\begin{aligned} G(g^\kappa_{g^\kappa_\beta(\eta)}) &= \{\langle \xi, \zeta \rangle \in \kappa^2 : g^\kappa_{g^\kappa_\beta(\eta)}(\xi) < g^\kappa_{g^\kappa_\beta(\eta)}(\zeta)\} \\ &= \{\langle \xi, \zeta \rangle \in \kappa^2 : (g^\kappa_\beta(\xi) < g^\kappa_\beta(\zeta) < g^\kappa_\beta(\eta)) \,\text{or}\, (g^\kappa_\beta(\zeta) < g^\kappa_\beta(\eta) \leqslant g^\kappa_\beta(\xi))\} \\ &= \{\langle \xi, \zeta \rangle \in \kappa^2 : \langle \zeta, \eta \rangle \in G(g^\kappa_\beta) \wedge (\langle \xi, \zeta \rangle \in G(g^\kappa_\beta) \,\text{or}\, \langle \xi, \eta \rangle \notin G(g^\kappa_\beta))\} \end{aligned}$$

and

$$\begin{aligned}(B_{\mathcal{A}})'_{g^\kappa_\beta(\eta)} &= \{\langle\xi,\zeta\rangle \in \kappa^2 : \zeta \in (X_{\mathcal{A}})'_{g^\kappa_{g^\kappa_\beta(\eta)}(\xi)}\} \\ &= \{\langle\xi,\zeta\rangle \in \kappa^2 : (\zeta \in (X_{\mathcal{A}})'_{g^\kappa_\beta(\xi)}) \wedge g^\kappa_\beta(\xi) < g^\kappa_\beta(\eta)\} \\ &= \{\langle\xi,\zeta\rangle \in \kappa^2 : (\zeta \in (X_{\mathcal{A}})'_{g^\kappa_\beta(\xi)}) \wedge \langle\xi,\eta\rangle \in G(g^\kappa_\beta)\} \\ &= \{\langle\xi,\zeta\rangle \in \kappa^2 : (\langle\xi,\zeta\rangle \in (B_{\mathcal{A}})'_\beta) \wedge \langle\xi,\eta\rangle \in G(g^\kappa_\beta)\}\end{aligned}$$

So these two sets, and hence the structure $(\mathcal{B}_{\mathcal{A}})'_{g^\kappa_\beta(\eta)}$, are definable over $(B_{\mathcal{A}})'_\beta$ from η. Thus:,

$$(\mathcal{B}_{\mathcal{A}})'_\beta \restriction \lambda \prec (\mathcal{B}_{\mathcal{A}})'_\beta \text{ implies that } (\mathcal{B}_{\mathcal{A}})'_{g^\kappa_\beta(\eta)} \restriction \lambda \prec (\mathcal{B}_{\mathcal{A}})'_{g^\kappa_\beta(\eta)}.$$

So $\lambda \in D_{g^\kappa_\beta(\eta)}$ (of course, $g^{-1}(g^\kappa_\beta(\eta)) < \lambda$). □

Lemma 4.21. *For any $\beta \leqslant \alpha$ and $\lambda \in D_\beta$, $\gamma = \pi_\lambda^{-1}(\beta)$ exists and $(X_{\mathcal{A}\restriction\lambda})'_\gamma = (X_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \cap \lambda$, $(B_{\mathcal{A}\restriction\lambda})'_\gamma = (B_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \cap \lambda^2$, and $(\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\gamma = (\mathcal{B}_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \restriction \lambda$.*

Proof. That $\pi_\lambda^{-1}(\beta)$ exists follows easily from the definitions of D_β and π_λ: $\alpha \in \operatorname{ran}(\pi_\lambda)$ for all λ, and if $\beta < \alpha$ then $g^{-1}(\beta) < \lambda$ and $\beta = g(g^{-1}(\beta)) = \pi_\lambda(g^\lambda(g^{-1}(\beta)))$.

We proceed by induction on β. The case $\beta = 0$ is immediate. Suppose $\beta > 0$, and the hypothesis holds for all ordinals strictly less than β.

$$\begin{aligned}(B_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \cap \lambda^2 &= \{\langle\xi,\eta\rangle \in \lambda^2 : \eta \in (X_{\mathcal{A}})'_{g^\kappa_{\pi_\lambda(\gamma)}(\xi)}\} \\ &= \{\langle\xi,\eta\rangle \in \lambda^2 : \eta \in (X_{\mathcal{A}})'_{\pi_\lambda(g^\lambda_\gamma(\xi))}\} \\ &= \{\langle\xi,\eta\rangle \in \lambda^2 : \eta \in (X_{\mathcal{A}\restriction\lambda})'_{g^\lambda_\gamma(\xi)}\} \\ &= (B_{\mathcal{A}\restriction\lambda})'_\gamma\end{aligned}$$

where the third line follows from the induction hypothesis, using the fact that, by Lemma 4.20, $\lambda \in D_\beta \backslash (\xi + 1) \subseteq D_{g^\kappa_\beta(\xi)} = D_{g^\kappa_{\pi_\lambda(\gamma)}(\xi)} = D_{\pi_\lambda(g^\lambda_\gamma(\xi))}$ for each $\xi < \lambda$. That $(\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\gamma = (\mathcal{B}_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \restriction \lambda$ follows immediately since $G(g^\lambda_\gamma) = G(g^\kappa_{\pi_\lambda(\gamma)}) \cap \lambda^2$.

To prove $(X_{\mathcal{A}\restriction\lambda})'_\gamma = (X_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \cap \lambda$, we must consider two cases. Suppose, first, that β is a limit. Then

$$\begin{aligned}(X_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \cap \lambda &= \Delta_{\xi<\lambda}(X_{\mathcal{A}})'_{g^\kappa_{\pi_\lambda(\gamma)}(\xi)} \cap \lambda \\ &= \Delta_{\xi<\lambda}(X_{\mathcal{A}})'_{\pi_\lambda(g^\lambda_\gamma(\xi))} \cap \lambda \\ &= \Delta_{\xi<\lambda}(X_{\mathcal{A}\restriction\lambda})'_{g^\lambda_\gamma(\xi)} \cap \lambda \\ &= (X_{\mathcal{A}\restriction\lambda})'_\gamma\end{aligned}$$

where, just as for $(B_{\mathcal{A}\restriction\lambda})'_\gamma$, the third line follows the induction hypothesis and Lemma 4.20.

Now suppose that $\beta = \delta + 1$, some δ. Then $g^{-1}(\delta)$ is definable in $(\mathcal{B}_{\mathcal{A}})'_\beta$ since it is the $G(g_\beta)$-largest ordinal. Hence $g^{-1}(\delta) \in (B_{\mathcal{A}})'_\beta \restriction \lambda$. Set $\gamma' = g^\lambda(g^{-1}(\delta))$, and note $\beta = \pi_\lambda(\gamma') + 1$.

$$\begin{aligned}
(X_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \cap \lambda &= (X_{\mathcal{A}})'_{\pi_\lambda(\gamma')+1} \cap \lambda \\
&= t_{(\mathcal{B}_{\mathcal{A}})'_{\pi_\lambda(\gamma')+1}}((X_{\mathcal{A}})'_{\pi_\lambda(\gamma')}) \cap \lambda \\
&= t_{(\mathcal{B}_{\mathcal{A}})'_{\pi_\lambda(\gamma)}}((X_{\mathcal{A}})'_{\pi_\lambda(\gamma')}) \cap \lambda \\
&= t_{(\mathcal{B}_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \restriction \lambda}((X_{\mathcal{A}})'_{\pi_\lambda(\gamma')} \cap \lambda) \text{ since } \lambda \in D_{\pi_\lambda(\gamma)} \\
&= t_{(\mathcal{B}_{\mathcal{A} \restriction \lambda})'_\gamma}((X_{\mathcal{A}})'_{\pi_\lambda(\gamma')} \cap \lambda) \text{ since we have shown } (\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\gamma = (\mathcal{B}_{\mathcal{A}})'_{\pi_\lambda(\gamma)} \restriction \lambda \\
&= t_{(\mathcal{B}_{\mathcal{A} \restriction \lambda})'_\gamma}((X_{\mathcal{A}\restriction\lambda})'_{\gamma'}) \\
&= (X_{\mathcal{A}\restriction\lambda})'_\gamma
\end{aligned}$$

where this time it is the penultimate line that uses the induction hypothesis and Lemma 4.20. □

Lemma 4.22. *For each $\beta \leqslant \alpha$*

i. $(X_{\mathcal{A}})'_\beta \in \mathcal{F}_\beta$

ii. $(X_{\mathcal{A}})'_\beta \subseteq (X_{\mathcal{A}})_\beta$

Proof. Follows immediately by induction on β using the definitions and the Basic Properties Lemma 4.2. □

Remark 4.23. *It follows from Lemma 4.7 and Lemma 4.22 that the filters $\mathcal{F}_\beta$ are generated by the sets $(X_{\mathcal{A}})'_\beta$. So the analogue of Corollary 4.8 holds.*

Theorem 4.24. *For each $\beta \leqslant \alpha$, let $f_\beta : \kappa \to \kappa$ be the canonical function with definition $f_\beta(\lambda) = \mathrm{otp}(g^\kappa_\beta``\lambda)$. Then*

$$(X_{\mathcal{A}})'_\beta =_{\mathcal{C}} \{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\}$$

Specifically,

$$(*) \qquad (X_{\mathcal{A}})'_\beta \cap D_\beta = \{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\} \cap D_\beta.$$

Proof. Note first that if $g(\gamma_0) = \gamma$ then

$$\begin{aligned}
f_\gamma(\lambda) &= \mathrm{otp}(g^\kappa_\gamma``\lambda) \\
&= \mathrm{otp}(g^\kappa_{g(\gamma_0)}``\lambda) \\
&= \mathrm{otp}(g^\kappa_{\pi_\lambda(g^\lambda(\gamma_0))}``\lambda) \\
&= \mathrm{otp}(\pi_\lambda \circ g^\lambda_{g^\lambda(\gamma_0)}``\lambda) \\
&= \mathrm{otp}(g^\lambda_{g^\lambda(\gamma_0)}``\lambda) \\
&= g^\lambda(\gamma_0)
\end{aligned}$$

The proof will be by induction on β. If $\beta = 0$ then:

$$(X_{\mathcal{A}})'_\beta = \kappa = \{\gamma < \kappa : 0 \leqslant o_{\mathcal{A}}(\gamma)\} = \{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\}.$$

Suppose that $\beta = \gamma + 1$. Since $G(g^\kappa_\beta)$ is a predicate of $(\mathcal{B}_{\mathcal{A}})'_\beta$, $\gamma_0 = g^{-1}(\gamma)$ is definable in $(\mathcal{B}_{\mathcal{A}})'_\beta$ (it is either the $G(g^\kappa_\beta)$-maximal ordinal, if this exists, or 0). Similarly for $\beta_0 = g^{-1}(\beta)$. $(\mathcal{B}_{\mathcal{A}})'_\gamma$ is definable over

$(\mathcal{B}_\mathcal{A})'_\beta$ (from γ_0, which is itself definable) so $D_\beta \subseteq D_\gamma$, and any set of g.i.'s for $(\mathcal{B}_\mathcal{A})'_\beta$ is also a set of g.i.'s for $(\mathcal{B}_\mathcal{A})'_\gamma$. Let

$$\eta \in (X_\mathcal{A})'_\beta \cap D_\beta = (X_\mathcal{A})'_{\gamma+1} \cap D_\beta = t_{(\mathcal{B}_\mathcal{A})'_\beta}((X_\mathcal{A})'_\gamma) \cap D_\beta.$$

We show $f_\beta(\eta) \leq o_\mathcal{A}(\eta)$ to establish the $(\subseteq)$ direction of $(*)$. Let $I \subseteq (X_\mathcal{A})'_\gamma$ be a set of g.i.'s for $(\mathcal{B}_\mathcal{A})'_\beta$ with $\sup(I) = \eta$. Then $I \subseteq D_\beta \subseteq D_\gamma$ so by the inductive hypothesis:

$$I \subseteq (X_\mathcal{A})'_\gamma \cap D_\gamma = \{\xi < \kappa : f_\gamma(\xi) \leqslant o_\mathcal{A}(\xi)\} \cap D_\gamma.$$

Let $\lambda \in I$. Since $f_\gamma(\lambda) \leqslant o_\mathcal{A}(\lambda)$, and both $f_\gamma(\lambda) = g^\lambda(\gamma_0)$ and $\mathcal{A} \restriction \lambda$ are uniformly definable in $\mathcal{A}$ from λ with parameter γ_0, there is a sequence $\langle I^\lambda_\xi : \xi < f_\gamma(\lambda)\rangle$ of sets of g.i.'s for $\mathcal{A}$, uniformly definable (in $\mathcal{A}$ from λ with parameter γ_0), such that $o_{\mathcal{A}\restriction\lambda}(I^\lambda_\xi) \geqslant \xi$ whenever $\xi < f_\gamma(\lambda)$.

In the notation of the Jensen Indiscernibles Lemma proof, set $\mathcal{D} = \mathcal{B}^{\mathcal{A},I}_\delta$, where $\delta = \min(I)$. Then, by the definability described in the previous paragraph:

$$\langle I^\delta_\xi : \xi < f_\gamma(\delta)\rangle \in \mathcal{D} \text{ and } \pi^{\mathcal{A},I}_{\delta,\lambda}(\langle I^\delta_\xi : \xi < f_\gamma(\delta)\rangle) = \langle I^\lambda_\xi : \xi < f_\gamma(\lambda)\rangle \text{ for all } \lambda \in I.$$

Let $\vec{J} = \pi^{\mathcal{A},I}_\eta(\langle I^\delta_\xi : \xi < f_\gamma(\delta)\rangle) = \langle J_\xi : \xi < \varepsilon\rangle$. Then:

$$\varepsilon = \pi^{\mathcal{A},I}_\eta(f_\gamma(\delta)) = \pi^{\mathcal{A},I}_\eta(g^\delta(\gamma_0)) = g^\eta(\gamma_0) = f_\gamma(\eta)$$

so $\vec{J}$ is a sequence such that every member is in $M(\mathcal{A}, I)$ and $o_{\mathcal{A}\restriction\eta}(J_\xi) \geqslant \xi$ for each $\xi < f_\gamma(\eta)$. So I witnesses that $o_\mathcal{A}(\eta) \geqslant f_\gamma(\eta) + 1 = f_\beta(\eta)$.

For the converse, let $\eta \in D_\beta$ be such that $f_\beta(\eta) \leqslant o_\mathcal{A}(\eta)$. Let $\langle J_\xi : \xi < f_\beta(\eta)\rangle$ witness this, and let $J = J_\varepsilon$ where $\varepsilon = f_\gamma(\eta)$ (note that by the definition of D_β, $f_\beta(\eta) = f_\gamma(\eta) + 1$). So $o_\mathcal{A}(J) = \varepsilon$ and $J_\xi \in M(\mathcal{A}, J)$ for each $\xi < \varepsilon$. Since $\gamma_0 = g^{-1}(\gamma) < \eta$ (by definability in $(\mathcal{B}_\mathcal{A})'_\beta$, since $\eta \in D_\beta$) and $M(\mathcal{A}, J) = M(\mathcal{A}, J\backslash(\gamma_0 + 1))$, we assume $\gamma_0 < \min(J)$.

It follows by elementarity (in the language of set theory) that for any $\delta \in J$,

$$\mathcal{B}^{\mathcal{A},J}_\delta \vDash (\forall \xi < g^\delta(\gamma_0))(\exists I)(\sup(I) = \delta \text{ and } o_\mathcal{A}(I) \geqslant \xi)$$

So, by absoluteness, $o_\mathcal{A}(\delta) \geqslant g^\delta(\gamma_0) = f_\gamma(\delta)$. To apply the induction hypothesis and complete the successor step, it is now only necessary to check that J is a set of g.i.'s for $(\mathcal{B}_\mathcal{A})'_\beta$, and hence that $J \subseteq D_\beta \subseteq D_\gamma$.

$\beta_0 = g^{-1}(\beta) < \min(J)$ (the inequality holds since $\gamma_0 < \min(J)$, and β_0 is definable in $\mathcal{A}$ from γ_0). Using Lemma 4.21 for the third equality:

$$(\mathcal{B}_\mathcal{A})'_\beta \restriction \eta = (\mathcal{B}_\mathcal{A})'_{g(\beta_0)} \restriction \eta = (\mathcal{B}_\mathcal{A})'_{\pi_\eta(g^\eta(\beta_0))} \restriction \eta = (\mathcal{B}_{\mathcal{A}\restriction\eta})'_{g^\eta(\beta_0)}.$$

However $(\mathcal{B}_{\mathcal{A}\restriction\eta})'_{g^\eta(\beta_0)} \in M(\mathcal{A}, J)$ because $(\mathcal{B}_{\mathcal{A}\restriction\eta})'_{g^\eta(\beta_0)}$ can be constructed absolutely from $G(g) \restriction \eta, \beta_0$, and H_η in any transitive ZF^--model (recall that $G(g)$ is a predicate of $\mathcal{A}$). Moreover, by Lemma 2.19, J is a set of g.i.'s for $(\mathcal{B}_{\mathcal{A}\restriction\eta})'_{g^\eta(\beta_0)} = (\mathcal{B}_\mathcal{A})'_\beta \restriction \eta \prec (\mathcal{B}_\mathcal{A})'_\beta$. So J is a set of g.i.'s for $(\mathcal{B}_\mathcal{A})'_\beta$, as required.

Now suppose that β is a limit ordinal, and let $\beta_0 = g^{-1}(\beta)$.

Let $\eta \in (X_\mathcal{A})'_\beta \cap D_\beta$. Now, $(X_\mathcal{A})'_\beta = \Delta_{\xi<\kappa}(X_\mathcal{A})'_{g^\kappa_\beta(\xi)}$ so $\eta \in (X_\mathcal{A})'_{g^\kappa_\beta(\xi)}$ for any $\xi < \eta$. By Lemma 4.20 and the induction hypothesis, we have

$$\eta \in (X_\mathcal{A})'_{g^\kappa_\beta(\xi)} \cap D_{g^\kappa_\beta(\xi)} \quad = \quad \{\gamma < \kappa : f_{g^\kappa_\beta(\xi)}(\gamma) \leqslant o_\mathcal{A}(\gamma)\} \cap D_{g^\kappa_\beta(\xi)}$$

for all $\xi < \eta$. However then, as required:

$$f_\beta(\eta) = \mathrm{otp}(\beta \cap g``\eta) = \sup_{\xi<\eta}(\mathrm{otp}(\beta \cap g``\xi)) = \sup_{\xi<\eta}(f_\beta(\xi)) \leqslant o_{\mathcal{A}}(\eta).$$

For the other direction, let $\eta \in \{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\} \cap D_\beta$. Then $f_{g^\kappa_\beta(\xi)}(\eta) \leqslant f_\beta(\eta) \leqslant o_{\mathcal{A}}(\eta)$ for all $\xi < \eta$ so

$$\begin{aligned} \eta \in \Delta_{\xi<\kappa}\{\gamma < \kappa : f_{g^\kappa_\beta(\xi)}(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\} \cap D_{g^\kappa_\beta(\xi)} &= \Delta_{\xi<\kappa}(X_{\mathcal{A}})'_{g^\kappa_\beta(\xi)} \cap D_{g^\kappa_\beta(\xi)} \\ &\subseteq (X_{\mathcal{A}})'_\beta \end{aligned}$$

So the induction is complete. □

Corollary 4.25. *Let κ be inaccessible, and let $\langle f_\beta : \beta < \kappa^+\rangle$ be any choice of canonical functions for κ. Then, for any $\beta < \kappa^+$ with $\beta > 0$, $\mathcal{F}_\beta$ is the filter generated by the sets $\{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\}$ as $\mathcal{A}$ ranges over all κ-structures.*

Consequently, κ is greatly Erdős iff $\{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\} \neq \varnothing$ for all $\beta < \kappa^+$ and κ-structures $\mathcal{A}$ such that $\mathcal{A} \vDash$ ZFC.

Proof. By the remark following Lemma 4.22, given any bijection $g : \kappa \to \alpha$, for each $\beta \leqslant \alpha$ $\mathcal{F}_\beta$ is generated by the sets $(X_{\mathcal{A}})'_\beta(g)$ as $\mathcal{A}$ ranges over κ-structures of the form $\langle V_\kappa, \in, A_1, \ldots, A_n, G(g)\rangle$. So, by Theorem 4.24 and normality of the filters, each $\mathcal{F}_\beta$ is generated by the sets $\{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\} \cap C$ as C ranges over clubs and $\mathcal{A}$ over κ-structures satisfying ZFC (unlike $(X_{\mathcal{A}})'_\beta(g)$, the sets $\{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\}$ are defined for all such $\mathcal{A}$).

So it suffices to find, for each $\mathcal{A}$ and C, a structure $\mathcal{A}'$ such that

$$\{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\} \cap C \supseteq \{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}'}(\gamma)\}.$$

Set $\mathcal{A}' = \langle \mathcal{A}, C\rangle$. Then, by induction on $\delta \leqslant \beta$ with $0 < \delta$, it is clear that

$$\{\gamma < \kappa : f_\delta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\} \cap C \supseteq \{\gamma < \kappa : f_\delta(\gamma) \leqslant o_{\mathcal{A}'}(\gamma)\},$$

since any set, I, of g.i.'s for $\mathcal{A}'$ must be a subset of C, hence $\sup(I) \in C$. □

Corollary 4.26. *Suppose κ is a stationary limit of Ramsey cardinals. Then κ is greatly Erdős.*

Proof. If $\gamma < \kappa$ is a Ramsey cardinal then for any κ-structure $\mathcal{A}$ (satisfying ZFC), $o_{\mathcal{A}}(\gamma) = \gamma^+$. By Corollary 4.25, $\mathcal{F}_\beta \not\ni \varnothing$ for all $\beta < \kappa^+$. □

We conclude this subsection with a minor generalisation that will be of use in Section 4.

Definition 4.27. *Let κ have uncountable cofinality and let $X \subseteq \kappa$. Let $\mathcal{A}$ be a κ-structure that satisfies* ZFC*, and suppose $I, J \subseteq X$ are cofinal sets of good indiscernibles for $\mathcal{A}$.*

Say $I <^X_{\mathcal{A}} J$ iff $I \in M(\mathcal{A}, J)$.

Define $o^X_{\mathcal{A}}(I)$ and $o^X_{\mathcal{A}} : \kappa \to \kappa$ analogously to $o_{\mathcal{A}}(I)$ and $o_{\mathcal{A}} : \kappa \to \kappa$.

Lemma 4.28. *Let $X \in \mathcal{F}$, where $\mathcal{F}$ is the minimal witness that κ is greatly Erdős. Let $\langle f_\beta : \beta < \kappa^+\rangle$ be any choice of canonical functions for κ. Then the filter $\mathcal{F}$ is generated by the sets $\{\gamma < \kappa : f_\beta(\gamma) \leqslant o^X_{\mathcal{A}}(\gamma)\}$ as $\mathcal{A}$ ranges over all κ-structures satisfying* ZFC *and β over ordinals less than κ^+.*

Proof. In the definition of $\langle (X_{\mathcal{A}\restriction\lambda})'_\beta : \beta \leqslant \alpha_\lambda\rangle$, $\langle (\mathcal{B}_{\mathcal{A}\restriction\lambda})'_\beta : \beta \leqslant \alpha_\lambda\rangle$, $\langle (B_{\mathcal{A}\restriction\lambda})'_\beta : \beta \leqslant \alpha_\lambda\rangle$ given above, we could have set $(X_{\mathcal{A}\restriction\lambda})'_0 = \lambda \cap X$. Just as above, it would then be the case that $(X_{\mathcal{A}})'_\beta \in \mathcal{F}_{\gamma+\beta}$ and $(X_{\mathcal{A}})'_\beta \subseteq (X_{\mathcal{A}})_\beta$ whenever $\gamma + \beta \leqslant \alpha$, where $X \in \mathcal{F}_\gamma$. Hence, for any $\beta < \kappa^+$, if $\mathcal{G}$ is the filter generated by the sets $(X_{\mathcal{A}})'_\beta$ as $\mathcal{A}$ ranges over κ-structures then $\mathcal{F}_{\gamma+\beta} \subseteq \mathcal{G} \subseteq \mathcal{F}_\beta$. So $\mathcal{F}$ is generated by the the sets $(X_{\mathcal{A}})'_\beta$ as $\mathcal{A}$ ranges over κ-structures and β ranges over ordinals less than κ^+.

The proof of Theorem 4.24 shows that $(X_{\mathcal{A}})'_\beta =_{\mathcal{C}} \{\varepsilon < \kappa : f_\beta(\varepsilon) \leqslant o^X_{\mathcal{A}}(\varepsilon)\}$, and the proof of Corollary 4.25 gives the required result. □

Lemma 4.29. *Suppose $\boldsymbol{V} = \boldsymbol{K}$, $\kappa > \lambda$, $\mathrm{cf}(\lambda) > \omega$, $X \subseteq \kappa$, and $\mathcal{A} = \langle J^E_\kappa, \in, E, \vec{A}, X\rangle$ is a κ-structure satisfying* ZFC. *If $\gamma = o^X_{\mathcal{A}}(\lambda)$ then there is a sequence $\langle I_\varepsilon : \varepsilon < \gamma\rangle$ of sets of g.i.'s for $\mathcal{A}$ cofinal in λ such that for any $\varepsilon < \varepsilon' < \gamma$, $I_\varepsilon <^X_{\mathcal{A}} I_{\varepsilon'}$.*

Moreover, $\langle I_\varepsilon : \varepsilon < \gamma\rangle$ may chosen such that whenever I is set of g.i.'s for $\mathcal{A}$ cofinal in λ, $\langle I_\varepsilon : \varepsilon < \eta\rangle \in M^{\mathcal{A},I}$, where $\eta = o^X_{\mathcal{A}}(I)$.

Proof. Just as Lemma 4.19. □

4.3. Consistency Strength

Lemma 4.30. *If κ is a Ramsey cardinal then the club filter witnesses that κ is greatly Erdős.*

Proof. In Example 4.3 it is observed that if κ is Ramsey then $t_{\mathcal{A}}(C)$ contains a club whenever C is club. This observation and the fact that the diagonal intersection of κ-many clubs is again club prove the successor and limit stages respectively of a proof by induction on β that $\mathcal{F}_\beta = \mathcal{C}$. The case $\beta = 0$ holds by definition. □

Remark 4.31. *Lemma 4.30 shows that there can be no result in the style of [12] showing that the filters are strictly increasing since it is possible that $\mathcal{F}_\beta = \mathcal{C}$ for all $\beta < \kappa^+$. The closest results to a hierarchy theorem in the sense of that paper are Theorem 4.24, Corollary 4.25, and Lemma 4.35.*

Lemma 4.32. *Let κ be greatly Erdős and suppose that $M \supseteq H_\kappa$ is a useful premouse. Then:*

$$\{\gamma < \kappa : \gamma \textit{ is greatly Erdős }\} \in U(M).$$

In particular, there are stationarily many completely ineffable, greatly Erdős cardinals below any Ramsey cardinal.

Proof. Since $U = U(M)$ is an amenable normal measure w.r.t. M, the ultrapower $N = \mathrm{Ult}(M, U)$ is well-founded up to $(\kappa^+)^M = \mathrm{On} \cap M = (\kappa^+)^N$.

Let $\mathcal{A} \in N$ be a κ-structure, and let $\langle f_\beta : \beta < (\kappa^+)^N\rangle \in N$ be a sequence of canonical functions. Then $\langle f_\beta : \beta < (\kappa^+)^N$ really is a sequence of canonical functions (in $\boldsymbol{V}$). Moreover, since $H^N_\kappa = H^M_\kappa = H_\kappa$:

$$\{\gamma < \kappa : f_\beta(\gamma) \leqslant (o_{\mathcal{A}}(\gamma))^N\} = \{\gamma < \kappa : f_\beta(\gamma) \leqslant o_{\mathcal{A}}(\gamma)\} \neq \varnothing.$$

So $N \models \kappa$ is greatly Erdős . So by Łoś Theorem, and normality of U, $\{\gamma < \kappa : \gamma$ is greatly Erdős$\}$ $\in U$.

Suppose now that κ is Ramsey and $C \subseteq \kappa$ is club. Then let $\mathcal{A} = \langle H_\kappa, \in, A, C\rangle$, where A is such that $J^A_\kappa = H_\kappa$, and let I be a cofinal set of g.i.'s for $\mathcal{A}$. Let $M = M(\mathcal{A}, I)$. Then $I \subseteq C$ (by the argument of Example 4.3), and $\{\gamma < \kappa : \gamma$ is greatly Erdős $\}$ must contain a final segment of I, since $\{\gamma < \kappa : \gamma$ is greatly Erdős $\} \in U(M)$ and I generates $U(M)$, so every element of I is greatly Erdős, by indiscernibility.

To see that every $\gamma \in I$ is completely ineffable, it is sufficient to show that $(Q^\kappa_\alpha)^N = (Q^\kappa_\alpha)^M \not\ni \kappa$ (that is, Q^κ_α as defined over M) for all $\alpha \in M$ (see Section 3.2.2 for definitions) - can then apply Łoś Theorem, as above.

We induct on $\alpha \in \{-1\} \cup \mathrm{On} \cap M$ to show that $U(M) \subseteq ((Q^\kappa_\alpha)^+)^M$ for all such α. This is non-trivial only if $\alpha = \beta+1$ for some $\beta \in \{-1\} \cup \mathrm{On} \cap M$, so suppose this is the case. Let $f \in M$, $f : [X]^2 \to 2$, $X \in U(M)$. Then, by Lemma 2.5 and the induction hypothesis, $Y_2^{f,M} \in U(M) \subseteq ((Q^\kappa_\beta)^+)^M$ is homogeneous for f. The induction is complete. □

Lemma 4.33. *If κ is a 2-weakly Erdős cardinal then κ is* almost Ramsey *(cf [26]). That is, for any $\gamma < \kappa$ and κ-structure, $\mathcal{A}$, there is a set of g.i.'s for $\mathcal{A}$ of order type at least γ.*

Proof. Let $\mathcal{A}$ be a κ-structure, let $\gamma < \kappa$, and let I, J be sets of g.i.'s for $\langle \mathcal{A}, \{\gamma\} \rangle$ such that $\lambda = \sup(I) = \sup(J)$ and $I \in M = M(\mathcal{A}, J)$. Then:

$$M \models \text{“There is a cofinal set of good indiscernibles for } \mathcal{A} \restriction \lambda \text{”}.$$

Hence:

$$\mathcal{B}^{\mathcal{A},J}_\delta \models \text{“ There is a cofinal set of good indiscernibles for } \mathcal{A} \restriction \delta\text{” for each } \delta \in J.$$

Pick $\delta \in J$ and let I' be such that $\mathcal{B}^{\mathcal{A},J}_\delta \models$ “I' is a cofinal set of good indiscernibles for $\mathcal{A} \restriction \delta$” This is absolute since the satisfaction relation for $\mathcal{A} \restriction \delta$ is a member of $\mathcal{B}^{\mathcal{A},J}_\delta$ (since $\mathcal{B}^{\mathcal{A},J}_\delta$ is a transitive ZF^--model containing $\mathcal{A} \restriction \delta$). Moreover, δ is regular, since $\delta \in J$, so I' is a set of g.i.'s for $\mathcal{A} \restriction \delta \prec \mathcal{A}$ with $\mathrm{otp}(I') = \delta > \gamma$. □

Lemma 4.34. *Suppose $\neg 0^\P$ and κ is $(\alpha+1)$-weakly Erdős. Then in $\boldsymbol{K}$, κ is α-weakly Erdős. Consequently, if κ is greatly Erdős then it is greatly Erdős in $\boldsymbol{K}$.*

Proof. We use the characterisation of Corollary 4.8. Let $\mathcal{A} \in \boldsymbol{K}$ be a κ-structure satisfying ZFC, w.l.o.g. $\mathrm{dom}(\mathcal{A}) = (H_\kappa)^{\boldsymbol{K}}$. Let $C \subseteq \kappa$ be club. Then $C \cap (X_\mathcal{A})_\alpha \in \mathcal{F}_\alpha$, so let $\gamma \in t_\mathcal{A}(C \cap (X_\mathcal{A})_\alpha)$ and let $I \subseteq \gamma \cap (C \cap (X_\mathcal{A})_\alpha)$ witness this. Since $I \subseteq \mathrm{Reg}$, we have shown $(X_\mathcal{A})_\alpha \cap \mathrm{Reg}$ is stationary. Since $\langle (X_\mathcal{A})_\beta : \beta < \alpha \rangle$ is decreasing (mod club), $(X_\mathcal{A})_\beta \cap \mathrm{Reg}$ is stationary for all $\beta < \alpha$. However (inductively) the Jensen Indiscernibles Lemma implies that $((X_\mathcal{A})_\beta)^{\boldsymbol{K}} \cap \mathrm{Reg} = (X_\mathcal{A})_\beta \cap \mathrm{Reg}$ for all $\beta < \kappa^+$. Hence $((X_\mathcal{A})_\alpha)^{\boldsymbol{K}} \neq \varnothing$. □

The following establishes some form of hierarchy lemma. We omit its proof.

Lemma 4.35. *Suppose $\boldsymbol{V} = \boldsymbol{K}$ and κ is an inaccessible cardinal. Let $\langle f_\beta : \beta < \kappa^+ \rangle$ be canonical functions for κ derived from surjections $\langle g_\beta : \kappa \to \beta : \beta < \kappa^+ \rangle$, i.e. $f_\beta(\lambda) = \mathrm{otp}(g_\beta``\lambda)$.*

Suppose $o_\mathcal{A}(\gamma) \geqslant f_\beta(\gamma)$, $\mathrm{cf}(\gamma) > \omega$, where $\mathcal{A}$ is a κ-structure with domain H_κ and $G(g_\beta)$ as a predicate. Let I be a set of g.i.'s for $\mathcal{A}$ with $\sup(I) = \gamma$ and $o_\mathcal{A}(I) = 0$. Then every $\eta \in I$ is $f_\beta(\eta)$-weakly Erdős.

In particular, if $\alpha < \beta < \kappa^+$ and κ is β-weakly Erdős then the set $\{\eta < \kappa : \eta$ is $f_\alpha(\eta)$-weakly Erdős$\}$ is stationary in κ.

5. Characterisations of Ramsey Cardinals

In this section we look at some characterisations of Ramsey cardinals: one old (5.1), one new (5.6), and one plausible but false ($\boldsymbol{Q}(\lambda)$).

5.1. Mitchell's Characterisation

In effect, Theorem 5.1 has been used several times already. The statement is made in the terminology we have established so differs somewhat to the original (the notion of a 'useful structure' is new).

Theorem 5.1. **(Mitchell, [18])** *The following are equivalent for κ with $\mathrm{cf}(\kappa) > \omega$:*

i. κ is a Ramsey cardinal

ii. for every $A \subseteq \kappa$ there is a pre-useful structure M with $\lambda(M) = \kappa$ and $A \in M$ such that $U(M)$ is ω-complete

iii. for every $A \subseteq \kappa$ there is a useful structure M with $\lambda(M) = \kappa$ and $A \in M$

Proof. (i⇒ii) Let I be a cofinal set of g.i.'s for $\mathcal{A} = \langle J^E_\kappa, \in, E, A\rangle$. Set $M = M^{\mathcal{A},I}$. ii holds by Lemma 2.23, since $\mathrm{cf}(\kappa) > \omega$.

(ii⇒iii) By Lemma 2.8, M is useful since $U(M)$ is ω-complete.

(iii⇒i) Given a κ-structure $\mathcal{A}$, there is a useful structure $M \ni \mathcal{A}$ with $\lambda(M) = \kappa$, by iii. Then $I(\mathcal{A}, M)$ is a set of g.i.'s for $\mathcal{A}$ with supremum κ. □

Useful structures do not always have ω-complete measures, but most natural examples will be iterable, so we may ask whether ω-completeness in clause ii of Theorem 5.1 could be weakened to iterability. Consider the following property:

$\boldsymbol{Q}(\lambda)$: *for every $A \subseteq \lambda$ there is a pre-useful structure M with $\lambda(M) = \lambda$ and $A \in M$ such that **M** is iterable by $U(M)$.*

We show that $\boldsymbol{Q}(\lambda)$ is not equivalent to λ being Ramsey. In fact, it is strictly weaker than the existence of an ω_1-Erdős cardinal. It follows, also, that an iterable pre-useful structure need not be useful.

Lemma 5.2. *Let $M \supseteq H_\lambda$ be a useful structure with $\lambda = \lambda(M)$ and ω-complete measure $U = U(M)$ (so $|H_\lambda| = \lambda \geqslant \omega_1$). Then $\{\alpha < \lambda : \boldsymbol{Q}(\alpha)\} \in U(M)$.*

In particular, if λ is an ω_1-Erdős cardinal then there are stationarily many $\alpha < \lambda$ such that $\boldsymbol{Q}(\alpha)$.

Proof. Let $\nu = (\lambda^+)^M = \mathrm{ht}(M)$. U is an amenable normal M-ultrafilter so the ultrapower $N = \mathrm{Ult}(M, U)$ is well-founded. By amenability, $(\lambda^+)^N = \nu$ and $(H_\nu)^N = (H_\nu)^M = M$. Suppose, w.l.o.g., that $M = M(H_\lambda, I)$, where $I = I(H_\lambda, M)$. Let $A \in P(\lambda) \cap M = P(\lambda) \cap N$. Let $\gamma \in I$ such that there is $\overline{A} \in M^{H_\lambda,I}_\gamma$ with $\pi^{H_\lambda,I}_\gamma(\overline{A}) = A$. Note that $M^{H_\lambda,I}_\gamma \in H_\lambda$ is iterable and useful, since it embeds into M. Hence:

$H_\lambda \models$"*There is a $\overline{M} \in H_{\gamma^+}$ such that $\overline{A} \in \overline{M}$ and $\overline{M}$ is iterable and pre-useful*".

However $\overline{A} \in M^{\mathcal{A},I}_\gamma$, so, by definition of $M^{H_\lambda,I}_\gamma$:

$M^{H_\lambda,I}_\gamma \models$"*There is a $\overline{M} \in H_{\gamma^+}$ such that $\overline{A} \in \overline{M}$ and $\overline{M}$ is iterable and pre-useful.*"

However then, by elementarity of $\pi^{A,I}_\gamma$:

$M = M^{H_\lambda,I} \models$*There is a $\overline{M} \in H_{\lambda^+}$ such that $\overline{A} \in \overline{M}$ and $\overline{M}$ is iterable and pre-useful*",

Since $(H_\alpha)^N = (H_\alpha)^M = M$,

$N \models$"*There is a $\overline{M} \in H_\nu$ such that $\overline{A} \in \overline{M}$ and $\overline{M}$ is iterable and pre-useful.*"

So $N \models \boldsymbol{Q}(\lambda)$. By Łoś Theorem, $\{\alpha < \lambda : \boldsymbol{Q}(\alpha)\} \in U$.

If λ is ω_1-Erdős and $C \subseteq \lambda$ is club then set $M = M(\langle H_{\gamma^*}, \in, C\rangle, I)$, where I is a set of good indiscernibles for $\langle H_{\gamma^*}, \in, C\rangle$ with $\mathrm{cf}(\mathrm{otp}(I)) > \omega$ and $\gamma^* = \sup(I)$.

Then $\{\alpha < \gamma^* : \boldsymbol{Q}(\alpha)\} \cap C \in U$. □

5.2. A Characterisation in Terms of Games

Donder and Levinski, [11], introduced games that characterised ω_1-Erdős cardinals. The following is similar to their G^{**} - see Sect 3.3 *op. cit.*

Definition 5.3. *Let λ be a limit ordinal, and let $\mathcal{A}$ be a λ-structure. Then $\widetilde{G}_\lambda(\mathcal{A})$ is the game with (up to) ω_1 moves where in turn ξ, player (I) plays an ordinal $\eta_\xi < \lambda$ and player (II) plays I_ξ such that*

i. I_ξ *is a set of good indiscernibles for* $\mathcal{A}$ *;*

ii. $\lim(\mathrm{otp}(I_\xi))$ *and* $\sup(I_\xi) \geqslant \eta_\xi$ *;*

iii. $I_\xi \trianglerighteq \bigcup_{\zeta<\xi} I_\zeta$.

The game terminates at $\xi < \omega_1$ iff (II) cannot move. (II) wins iff the game continues for ω_1 moves.

Their arguments show:

Theorem 5.4. *λ is ω_1-Erdős iff for no λ-structure $\mathcal{A}$ does I have a winning strategy for $\widetilde{G}_\lambda(\mathcal{A})$.*

The game may not be determined, so it is natural to ask when *II* has a winning strategy. We prove in Lemma 5.6 that this occurs in $\boldsymbol{K}$ exactly when λ is Ramsey.

Lemma 5.5. *Suppose $\neg 0^\P$. Suppose (II) has a winning strategy in $\widetilde{G}_\kappa(\mathcal{A})$ for all κ-structures $\mathcal{A}$. Then κ is Ramsey in $\boldsymbol{K}$.*

Remark: It is possible to show that *(II)* has a winning strategy then she has one that also always plays indiscernible sets of minimal $o_{\mathcal{A}}$ order. We shall not use that fact to prove the lemma, but use something faster.

Proof. Let $\mathbb{P} = \langle \{f : f \text{ is an injection } f : \alpha \to \kappa, \text{ some } \alpha < \omega_1\}, \subseteq\rangle$. Let G be generic for $\mathbb{P}$, and let $F = \bigcup G$. So $F : \omega_1 \to \kappa$ is a bijection.

Let $\mathcal{A} \in \boldsymbol{K}^{\boldsymbol{V}[G]} = \boldsymbol{K}^{\boldsymbol{V}}$ and let σ be a winning strategy for $\widetilde{G}_\kappa(\mathcal{A})$ in $\boldsymbol{V}$. Define a play, p, of $\widetilde{G}_\kappa(\mathcal{A})$ by

$$\begin{aligned} p(\delta) &= F(\varepsilon) && \text{if } \delta = 2\cdot\varepsilon, \text{ some } \varepsilon; \\ p(\delta) &= \sigma'(p \restriction \delta) && \text{if } \delta = 2\cdot\varepsilon{+}1, \text{ some } \varepsilon, \end{aligned}$$

and note that this is well-defined since each partial play $p \restriction \delta$ is in $\boldsymbol{V}$, since $\mathbb{P}$ is countably closed (so $({}^{\omega}\boldsymbol{V})^{\boldsymbol{V}[G]} \subseteq \boldsymbol{V}$).

Let $I = \bigcup_{\varepsilon<\omega_1} p(2\cdot\varepsilon{+}1)$. Then I is a set of g.i.'s for $\mathcal{A}$ with supremum κ, and, by Jensen Indiscernibles Lemma, there is $I' \supseteq I$ with $I' \in \boldsymbol{K}^{\boldsymbol{V}[G]} = \boldsymbol{K}^{\boldsymbol{V}}$ such that I' is a set of g.i.'s for $\mathcal{A}$, since the fact that $\mathbb{P}$ is ω-closed implies that $\mathrm{cf}^{\boldsymbol{V}[\boldsymbol{G}]}(\kappa) > \omega$. □

Theorem 5.6. *If $\boldsymbol{V} = \boldsymbol{K}$ then II has a winning strategy in $\widetilde{G}_\lambda(\mathcal{A})$ for all λ-structures $\mathcal{A}$ iff λ is Ramsey.*

Proof. The direction from left to right is proven in Lemma 5.5. For right to left, fix $\mathcal{A}$ and a cofinal set of g.i.'s, I, for $\mathcal{A}$. Then set $\sigma(p) = I$ for all plays p. σ is clearly a winning strategy. □

6. Strengthenings of Chang's Conjecture

In his book, [23], Shelah proves that if Namba forcing is semi-proper (see [17], p601, Thm 2.5(2) for definitions and properties) then a strong form of Chang's Conjecture holds, as follows:

Definition 6.1. *Strong Chang's Conjecture* (SCC) *holds iff whenever χ is a regular cardinal, $\chi > 2^{\aleph_2}$, M^* is any structure extending $\langle H_\chi, \in\rangle$, and $N \prec M^*$ is countable, for arbitrarily large $\alpha < \omega_2$ there is N_α with $N \prec N_\alpha \prec M^*$, $\alpha \in N_\alpha$, and $N_\alpha \cap \omega_1 = N \cap \omega_1$.*

Magidor (unpublished, and in private conversation) has shown that the semi-properness of Namba forcing is equiconsistent with the existence of a measurable cardinal.

To make some further definitions, it will be useful to have a variant of the notion of κ-structure that allows infinitely many predicates, and has a fixed domain. Say that $\mathcal{A}$ is a H_κ-structure iff $\mathcal{A}$ has the form $\langle H_\kappa, \in, A_0, \ldots, A_n, \ldots\rangle_{n<\omega}$, for some $A_0, \ldots, A_n, \ldots \subseteq H_\kappa (n < \omega)$.

Definition 6.2. CC^+ *holds iff for any H_{ω_2}-structure, $\mathcal{A}$, there is an H_{ω_2}-structure, $\mathcal{B}$, extending $\mathcal{A}$, such that for any countable $X \prec \mathcal{B}$ and $\gamma < \omega_2$ there is $Y \prec \mathcal{B}$ with $Y \supseteq X$, $X \cap \omega_1 = Y \cap \omega_1$, and* $\sup(Y \cap \omega_2) > \gamma$.

It is possible to show directly that if SCC holds then CC^+ holds. In view of the fact that we shall show below that CC^+ is weaker than measurability, and in the light of Magidor's proof mentioned above, we shall omit the proof of this assertion.

6.1. Successors in the Core Model under CC^+

The Weak Covering Lemma for $\boldsymbol{K}$ shows that if λ is a singular cardinal then $(\lambda^+)^{\boldsymbol{K}} = \lambda^+$. This holds for a large range of core models, in particular those built below assuming there is no inner model of a Woodin cardinal. In general, the same need not be true of successors of regular cardinals, but there are some special cases: it is well-known that if λ is weakly compact then $(\lambda^+)^{\boldsymbol{K}} = \lambda^+$, for such $\boldsymbol{K}$; and, this is also true if λ is Jónsson [27].

In this section we prove that it is not only large cardinal properties that imply this form of covering at a regular cardinal. If CC^+ holds, then it is also true for $\lambda = \omega_2$.

6.1.1. Equivalents and variants

We follow [11] in defining the game below. In that paper, they consider ω_2-structures $\mathcal{A}$, and the consequences of *I* not having a winning strategy. We consider the stronger property that *II* has a winning strategy; we also allow for taller structures.

Definition 6.3. *Let $\mathcal{A}$ be an α-structure for some $\alpha \geq \omega_2$. Then the game $G^+(\mathcal{A})$ has (up to) ω_1 moves, and in the ξth move*

I plays an ordinal $\lambda_\xi < \omega_2$

II plays a countable set $X_\xi \prec \mathcal{A}$ such that $\sup(X_\xi) > \lambda_\xi$ *and for all $\zeta < \xi$, $X_\xi \cap \omega_1 = X_\zeta \cap \omega_1$ and $X_\xi \cap \omega_2 \trianglerighteq X_\zeta \cap \omega_2$.*

The game terminates if II cannot move. II wins if the game continues for ω_1 moves, otherwise I wins.

We extend Definition 6.3 slightly to a form that will be more useful for applications.

Lemma 6.4. *II has a winning strategy in $G^+(\mathcal{A})$ for all ω_2-structures $\mathcal{A}$ iff for all $\alpha \geqslant \omega_2$, II has a winning strategy in $G^+(\mathcal{A})$ for all α-structures $\mathcal{A}$.*

Proof. We proceed much as in Lemma 3.28.

To prove the nontrivial implication, let $\alpha \geqslant \omega_2$ and let $\mathcal{A} = \langle J_\alpha^{\vec{A}}, \in, \vec{A}, \ldots\rangle$ be an α-structure. $C = \{\gamma < \omega_2 : \mathrm{Hull}^{\mathcal{A}}(\gamma) \cap \omega_2 = \gamma\}$ is closed unbounded in ω_2. For any $\gamma, \delta \in C$ with $\gamma \leqslant \delta$, let $\pi_\gamma : \mathcal{A}_\gamma \prec \mathcal{A}$ be the inverse of the transitive collapse of $\mathrm{Hull}^{\mathcal{A}}(\gamma)$ and $\pi_{\gamma\delta} = \pi_\delta^{-1} \circ \pi_\gamma : \mathcal{A}_\gamma \prec \mathcal{A}_\delta$. Let $\mathcal{A}' = \langle J_{\omega_2}^{C,\Pi,A,B}, \in, C, \Pi, A, B\rangle$ where $\Pi = \langle \pi_{\gamma\delta} : \gamma, \delta \in C, \gamma \leqslant \delta\rangle$, $A = \langle \mathcal{A}_\gamma : \gamma \in C\rangle$, and $B = \langle g_\gamma : \gamma < \omega_2\rangle$, where each $g_\gamma : |\gamma| \to \gamma$ is a bijection.

We shall show that whenever $X \prec \mathcal{A}'$, $\mathrm{Hull}^{\mathcal{A}}(X) \cap \omega_2 \subseteq X$. The result then follows for then if σ' is a winning strategy for *II* in $G^+(\mathcal{A}')$ then σ given by $\sigma(p) = \mathrm{Hull}^{\mathcal{A}}(\sigma'(p'))$, where p' is the play p, but with all *II*'s moves replaced with their Skolem hulls in $\mathcal{A}$, is a winning strategy for *II* in $G^+(\mathcal{A})$.

Let $X \prec \mathcal{A}'$ and let $\varepsilon \in \mathrm{Hull}^{\mathcal{A}}(X) \cap \omega_2$. Then $\varepsilon = f_\varphi^{\mathcal{A}}(a_0, \ldots, a_{n-1})$, where $f_\varphi^{\mathcal{A}}$ is one of the (uniformly definable) Skolem functions of $\mathcal{A}$, and $a_0, \ldots, a_{n-1} \in X$. $C \cap X$ is unbounded in $\omega_2 \cap X$ since $X \prec \mathcal{A}'$, so there is $\delta \in C \cap X$ such that $\varepsilon, a_0, \ldots, a_{n-1} \in \mathcal{A}' \upharpoonright \delta$. However then

$$\varepsilon = \pi_\delta^{-1}(\varepsilon) = \pi_\delta^{-1}(f_\varphi^{\mathcal{A}}(a_0, \ldots, a_{n-1})) = f_\varphi^{\mathcal{A}_\delta}(\pi_\delta^{-1}(a_0), \ldots, \pi_\delta^{-1}(a_{n-1})) = f_\varphi^{\mathcal{A}_\delta}(a_0, \ldots, a_{n-1}) \in X$$

since $\delta, a_0, \ldots, a_{n-1} \in X$. □

We now prove a useful equivalent of CC^+, then a characterisation in terms of the games $G^+(\mathcal{A})$.

Lemma 6.5. CC^+ *holds iff for any H_{ω_2}-structure, $\mathcal{A}$, there is an H_{ω_2}-structure, $\mathcal{B}$, extending $\mathcal{A}$, such that for any countable $X \prec \mathcal{B}$ and $\gamma < \omega_2$ there is $Y \prec \mathcal{B}$ with $Y \cap \omega_2 \trianglerighteq X \cap \omega_2$ and* $\sup(Y \cap \omega_2) > \gamma$.

Proof. Assume CC^+. Let $\mathcal{A}$ be an H_{ω_2}-structure. Then let $B = \langle g_\gamma : \gamma < \omega_2\rangle$, where each $g_\gamma : |\gamma| \to \gamma$ is a bijection, as in the previous proof. Let $\mathcal{A}' = \mathcal{A} \frown \langle B\rangle$, and apply CC^+ to $\mathcal{A}$ to obtain $\mathcal{B}$.

Let $X \prec \mathcal{B}$ be countable and $\gamma < \omega_2$. There is $Y \prec \mathcal{B}$ with $Y \supseteq X$, $X \cap \omega_1 = Y \cap \omega_1$, and $\sup(Y \cap \omega_2) > \gamma$. We require $Y \cap \omega_2 \trianglerighteq X \cap \omega_2$. Suppose not. Then there exist $\eta < \eta' < \omega_2$ such that $\eta' \in X$ but $\eta \in Y \backslash X$. Then $g_{\eta'}^{-1}(\eta) \in Y \cap \omega_1 = X \cap \omega_1$ so $\eta = g_{\eta'}(g_{\eta'}^{-1}(\eta)) \in X$ - contradiction. □

Theorem 6.6. CC^+ *holds iff II has a winning strategy in $G^+(\mathcal{A})$ for all ω_2-structures $\mathcal{A}$.*

Proof. The implication from left to right follows immediately from Lemma 6.5.

Let $\kappa = \omega_2$. Let $\mathcal{A} = \langle H_\kappa, \in, A_0, \ldots, A_n, \ldots\rangle_{n<\omega}$ be an H_{ω_2}-structure. We construct a sequence of ω-many $|H_{\omega_2}|$-structures inductively, using Lemma 6.4. Set $\mathcal{A}_0 = \langle H_\kappa, \in, A_0\rangle$. Suppose $\mathcal{A}_n$ has been defined. For each $x \in [H_\kappa]^{<\omega}$ let $\sigma_{\mathcal{A}_n, x}$ be a winning strategy for *II* in $G^+(\mathcal{A}_n \frown \langle x\rangle)$, and let:

$$\sigma_{\mathcal{A}_n}^* = \{\langle x, y\rangle : x \in [\mathcal{A}_n]^{<\omega} \text{ and } y \in \sigma_{\mathcal{A}_n, x}\};\ \mathcal{A}_{n+1} = \langle H_\kappa, \in, A_0, \sigma_{\mathcal{A}_0}^*, \ldots, A_n, \sigma_{\mathcal{A}_n}^*, A_{n+1}\rangle.$$

Let $\mathcal{B} = \langle H_\kappa, \in, A_0, \sigma_{\mathcal{A}_0}^*, \ldots, A_n, \sigma_{\mathcal{A}_n}^*, \ldots\rangle_{n<\omega}$ be the H_{ω_2}-structure whose predicates are exactly those that occur in $\mathcal{A}_n$, for some $n < \omega$. We show that $\mathcal{B}$ witnesses CC^+.

Let $X \prec \mathcal{B}$ be countable, $X = \{x_n : n < \omega\}$, and let $\gamma < \omega_2$. Then let

$$Y = \bigcup_{n<\omega} \sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0, \sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0\rangle), \gamma\rangle).$$

Now, $\sup(Y \cap \omega_2) > \gamma$ follows immediately, and

$$Y \cap \omega_1 = \bigcup_{n<\omega} \sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0, \sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0\rangle), \gamma\rangle) \cap \omega_1 = \bigcup_{n<\omega} \sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0\rangle) \cap \omega_1 \subseteq X$$

since $\sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0\rangle) \in X$ for each n (using $\sigma_{\mathcal{A}_n}^*$), and $X \vDash \sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0\rangle)$ is countable , and so $\sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0\rangle) \subseteq X$. Note that $X \subseteq Y$ since $x_0, \ldots, x_n \in \sigma_{\mathcal{A}_n, \{x_0, \ldots, x_n\}}(\langle 0\rangle)$ for each $n < \omega$.

It remains to show that $Y \prec \mathcal{B}$. If $\psi \in \mathcal{L}_{\mathcal{B}}$ and $a_0, \ldots, a_{n-1} \in Y$ then there is $k < \omega$ such that $\psi \in \mathcal{L}_{\mathcal{A}_k}$ and $a_0, \ldots, a_{n-1} \in \sigma_{\mathcal{A}_k, \{x_0, \ldots, x_k\}}(\langle 0, \sigma_{\mathcal{A}_k, \{x_0, \ldots, x_k\}}(\langle 0 \rangle), \gamma \rangle)$ so

$$\begin{aligned} \mathcal{B} \vDash (\exists x)\psi(x, a_0, \ldots, a_{n-1}) \quad &\text{iff} \quad \mathcal{A}_k \vDash (\exists x)\psi(x, a_0, \ldots, a_{n-1}) \\ &\text{iff} \quad \sigma_{\mathcal{A}_k, y}(\langle 0, \sigma_{\mathcal{A}_k, y}(\langle 0 \rangle), \gamma \rangle) \vDash (\exists x)\psi(x, a_0, \ldots, a_{n-1}), \\ &\qquad \text{where } y = \{x_0, \ldots, x_k\}. \end{aligned}$$

Let $a \in \sigma_{\mathcal{A}_k, y}(\langle 0, \sigma_{\mathcal{A}_k, y}(\langle 0 \rangle), \gamma \rangle) \subseteq Y$ such that $\sigma_{\mathcal{A}_k, y}(\langle 0, \sigma_{\mathcal{A}_k, y}(\langle 0 \rangle), \gamma \rangle) \vDash \psi(a, a_0, \ldots, a_{n-1})$. Then, $\mathcal{A}_k \vDash \psi(a, a_0, \ldots, a_{n-1})$, so $\mathcal{B} \vDash \psi(a, a_0, \ldots, a_{n-1})$, as required. □

Theorem 6.7. **($\neg 0^{\P}$)** *Suppose that* CC^+ *holds. Then* $\omega_3 = (\lambda^+)^{\boldsymbol{K}}$, *where* $\lambda = \omega_2$.

Proof. Our proof is based on [25]. Suppose for a contradiction that $\nu = (\lambda^+)^{\boldsymbol{K}} < \lambda^+$, and let $F : \lambda \to \nu$ be monotone cofinal (by the Covering Lemma for $\boldsymbol{K}$ note that $\mathrm{cf}(\nu) \geqslant \lambda$) and $\hat{M} = \boldsymbol{K}|\nu \frown \langle F \rangle$ (recall that this is the expansion of the structure $\boldsymbol{K}|\nu$ to include the additional predicate F).

Let σ be a winning strategy for *II* in $G^+(\hat{M})$ of $\boldsymbol{V}$ (this exists by Theorem 6.6 and Lemma 6.4).

Let $\mathbb{P} = \mathrm{Col}(\omega_1, \{\lambda\})$. Forcing with $\mathbb{P}$ is countably closed, so adds no countable sequences of sets in $\boldsymbol{V}$. $\mathbb{P}$ preserves all cardinals except λ, which has cardinality, and cofinality, ω_1 in the forcing extension. (See [17].) Assume G is $\mathbb{P}$-generic, and let $f \in \boldsymbol{V}[G]$ such that $f : \omega_1 \to \lambda$ is monotone cofinal. Define a play, p, of $G^+(\hat{M})$ in $\boldsymbol{V}[G]$ by

$$\begin{aligned} p(\delta) &= f(\varepsilon) \text{ if } \delta = 2 \cdot \varepsilon, \text{ some } \varepsilon < \omega_1 \\ p(\delta) &= \sigma(p \restriction \delta) \text{ if } \delta = 2 \cdot \varepsilon{+}1, \text{ some } \varepsilon < \omega_1 \end{aligned}$$

and note that this is well-defined since each partial play $p \restriction \delta$ is in $({}^{\omega}\boldsymbol{V})^{\boldsymbol{V}[G]} \subseteq \boldsymbol{V}$.

Let $Y = \bigcup_{\varepsilon < \omega_1} p(2 \cdot \varepsilon{+}1) \prec \hat{M}$. Then, since $p(2 \cdot \varepsilon{+}1) \cap \lambda \trianglelefteq p(2 \cdot \varepsilon'{+}1) \cap \lambda$ whenever $\varepsilon < \varepsilon' < \omega_1$, $\mathrm{otp}(Y \cap \lambda) = \omega_1$.

There is thus an inverse of a transitive collapse map: $\overline{j} : \overline{M} \prec \hat{M}$ such that $\mathrm{crit}(\overline{j}) < \omega_1$, $\overline{j}(\omega_1) = \lambda$, and $\overline{j}``\omega_1$ is cofinal in $\lambda$. Since $F : \lambda \to \nu$ is a predicate of $\hat{M}$, $\overline{j}``\mathrm{ht}(\overline{M})$ is cofinal in ν.

Let $j : M \prec \boldsymbol{K}|\nu$ be the map corresponding to $\overline{j}$ between the structures in the language without F, and let $\overline{\nu} = \mathrm{ht}(M)$ and $\kappa = \omega_1$. By Theorem 2.26 there is a mouse N such that $|N| < \omega_1$ and $N >^* M$. Let $\mathcal{I}_M = \langle M_i, \pi^M_{ij} : i \leqslant j \leqslant \theta \rangle$ and $\mathcal{I}_N = \langle N_i, \pi^N_{ij} : i \leqslant j \leqslant \theta \rangle$ be the M-side and N-side respectively of the coiteration of the two mice, and let $\langle \kappa_i, \nu_i : i < \theta \rangle$ be the sequence of critical points and indices. Note that each $M_i \models ZFC^-$, and thus the ultrapowers are being taken on the M side by using functions *within* the structures.

By standard coiteration arguments again, one has:

i. $\pi^M_{0\kappa}(\kappa) = \kappa$ (so $\pi^M_{0\kappa}``\kappa \subseteq \kappa$);

ii. There is a cub $D \subseteq \kappa$ so that if $i < j \in D$, then $\pi^N_{i,j}(\kappa_i) = \kappa_j$; $\pi^N_{i,\kappa}(\kappa_i) = \kappa$; $\pi^M_{0,i}``\kappa_i \subseteq \kappa_i$;

iii. $N_\kappa | \mathrm{ht}(M_\kappa) = M_\kappa = M_\theta$.

The ultrapower maps on the M side are all Σ_0 and cofinal, thus π^M_{0i} is cofinal in M_i (and hence $\mathrm{cf}(\mathrm{ht}(M_i)) = \kappa$) for each $i \leqslant \kappa$. This persists into the limit stages.

Similarly, if $i^* \in D$ then $\mathrm{cf}((\kappa^+)^{N_\kappa}) = \mathrm{cf}((\kappa^+_{i^*})^{N_{i^*}}) < \kappa$. So $(\kappa^+)^{N_\kappa} \neq \mathrm{ht}(M_\kappa)$ as their cofinalities differ. Hence $(\kappa^+)^{N_\kappa} > \mathrm{ht}(M_\kappa)$, as otherwise there would be a truncation on the M-side of the coiteration. (To justify this, note that were $\nu = (\kappa^+)^{N_\kappa} < \mathrm{ht}(M_\kappa)$ but there were no truncation at this stage on the M side, then we could only have that in M_κ that some $\tau < \kappa$ had an extender indexed by ν. However this is absurd as then we have some $\tau < \kappa$ with $M_\kappa \models$ "$o(\tau) \geq \kappa$." However then the same would hold in M which is impossible by Theorem 2.26.) Let $\widetilde{M} = N_\kappa \| \alpha$, where $\alpha < (\kappa^+)^{N_\kappa}$ is the least ordinal such that $\omega\rho^\omega_{N_\kappa\|\alpha} = \kappa < \overline{\nu} \leqslant \alpha$, where $\overline{\nu} = \mathrm{ht}(M_\kappa)$. Set $\sigma = \pi^N_{0\kappa}$, $\pi = \pi^M_{0\kappa}$, $M' = M_\kappa$, and let $j' : M' \to K' = K'_\kappa$ and $\mathcal{I}' = \langle K'_i, \pi'_{ij} : i \leqslant j \leqslant \kappa \rangle$ arise from the application of the standard copying construction (*cf.* Lemma 4.3.1 [29]) to j and $\mathcal{I}_M$. Set $\pi' = \pi'_{0\kappa}$. See Figure 1 again (in which $\sigma = \pi^N_{0,\kappa} : N \to N_\kappa$).

Since each measure $E_i^{K'_i}$ used in the iteration $\mathcal{I}'$ is total in K'_i and indexed below the regular cardinal $\lambda = \pi'_{0i}(\lambda)$ of K'_i, the iteration $\mathcal{I}'' = \langle K''_i, \pi''_{ij} : i \leqslant j \leqslant \kappa \rangle$ of $\boldsymbol{K}$ using the same indices is a well-defined simple normal iteration and for each $i \leqslant \kappa$, $\lambda = \pi'_{0i}(\lambda)$ and $K'_i = K''_i | (\lambda^+)^{K''_i}$. Set $\pi'' = \pi''_{0\theta}$.

Note that $j' : M' \to K'$ is cofinal by commutativity of Figure 1 and the fact that π' and j are cofinal. By an argument similar to Lemma 3.17 of [26] (done there in the $\neg 0^{\text{sword}}$ setting but which is the same here), (or see [29] 3.6.7 & 3.6.13) the canonical extension $\tilde{j} : \widetilde{M} \to M^*$ of j' to $\widetilde{M}$ exists, M^* is a mouse, and since $\widetilde{M}$ is sound (it is a proper truncation of a mouse) $\tilde{j}$ is $\Sigma_1^{(m)}$-elementary, where $\omega\rho^{m+1}_{\widetilde{M}} \leqslant \kappa < \omega\rho^m_{\widetilde{M}}$. One consequence is that $\omega\rho^{m+1}_{M^*} \leqslant \lambda < \omega\rho^m_{M^*}$.

Since it is a simple iterate of $\boldsymbol{K}$, $\boldsymbol{K}''$ is a universal weasel, hence $M^* <^* \boldsymbol{K}''$. Let M^{**} and K^{**} be the final mice in the coiteration of M^* with $\boldsymbol{K}''$. The coiteration is above $\nu'' =_{\mathrm{df}} (\lambda^+)^{\boldsymbol{K}''} = \mathrm{ht}(K') = \sup(j\text{“}M')$ since $M^*|\nu'' = K' = \boldsymbol{K}''|\nu''$, (and so $P(\lambda) \cap M^* = P(\lambda) \cap \boldsymbol{K}''$).

However, as we have stated, $\omega\rho^{m+1}_{M^*} \leqslant \lambda$ by $\Sigma_1^{(m)}$-elementarity of $\tilde{j}$. Further M^* is sound above λ. Hence we may let $A \in P(\lambda) \cap \boldsymbol{\Sigma}_1^{(m)}(M^*)$ code a wellordering of length $On \cap M^* \geq \lambda^{+\boldsymbol{K}''}$. However

$$P(\lambda) \cap \boldsymbol{\Sigma}_1^{(m)}(M^{**}) \subseteq P(\lambda) \cap \boldsymbol{\Sigma}_1^{(m)}(K^{**}) \subseteq P(\lambda) \cap \boldsymbol{K}'',$$

by the $\Sigma_1^{(m)}$ preservation properties of our iteration maps; which is absurd. Contradiction! $\Box$

Since $\Box_\lambda$ holds in $\boldsymbol{K}$, which is absolute to $\boldsymbol{V}$, we conclude that the theorem of the abstract holds:

Corollary 6.8. *If* $\mathrm{CC}^+ \wedge \neg\Box_{\omega_2}$ *then* $0^\P$ *exists.*

6.2. The Consistency Strength of CC^+

By the above remarks SCC is known to be equiconsistent to the existence of a measurable cardinal. The consistency strength of Chang's Conjecture is exactly that of one ω_1-Erdős cardinal, by work of Silver (unpublished) and Donder ([9]). Below, we shall see that CC^+ lies properly in the gap between measurability and the ω_1-Erdős property.

6.2.1. An Upper Bound

Theorem 6.9. *If* κ *is Ramsey and* $\mathbb{P} = \mathrm{Col}(\omega_1, <\kappa)$ *then* $\Vdash_{\mathbb{P}} \mathrm{CC}^+$.

Our proof will be based on a proof of the following result of Baumgartner from [3]:

Theorem 6.10. ([3]) *If* κ *is an* ω_1*-Erdős cardinal and* $\mathbb{P} = \mathrm{Col}(\omega_1, <\kappa)$ *then* $\Vdash_{\mathbb{P}} \omega_2$ *is* ω_1*-remarkable .*

Where:

Definition 6.11. *A cardinal* κ *is* λ-remarkable *iff for any* κ*-structure* $\mathcal{A}$*, there is a set* $I \subseteq \kappa$ *of ordertype* λ *such that whenever* $\tau(v_0, \ldots, v_{n-1}, v_n, \ldots, v_{n+m-1})$ *is an* $\mathcal{A}$*-term (i.e., a function* $\tau : ^{n+m}\mathcal{A} \to \mathcal{A}$ *that is definable over* $\mathcal{A}$*) and for any* $\gamma_0, \ldots, \gamma_{2m-1} \in I$ *and* $\alpha_0, \ldots, \alpha_{n-1} < \min\{\gamma_0, \ldots, \gamma_{2m-1}\}$ *such that* $\gamma_0 < \ldots < \gamma_{m-1}$ *and* $\gamma_m < \ldots < \gamma_{2m-1}$,

$$\tau(\alpha_0, \ldots, \alpha_{n-1}, \gamma_0, \ldots, \gamma_{m-1}) = \tau(\alpha_0, \ldots, \alpha_{n-1}, \gamma_m, \ldots, \gamma_{2m-1}).$$

We define some notation and quote some results from that paper (rephrased, somewhat).

Notation 6.12. *Suppose* $\mathbb{P} = \mathrm{Col}(\omega_1, < \kappa)$ *and* $p \Vdash_{\mathbb{P}}$ *"* $\dot{\mathcal{A}}$ *is a* κ*- structure." Denote*

$$\mathcal{B}(\dot{\mathcal{A}}) = \langle H_\kappa, \in, \lhd, \mathbb{P}, \Vdash_\varphi, \dot{\mathcal{A}}\rangle,$$

where $\lhd$ *is a well-ordering of* H_κ*,* φ *ranges over all formulae of the language of* $\dot{\mathcal{A}}$*, and* $\Vdash_\varphi$ *is the relation*

$$\{\langle q, a_0, \ldots, a_{n-1}\rangle : q \Vdash \dot{\mathcal{A}} \models \varphi(a_0, \ldots, a_{n-1})\}$$

(so $\mathcal{B}(\dot{\mathcal{A}})$ *is a* H_κ*-structure). If* $I \subseteq \mathrm{On}$ *then let*

$$[[I]]^{<\omega_1} = \{X \subseteq I : \mathrm{otp}(X) \textit{ is limit ordinal } < \omega_1\} \cup \{\varnothing\}.$$

Definition 6.13. (Baumgartner, [3]) *Suppose* $\mathbb{P} = \mathrm{Col}(\omega_1, < \kappa)$ *and* $p \Vdash_{\mathbb{P}} \dot{\mathcal{A}}$ is aκ*-structure . Let* $\mathcal{B} = \mathcal{B}(\dot{\mathcal{A}})$*, and* $H \prec \mathcal{B}$*. Then*

- G *is* $\mathbb{P}$-generic over H *if* $G \cap (\mathbb{P} \cap H)$ *is a filter that meets every dense subset,* D*, of* $\mathbb{P}$ *with* $D \in \mathrm{Def}^{\mathcal{B}}(H)$
- p *is* $\mathbb{P}, H$-generic *if* $G = \{q \in \mathbb{P} \cap H : p \leqslant q\}$ *is* $\mathbb{P}$*-generic over* H

Lemma 6.14. (Baumgartner, [3], Lemma 5.3) *Let* $\mathbb{P}$*,* $\dot{\mathcal{A}}$*,* p*, and* $\mathcal{B}$ *be as in Definition 6.13, with* $\alpha < \omega_1$*, and let* G *be* $\mathbb{P}$*-generic. Suppose* $J \in \boldsymbol{V}[G]$ *is a set of g.i.'s for* $\mathcal{B}$*, and* G *is* $\mathbb{P}$*-generic over* $\mathrm{Hull}^{\mathcal{B}}(J \cup \alpha)$*. Then* $\mathrm{Hull}^{(\dot{\mathcal{A}})_G}(J \cup \alpha) \cap \kappa = \mathrm{Hull}^{\mathcal{B}}(J \cup \alpha) \cap \kappa$*.*

Lemma 6.15. (Baumgartner, [3], Lemmas 5.4-5.7) *Let* $\mathbb{P}$*,* $\dot{\mathcal{A}}$*,* p*, and* $\mathcal{B}$ *be as in Definition 6.13 again with* $\alpha < \omega_1$*. If* I *is a set of g.i.'s for* $\mathcal{B}$*,* $\mathrm{cf}(\sup(I)) > \omega$*,* $X \in [[I]]^{<\omega_1}$ *then*

- *the set* $\{q \in \mathbb{P} : (\exists Y \in [[I]]^{<\omega_1})(q \textit{ is } \mathbb{P}, \mathrm{Hull}^{\mathcal{B}}(Y \cup \alpha)\textit{- generic })\}$ *is dense;*
- *if* p *is* $\mathbb{P}, \mathrm{Hull}^{\mathcal{B}}(X \cup \alpha)$*-generic then, for any* $\gamma < \sup(I)$*, the set*

  $$\{q \leqslant p : (\exists Y \in [[I]]^{<\omega_1})(q \textit{ is } \mathbb{P}, \mathrm{Hull}^{\mathcal{B}}(Y \cup \alpha)\textit{-generic }, \sup(Y) > \gamma, \textit{ and } X \trianglelefteq Y)\}$$

  is dense below p*.*

Remark 6.16. *The ordinal* γ *does not occur in the version of Lemma 6.15 in [3] however the proof requires only an (obvious) trivial change. Also the cited lemma does not explicitly mention an* α*, but these countably many ordinals can be admitted to the structures and named in the language. In [3], Lemma 6.14 has the additional hypothesis that* J *is uncountable, but this is not used.*

Proof. **(Theorem 6.9)** Suppose G is $\mathbb{P}$-generic. It is standard that $\omega_2^{\boldsymbol{V}[G]} = \kappa$, and $({}^{\omega}\boldsymbol{V})^{\boldsymbol{V}[G]} \subseteq \boldsymbol{V}$. We show that, in $\boldsymbol{V}[G]$, *II* has a winning strategy in $G^+(\mathcal{A})$ for all ω_2-structures $\mathcal{A}$.

Let $\mathcal{A} = (\dot{\mathcal{A}})_G \in \boldsymbol{V}[G]$ be an ω_2-structure, and set $\mathcal{B} = \mathcal{B}(\dot{\mathcal{A}}) \in \boldsymbol{V}$. Since κ is Ramsey in $\boldsymbol{V}$, there is a set I of g.i.'s for $\mathcal{B}$, cofinal in κ. We describe *II*'s winning strategy in $G^+(\mathcal{A})$:

Assume that in the ξth move, each of *II*'s previous moves (if any) has the form $X_\zeta = \mathrm{Hull}^{(\dot{\mathcal{A}})_G}(I_\zeta)$ for some $I_\zeta \in [[I]]^{<\omega_1}$, with $I_\zeta \trianglelefteq I_{\zeta'}$ whenever $\zeta < \zeta' < \xi$, and G is $\mathbb{P}$-generic over $X_\zeta = \mathrm{Hull}^{(\dot{\mathcal{A}})_G}(I_\zeta)$. Suppose that *I*'s ξth move is $\lambda_\xi = \lambda$. Now, it is immediate from the definitions that G is $\mathbb{P}$-generic over

$$X^*_\xi =_{\mathrm{df}} \bigcup_{\zeta<\xi} X_\zeta = \bigcup_{\zeta<\xi} \mathrm{Hull}^{(\dot{\mathcal{A}})_G}(I_\zeta) = \mathrm{Hull}^{(\dot{\mathcal{A}})_G}(I^*_\xi),$$

where $I^*_\xi =_{\rm df} \bigcup_{\zeta<\xi} I_\zeta$. So, by Lemma 6.15, there is $Y \in [[I]]^{<\omega_1}$ such that G is $\mathbb{P}$-generic over $X =_{\rm df} \mathrm{Hull}^{(\dot{\mathcal{A}})_G}(Y)$, $\sup(Y) > \lambda$, and $Y \trianglerighteq I^*_\xi$.

By Lemma 6.14, for all $\zeta < \xi$:

$$X \cap \omega_1 = \mathrm{Hull}^{(\dot{\mathcal{A}})_G}(Y) \cap \omega_1 = \mathrm{Hull}^{\mathcal{B}}(Y) \cap \omega_1 = \mathrm{Hull}^{\mathcal{B}}(I_\zeta) \cap \omega_1 = \mathrm{Hull}^{(\dot{\mathcal{A}})_G}(I_\zeta) \cap \omega_1 = X_\zeta \cap \omega_1.$$

II's play is $X_\xi = X$. It is now clear (inductively) that the strategy defined above is a winning strategy. □

6.2.2. Equiconsistency

Theorem 6.17. ($\neg 0^\P$). *Set $\lambda = \omega_2$. If* CC^+ *holds then λ is Ramsey in $\boldsymbol{K}$.*

Proof. Let $\mathcal{A} \in \boldsymbol{K}$ be a λ-structure. We shall show that there is a forcing extension containing a set of g.i.'s for $\mathcal{A}$, and apply the Jensen Indiscernibles Lemma and the fact that $\boldsymbol{K}$ of the forcing extension is the same as $\boldsymbol{K}$ of the the ground model.

Note that by Theorem 6.7 $(\lambda^+)^{\boldsymbol{K}} = \omega_3^{\boldsymbol{V}}$. Let $M_0 \prec \langle \boldsymbol{K}|\omega_3, \in, \mathcal{A}, \{\omega_1\}\rangle$ be a substructure with $|M_0| = \lambda_2 \wedge \mathrm{Trans}(M_0) \wedge \mathrm{cf}^{\boldsymbol{V}}(On^{M_0}) = \lambda$. Then $|M_0| = \boldsymbol{K}|\bar{\lambda}$ for some $\bar{\lambda} \in (\lambda, \omega_3)$ and $\mathrm{cf}^{\boldsymbol{V}}(\bar{\lambda}) = \lambda$. Let $g : \lambda \to \bar{\lambda}$ be a monotone cofinal map witnessing this.

Let $M' = \langle M_0, \in, g\rangle$. Let σ be a winning strategy for *II* in $G^+(M')$ of $\boldsymbol{V}$ (this exists by Theorem 6.6 and Lemma 6.4).

Let $\mathbb{P} = \mathrm{Col}(\omega_1, \{\lambda\})$. Just as in Theorem 6.7 there is j, Y, M with $j : M \cong Y \prec M'$, with $\mathcal{A} \in \mathrm{ran}(j)$, and if $\mu = \mathrm{crit}(j) < \omega_1$, then $\omega_1 \cap Y = \mu$, $j(\mu) = \omega_1$ and $j(\omega_1) = \lambda$. Moreover, by the presence of g $On \cap M$ has cofinality ω_1. Lastly $M \models$ "$\omega_1^{\boldsymbol{V}}$ is the largest cardinal."

We now apply Theorem 2.26 (in $\boldsymbol{V}[G]$) with κ as ω_1 to conclude that there is a mouse N that iterates past M. Let $\widetilde{N}, \widetilde{M}$ be the models at the ω_1'st stage of this coiteration, with maps $i : N \longrightarrow \widetilde{N}$, $k : M \longrightarrow \widetilde{M}$. Then we shall have that k"$\omega_1 \subseteq \omega_1$, and k"$On^M \subseteq On^{\widetilde{M}}$. On the N side let the critical points of the coiteration be κ_i for $i < \omega_1$. Then we shall also have $\omega = \mathrm{cf}((\omega_1^+)^{\widetilde{N}})$ (as $(\omega_1^+)^{\widetilde{N}}$ is the supremum of the continuous image of a successor cardinal κ_i^+ of critical points κ_i. Whilst ω_1 remains $\mathrm{cf}((\omega_1^+)^{\widetilde{M}})$ since $(\omega_1^+)^{\widetilde{M}} = On^{\widetilde{M}}$. Since in this coiteration there can be no truncation on the M side, by the Dodd-Jensen Lemma, this ensures that $(\omega_1^+)^{\widetilde{N}} > (\omega_1^+)^{\widetilde{M}}$. Hence we see that the very next ultrapower (if any) to be taken on the N side will be to truncate $\widetilde{N}$.

Let $\delta_{\tilde{M}} = \mathrm{On}^{\tilde{M}} = (\omega_1^+)^{\tilde{M}}$ and $\delta_{\tilde{N}} = (\omega_1^+)^{\tilde{N}}$. Then $\delta_{\tilde{N}} > \delta_{\tilde{M}}$. It suffices to find a set $I_{\tilde{\mathcal{A}}}$ of g.i. unbounded in ω_1 for $\tilde{\mathcal{A}}$. Because then we may take the copy construction to copy up the iteration $k : M \longrightarrow \tilde{M}$ *via* j to yield a map $\tilde{j} \supseteq j$ with $\tilde{j} : \tilde{M} \longrightarrow \widetilde{M_0}$ with copy iteration $\tilde{k} : M' \longrightarrow \widetilde{M}_0$. Recall that $|M'|$ is $K|\bar{\lambda}$, and so the ultapowers used in the copied iteration yielding $\tilde{k}$ are all total, and on critical points below λ. Hence we can construe this iteration as a simple one of $\tilde{k} : K \longrightarrow \tilde{K}$. However then $\tilde{j}$"$I_{\tilde{\mathcal{A}}}$ is a set of g.i.'s for $\tilde{j}(\tilde{\mathcal{A}}) = \tilde{k}(\mathcal{A})$ which is unbounded in λ. By Theorem 2.3, the Jensen Indiscernibles Lemma for the universal weasel $\tilde{K}$, there is a set $I \supseteq \tilde{j}$"$I_{\tilde{\mathcal{A}}}$ also of g.i's for $\tilde{k}(\mathcal{A})$, with $I \in \tilde{K}$. By elementarity of $\tilde{k}$ there is such a set in K for $\mathcal{A}$, and so we are done.

Case 1 The coiteration of $\tilde{N}$ with $\tilde{M}$ continues with a truncation of $\tilde{N}$ to $\tilde{N}|\eta$, so that $\delta_{\tilde{M}} = (\omega_1^+)^{\tilde{N}|\eta}$, followed by an ultrapower of $\tilde{N}|\eta$.

The topmost filter F of $\tilde{N}|\eta$ which measures $\mathcal{P}(\omega_1)^{\tilde{M}}$ is weakly amenable with $\tilde{\mathcal{A}} \in \tilde{N}|\eta$. However then, as $\mathrm{cf}(\delta_{\tilde{M}}) = \omega_1$, a set of g.i.'s $I_{\tilde{\mathcal{A}}}$ can be constructed *inside* $\tilde{N}|\eta$ from F, that is $I_{\tilde{\mathcal{A}}} \in \mathcal{P}(\omega_1)^{\tilde{M}}$ in this case.

Case 2 $\tilde{M}$ is already a proper initial segment of $\tilde{N}$.

Suppose there was a proper initial segment $\tilde{N}|\eta$, so that $\delta_{\tilde{M}} < (\omega_1^+)^{\tilde{N}|\eta}$ but with $\mathrm{cf}((\omega_1^+)^{\tilde{N}|\eta}) > \omega$. Then we could argue as in *Case 1* to find a suitable $I_{\tilde{\mathcal{A}}} \in \mathcal{P}(\omega_1)^{\tilde{N}|\eta} \cap F$. The difference now is only that $I_{\tilde{\mathcal{A}}} \notin \mathcal{P}(\omega_1)^{\tilde{M}}$. However the existence of $I_{\tilde{\mathcal{A}}}$ is sufficient to still complete the plan above of finding a set of g.i.'s for $\mathcal{A}$.

So consider the possibility that the supposition fails and there is no such $\eta < \mathrm{On}^{\tilde{N}}$ as above. There is a cub set $C \subseteq \omega_1$ of critical points of the coiteration with $\tilde{N}$ as the direct limit of N_i with the measures used in the ultrapower at that stage on $\kappa_i \in C$. For such i we have, on a tail of $i \in C$, some fixed r so that $\omega\rho^r_{N_i} \leq \kappa_i$. Either we have in the direct limit $\tilde{N}$ that $\delta_{\tilde{N}} < \mathrm{On}^{\tilde{N}}$ and there is thus a normal measure with critical point $\omega_1^{\boldsymbol{V}}$ in $\tilde{N}$ to provide a set of g.i.'s for $\tilde{A}$ inside $\tilde{N}$; we may then finish as before. Or else $\delta_{\tilde{N}} = \mathrm{On}^{\tilde{N}}$ and the topmost measure F is of "measure of order zero" type. Moreover N_i has topmost measure F_i say on κ_i, also of measure of order zero type, and it is this filter and its images that have been used to form the ultrapowers $\mathrm{Ult}(N_i, F_i)$ for $i \in C$. As the same filter has been used along C to form the direct limit, the critical points form a set of g.i.'s for $\tilde{M}$ and so *a fortiori* for $\tilde{\mathcal{A}}$. Again we finish in the same way. $\square$

6.3. A Milder Strengthening of Chang's Conjecture

If, as in the forcing extension of Theorem 6.10, ω_2 is ω_1-remarkable then the set

$$\{\alpha < \omega_1 : (\exists X)(X \prec \mathcal{A}, |X| = \omega_1 \text{ and } X \cap \omega_1 = \alpha\}$$

is closed unbounded in ω_1 for any ω_2-structure $\mathcal{A}$. We look at whether

$$\{\alpha < \omega_2 : (\exists X)(X \prec \mathcal{A}, |X| = \omega_1, X \cap \omega_1 \in \omega_1, \text{ and } \sup(X) = \alpha\}$$

can be ω_1-closed unbounded.

Definition 6.18. *Say that the* Intermediate Chang's Conjecture (ICC) holds *iff for any ω_2-structure $\mathcal{A}$, and stationary sets $S_0 \subseteq \omega_1$ and $S_1 \subseteq \omega_2 \cap \mathrm{Cof}_{\omega_1}$, $\mathcal{A}$ has a substructure $\mathcal{B} \prec \mathcal{A}$ such that* $\sup(\mathcal{B} \cap \omega_1) \in S_0$ *and* $\sup(\mathcal{B} \cap \omega_2) \in S_1$.

The original motivation to consider the principle ICC came from the theory of mutually stationary sets, first developed in [13]. This theory in general aims to generalise some of the properties of stationary sets to sequences of sets, typically below a singular cardinal. This is just a junior two cardinal version.

6.3.1. Upper Bounds on Consistency Strength

Definition 6.19. *A cardinal κ is* virtually Ramsey *iff for any κ-structure $\mathcal{A}$, there is a set $C \subseteq \kappa$ such that $C \subseteq t_{\mathcal{A}}(\kappa)$, C is unbounded and $(> \omega)$-closed. (Recall Definition 4.1.)*

Remark 6.20. *Any Ramsey cardinal is virtually Ramsey: just take $C = \lim(I)$, where I is a cofinal set of good indiscernibles for $\mathcal{A}$.*

In fact an elementary application of Π^1_1 indescribability yields the non-trivial direction of the following:

Lemma 6.21. *κ is virtually Ramsey and weakly compact if and only if κ is Ramsey.*

Lemma 6.22. *If κ is virtually Ramsey and $\mathbb{P} = \mathrm{Col}(\omega_1, < \kappa)$, $\Vdash_{\mathbb{P}}$ ICC.*

Proof. The proof of Lemma 6.22 is similar to that of Theorem 6.9. We shall use the slightly modified versions of Baumgartner's lemmas above. Suppose G is $\mathbb{P}$-generic. Let $\mathcal{A} = (\dot{\mathcal{A}})_G \in \boldsymbol{V}[G]$ be an ω_2-structure, and set $\mathcal{B} = \mathcal{B}(\dot{\mathcal{A}}) \in \boldsymbol{V}$. Further, let $S_0 = (\dot{S}_0)_G \subseteq \omega_1$ and $S_1 = (\dot{S}_1)_G \subseteq \omega_2 \cap \mathrm{Cof}_{\omega_1}$ be stationary sets in $\boldsymbol{V}[G]$. Now, there exists an unbounded $(> \omega)$-closed set $C \subseteq \kappa$ such that

$$(\forall \gamma \in C)(\exists I \subseteq \gamma)(I \text{ is a set of g.i.'s for } \mathcal{B}, \text{ cofinal in } \gamma).$$

Let $\delta \in C \cap S_1$, and I a set of g.i.'s for $\mathcal{B}$, cofinal in δ, w.l.o.g. $I \cap \omega_1 = \varnothing$.

Since I is a set of g.i.'s, the set $C_0 = \{\gamma < \omega_1 : \mathrm{Hull}^{\mathcal{B}}(I \cup \gamma) \cap \omega_1 = \gamma\}$ contains a club. Let $\alpha \in C_0 \cap S_0$.

Using Lemmas 6.14 and 6.15, it follows that there is $J \subseteq I$, $J \in \boldsymbol{V}[G]$ with $\sup(J) = \delta$ and $\mathrm{Hull}^{(\dot{\mathcal{A}})_G}(J \cup \alpha) \cap \kappa = \mathrm{Hull}^{\mathcal{B}}(J \cup \alpha) \cap \kappa$. Hence $Y = \mathrm{Hull}^{\mathcal{A}}(J \cup \alpha) \prec \mathcal{A}$ satisfies $\sup(Y \cap \omega_1) = \alpha \in S_0$ and $\sup(Y \cap \omega_2) = \delta \in S_1$. □

6.3.2. A Lower Bound on Consistency Strength

We give an easy equivalent of ICC, followed by a lower bound on its consistency strength.

Lemma 6.23. ICC *holds iff for any α-structure, $\mathcal{A}$, with $\alpha \geqslant \omega_2$, and stationary sets $S_0 \subseteq \omega_1$ and $S_1 \subseteq \omega_2 \cap \mathrm{Cof}_{\omega_1}$, $\mathcal{A}$ has a substructure $\mathcal{B} \prec \mathcal{A}$ such that* $\sup(\mathcal{B} \cap \omega_1) \in S_0$ *and* $\sup(\mathcal{B} \cap \omega_2) \in S_1$. *Moreover, $\mathcal{B}$ may be chosen such that* $\mathrm{otp}(\mathcal{B} \cap \omega_2) = \omega_1$.

Proof. Just as in Lemmas 3.28 and 6.4. To prove the nontrivial implication, assume ICC and let α, $\mathcal{A} = \langle J_\alpha^{\vec{A}}, \in, \vec{A}, \ldots\rangle$, S_0, and S_1 be as above. $C \subseteq \{\gamma < \omega_2 : \mathrm{Hull}^{\mathcal{A}}(\gamma) \cap \omega_2 = \gamma\}$ is closed unbounded in ω_2. For any $\gamma, \delta \in C$ with $\gamma \leqslant \delta$, let $\pi_\gamma : \mathcal{A}_\gamma \prec \mathcal{A}$ be the inverse transitive collapse of $\mathrm{Hull}^{\mathcal{A}}(\gamma)$ and $\pi_{\gamma\delta} = \pi_\delta^{-1} \circ \pi_\gamma : \mathcal{A}_\gamma \prec \mathcal{A}_\delta$. Let $\mathcal{A}' = \langle J_{\omega_2}^{C,\Pi,A,B}, \in, C, \Pi, A, B\rangle$ where $\Pi = \langle \pi_{\gamma\delta} : \gamma, \delta \in C, \gamma \leqslant \delta\rangle$, $A = \langle \mathcal{A}_\gamma : \gamma \in C\rangle$, and $B = \langle g_\gamma : \gamma < \omega_2\rangle$, where each $g_\gamma : |\gamma| \to \gamma$ is a bijection.

By ICC there is $Y \prec \mathcal{A}'$ such that $\sup(Y \cap \omega_1) \in S_0$ and $\sup(Y \cap \omega_2) \in S_1$. Let $\gamma = \sup(Y \cap \omega_2)$. Set $\mathcal{B} = \mathrm{Hull}^{\mathcal{A}}(Y)$. It is sufficient to show that $\mathcal{B} \cap \omega_2 = Y \cap \omega_2$. Let $\varepsilon = f_\varphi^{\mathcal{A}}(a_0, \ldots, a_{n-1}) \in \mathcal{B} \cap \omega_2$, where $f_\varphi^{\mathcal{A}}$ is one of the (uniformly definable) Skolem functions of $\mathcal{A}$, and $a_0, \ldots, a_{n-1} \in Y$. $C \cap Y$ is unbounded since $Y \prec \mathcal{A}'$, so there is $\delta \in C \cap Y$ such that $\varepsilon, a_0, \ldots, a_{n-1} \in \mathcal{A}' \restriction \delta$. However then

$$\varepsilon = \pi_\delta^{-1}(\varepsilon) = \pi_\delta^{-1}(f_\varphi^{\mathcal{A}}(a_0, \ldots, a_{n-1})) = f_\varphi^{\mathcal{A}_\delta}(\pi_\delta^{-1}(a_0), \ldots, \pi_\delta^{-1}(a_{n-1})) = f_\varphi^{\mathcal{A}_\delta}(a_0, \ldots, a_{n-1}) \in Y$$

since $\delta, a_0, \ldots, a_{n-1} \in Y$.

For the final sentence of the lemma, note first that $\mathrm{otp}(Y \cap \omega_2) \geqslant \omega_1$ since $\mathrm{cf}(Y \cap \omega_2) = \omega_1$. However if $\gamma \in Y \cap \omega_2$, $g_\gamma \restriction \min\{|\gamma|, Y \cap \omega_1\} : \min\{|\gamma|, Y \cap \omega_1\} \to Y \cap \gamma$ is a bijection, so $\mathrm{otp}(Y \cap \gamma) < \omega_1$. Hence $\mathrm{otp}(Y \cap \omega_2) = \omega_1$, as required. □

Theorem 6.24. *If $\lambda = \omega_2$ and* ICC *holds then $\boldsymbol{K} \vDash$ "λ is greatly Erdős."*

Proof. We may assume that $\boldsymbol{K} \vDash D =_{df} \{\alpha < \lambda : \alpha \text{ is measurable}\}$ is nonstationary. Otherwise, the theorem follows from Corollary 4.26 (since any measurable cardinal is Ramsey).

Let $\mathcal{A} = \langle J_\lambda^{\vec{A}}, \in, \vec{A}, \ldots\rangle \in \boldsymbol{K}$ be a λ-structure. The idea here is to show inductively that, in the notation of Corollary 4.11, $(X_{\mathcal{A}})_\beta$ contains a ω_1-club for each $\beta < \lambda^+$. Then the result would follow from that corollary. At successor stages of the induction, via the argument of Theorem 6.17, we obtain indiscernibles, I, for $\mathcal{A}$ and use the fact that $(X_{\mathcal{A}})_\beta$ contains a club to ensure that $I \subseteq (X_{\mathcal{A}})_\beta$.

This strategy is complicated by the fact that ICC does not give a club, but only an ω_1-club, of suprema of Chang substructures. We shall prove that for each $\beta < \nu =_{\mathrm{df}} (\lambda^+)^{\boldsymbol{K}}$, there is an ω_1-club $C_\beta \subseteq \lambda$ such that $C_\beta \subseteq (X_{\mathcal{A}})_\beta$ and for each $\gamma \in C_\beta$ there is a club $C_{\beta,\gamma} \subseteq \gamma$ such that $C_{\beta,\gamma} \subseteq (X_{\mathcal{A}})_\beta$. As above, the result then follows from Corollary 4.11. Suppose this holds for all $\delta < \beta$.

If $\beta = 0$, the result is trivial. Suppose $\beta = \delta + 1$, so $((X_{\mathcal{A}})_\beta)^{\boldsymbol{K}} = (t_{\mathcal{A}}(((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}))^{\boldsymbol{K}}$. Let

$$\mathcal{A}_\beta = \langle J_\nu^{E,\vec{A},B,D,((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}},\overline{C}_\beta}, \in, E, \vec{A}, B, ((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}, \overline{C}_\beta\rangle,$$

where $\vec{A} = \langle \vec{A}, \ldots\rangle$, $\overline{C}_\beta = \{\langle \xi, \zeta\rangle : \xi \in C_\delta \text{ and } \zeta \in C_{\delta,\xi}\}$ and $B = \langle g_\gamma : \gamma < \omega_2\rangle$, where each $g_\gamma : \mathrm{cf}(\gamma) \to \gamma$ is monotone cofinal. By ICC and Lemma 6.23, there is an ω_1-club $C \subseteq \omega_2$ such that for every $\alpha \in C$ there is $Y_\alpha \prec \mathcal{A}_\beta$ such that $\sup(Y_\alpha \cap \lambda) = \alpha$ and $\mathrm{otp}(Y_\alpha \cap \lambda) = \omega_1$.

Let $\alpha \in C$, $j_0 : \mathcal{A}'_\beta \cong Y = Y_\alpha \prec \mathcal{A}_\beta$ be the inverse of the transitive collapse, and let $j = j_0 \restriction (\boldsymbol{K})^{\mathcal{A}'_\beta} : (\boldsymbol{K})^{\mathcal{A}'_\beta} \longrightarrow \boldsymbol{K}|\nu$. If we set $\bar{D} = j_0^{-1}$ " D then it is club in ω_1 and consists of ordinals that are non-measurable in $(\boldsymbol{K})^{\mathcal{A}'_\beta}$. Consider the coiteration of the latter with $\boldsymbol{K}$. Assume for the moment that the $(\boldsymbol{K})^{\mathcal{A}'_\beta}$ side of the coiteration is trivial and no ultrapowers are taken. Then, as we have seen above, there is a countable mouse N_0 that iterates past $(\boldsymbol{K})^{\mathcal{A}'_\beta}$. By virtue of the fact that no $\gamma \in \bar{D}$ is measurable in $(\boldsymbol{K})^{\mathcal{A}'_\beta}$, on a club subset of points $C' \subseteq \bar{D}$ the ultrapower taken must be by the same measure/filter and its images. As $N_0 \; {}^*\!\!> (\boldsymbol{K})^{\mathcal{A}'_\beta}$ we have that $\mathcal{A}'_\beta \in N_{\omega_1}$. Consequently, C' generates a set of g.i.'s for $\mathcal{A}'_\beta$. By shortening C' if necessary then we shall assume all of C' form g.i.'s.

Thus j"C' is a set of good indiscernibles for $\mathcal{A} \frown \langle((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}\rangle$. We show that j"$C' \cap ((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}} \neq \varnothing$ (hence, by indiscernibility, j"$C' \subseteq ((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}$). Let $\eta \in \lim(j\text{"}C') \cap C_{\delta,\alpha} \cap \lim(C_\delta \cap Y) \cap \alpha$ (the predicate $\overline{C}_\beta$ ensures that $C_\delta \cap Y$ is unbounded in α), and let $\eta' = \min((Y \cap \lambda)\backslash\eta) \geqslant \eta$. Note that $\eta' \in j$"C' since C' is closed. If $\mathrm{cf}(\eta') = \omega$ then, using the predicate B, Y must be unbounded in η', so $\eta' = \eta \in (j\text{"}C') \cap C_{\delta,\alpha} \subseteq j\text{"}C' \cap ((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}$. Now suppose $\mathrm{cf}(\eta') = \omega_1$. Since $C_\delta \cap Y$ is unbounded in η, it follows by elementarity that C_δ is unbounded in η'. However C_δ is ω_1-closed so $\eta' \in j\text{"}C' \cap C_\delta \subseteq j\text{"}C' \cap ((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}$.

By the Jensen Indiscernibles Lemma, there is $I \in \boldsymbol{K} \cap P(\alpha)$, a set of g.i.'s for $\mathcal{A} \frown \langle((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}\rangle$ with $I \supseteq j$"C'. Since we have just shown j"$C' \subseteq ((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}$ we conclude by indiscernibility that $I \subseteq ((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}$, thence $\alpha \in (t_{\mathcal{A}}(((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}))^{\boldsymbol{K}} = ((X_{\mathcal{A}})_\beta)^{\boldsymbol{K}}$. Moreover, $\lim(I) \subseteq (t_{\mathcal{A}}(((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}))^{\boldsymbol{K}} = ((X_{\mathcal{A}})_\beta)^{\boldsymbol{K}}$. Thus, setting $C_\beta = C$, and $C_{\beta,\alpha} = \lim(I)$, the successor stage is complete.

Now note how little of the above argument is affected by allowing $(\boldsymbol{K})^{\mathcal{A}'_\beta}$ to move in the coiteration: on a club subset of C' (which we shall continue to call C') we shall have the ordinals are inaccessible but not measurable in $(\boldsymbol{K})^{\mathcal{A}'_\beta}$ and its iterates; these are thus unmoved by the ultrapowers on this side. We may form the copy construction of this iteration, $\pi'' : \boldsymbol{K} \to \boldsymbol{K}''$ (using again the notation of Figure 1) in a way that we hope is now familiar. We let $\pi''(\mathcal{A} \frown \langle((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}\rangle) = \mathcal{A}'' \frown \langle((X_{\mathcal{A}''})_{\delta''})^{\boldsymbol{K}''}\rangle$. The argumentation of the last two paragraphs is then transferred to this latter structure, obtaining g.i.'s $I'' \in \boldsymbol{K}'' \cap j'(\alpha)$ for it, with $I'' \supseteq j'$ " C'. We deduce the existence of a $C''_{\beta,j'(\alpha)} = \lim(I'') \in \boldsymbol{K}''$ which is contained in $((X_{\mathcal{A}''})_{\delta''})^{\boldsymbol{K}''}$. By elementarity we have such a $C_{\beta.\alpha} \in \boldsymbol{K}$ contained in $((X_{\mathcal{A}})_\delta)^{\boldsymbol{K}}$ as required.

Suppose β is a limit ordinal. This case is essentially a generalisation of the successor stage. Fix a surjection $g : \lambda \to \beta$ with $g \in \boldsymbol{K}$ such that $((X_{\mathcal{A}})_\beta)^{\boldsymbol{K}} = \Delta_{\delta<\lambda}((X_{\mathcal{A}})_{g(\delta)})^{\boldsymbol{K}}$. Setting $\vec{A} = \langle \vec{A}, \ldots\rangle$, let:

$$\mathcal{A}_\beta = \langle J_\nu^{E,\vec{A},B,X,((X_{\mathcal{A}})_\beta)^{\boldsymbol{K}},\overline{C}_\beta}, \in, E, \vec{A}, B, D, X, ((X_{\mathcal{A}})_\beta)^{\boldsymbol{K}}, \overline{C}_\beta\rangle,$$

$$\overline{C}_\beta = \{\langle \delta, \xi, \zeta\rangle : \delta < \lambda, \xi \in C_{g(\delta)}, \text{ and } \zeta \in C_{g(\delta),\xi}\},$$

$$X = \{\langle \delta, \xi\rangle : \delta < \lambda \text{ and } \xi \in ((X_{\mathcal{A}})_{g(\delta)})^{\boldsymbol{K}}\},$$

and let B be as before.

We shall first prove that the induction hypothesis holds for $\beta+1$, then infer that it holds for β. As above, by ICC and Lemma 6.23, there is an ω_1-club $C \subseteq \omega_2$ such that for every $\alpha \in C$ there is $Y_\alpha \prec \mathcal{A}_\beta$ such that $\sup(Y_\alpha \cap \lambda) = \alpha$ and $\operatorname{otp}(Y_\alpha \cap \lambda) = \omega_1$.

Again, let $\alpha \in C$, let $j_0 : \mathcal{A}'_\beta \cong Y = Y_\alpha \prec \mathcal{A}_\beta$ be the inverse transitive collapse, and let $j = j_0 \restriction (\boldsymbol{K})^{\mathcal{A}'_\beta} : (\boldsymbol{K})^{\mathcal{A}'_\beta} \prec \boldsymbol{K}|\nu$. Again, to keep the notation simpler, we shall assume that $(\boldsymbol{K})^{\mathcal{A}'}_\beta$ does not move in its coiteration with $\boldsymbol{K}$.

As for the successor case, there is a club $C' \subseteq \omega_1$ such that $j``C'$ is a set of good indiscernibles for $\mathcal{A} \frown \langle X \rangle$. In this case again, we aim to show that $j``C' \cap ((X_\mathcal{A})_\beta)^{\boldsymbol{K}} \neq \varnothing$.

Let $C^* = \Delta_{\xi \in Y \cap \lambda} C_{g(\xi),\alpha}$, and note that this set is club in α since $\operatorname{otp}(Y \cap \lambda) = \omega_1 = \operatorname{cf}(\alpha)$. Let

$$\eta \in \lim(j``C') \cap C^* \cap \lim(C^{**} \cap Y) \cap \alpha,$$

where $C^{**} = \Delta_{\xi<\lambda} C_{g(\xi)}$ (C^{**} is definable over $\mathcal{A}_\beta$, using the predicate $\overline{C}_\beta$, so is unbounded in α). Let $\eta' = \min((Y \cap \lambda)\backslash\eta) \geqslant \eta$. Again, $\eta' \in j``C'$.

Just as previously, if $\operatorname{cf}(\eta') = \omega$, then $\eta' = \eta \in (j``C') \cap C^*$. However then, $\eta' \in C_{g(\xi),\alpha} \subseteq ((X_\mathcal{A})_{g(\xi)})^{\boldsymbol{K}}$ for all $\xi \in Y \cap \eta$, so $Y_\alpha \vDash \eta' \in ((X_\mathcal{A})_\beta)^{\boldsymbol{K}}$, hence $\eta' \in ((X_\mathcal{A})_\beta)^{\boldsymbol{K}}$.

If $\operatorname{cf}(\eta') = \omega_1$ then, since $C^{**} \cap Y$ is unbounded in η, it follows by elementarity that C^{**} is unbounded in η'. However C^{**} is ω_1-closed so

$$\eta' \in j``C' \cap C^{**} = j``C' \cap \Delta_{\xi<\lambda} C_{g(\xi)} \subseteq j``C' \cap \Delta_{\xi<\lambda}((X_\mathcal{A})_{g(\xi)})^{\boldsymbol{K}} = j``C' \cap ((X_\mathcal{A})_\beta)^{\boldsymbol{K}}.$$

Just as in the successor stage, it now follows that there exists an ω_1-club set $C_{\beta+1} \subseteq ((X_\mathcal{A})_{\beta+1})^{\boldsymbol{K}}$ such that for each $\alpha \in C_{\beta+1}$, there is a club subset of α, $C_{\beta+1,\alpha} \subseteq ((X_\mathcal{A})_{\beta+1})^{\boldsymbol{K}}$.

To see that the same holds for β, recall that $\langle ((X_\mathcal{A})_\xi)^{\boldsymbol{K}} : \xi < \nu \rangle$ is decreasing modulo club. Let $D \subseteq \lambda$ be a closed unbounded set such that $((X_\mathcal{A})_\beta)^{\boldsymbol{K}} \supseteq D \cap ((X_\mathcal{A})_{\beta+1})^{\boldsymbol{K}}$. Setting $C_\beta = C_{\beta+1} \cap \lim(D)$ and $C_{\beta,\alpha} = C_{\beta+1,\alpha} \cap D$ for each $\alpha \in C_\beta$ concludes the argument.

If $(\boldsymbol{K})^{\mathcal{A}'_\beta}$ does move in the coiteration, we appeal to the same thinking as in the successor case using the copied iterate $\boldsymbol{K}''$ of $\boldsymbol{K}$, and the reasoning is identical. This concludes the proof. □

6.4. *Closing the Gap: conclusions and open questions*

We have now established that:

$\operatorname{Con}(\mathrm{ZFC} + \mathrm{CC}^+) \leftrightarrow \operatorname{Con}(\mathrm{ZFC} + (\exists\kappa)(\kappa \text{ Ramsey })) \rightarrow$
$\operatorname{Con}(\mathrm{ZFC} + (\exists\kappa)(\kappa \text{ virtually Ramsey }) \rightarrow$
$\operatorname{Con}(\mathrm{ZFC} + \mathrm{ICC}) \rightarrow$
$\operatorname{Con}(\mathrm{ZFC} + (\exists\kappa)(\kappa \text{ greatly Erdős})).$

However all we have shown in this paper in Section 4 is that

$\operatorname{Con}(\mathrm{ZFC} + (\exists\kappa)(\kappa \text{ greatly Erdős})) \nrightarrow Con(\mathrm{ZFC} + (\exists\kappa)(\kappa \text{ Ramsey})),$

so there is a gap here: we know that we cannot reverse *all* these arrows, but can we reverse any?

We don't have an answer. These questions may hinge on the consistency strength of the existence of a virtually Ramsey cardinal but, as it stands, it isn't clear that we gained anything starting with a virtually Ramsey, rather than Ramsey when proving Lemma 6.22. We conclude this section with some results on virtually Ramsey cardinals that may hint at a solution.

Lemma 6.25. *Let κ be a cardinal.*

i. If κ is Ramsey then κ is virtually Ramsey.

ii. If κ is virtually Ramsey then κ is greatly Erdős.

Proof. i. was proven in Remark 6.20. For ii., the same proof as Lemma 4.30, with $(>\omega)$-clubs in place of clubs, shows that if κ is virtually Ramsey then κ is greatly Erdős. □

Lemma 6.26. *Suppose $\boldsymbol{V} = \boldsymbol{K}$, where $\boldsymbol{K}$ is built from non-overlapping extenders, and κ is a regular cardinal. Let $\mathbb{P}$ be an ω-closed, κ-c.c. forcing, and suppose $\boldsymbol{V}[G] \models$ ICC and $\omega_2^{\boldsymbol{V}[G]} = \kappa$, where G is $\mathbb{P}$-generic. Then $\boldsymbol{V} \models \kappa$ is virtually Ramsey.*

In particular, this holds if $\mathbb{P} = \mathrm{Col}(\omega_1, \kappa)$ and κ is inaccessible.

Proof. (Sketch) We work in the forcing extension. Let $\mathcal{A} \in \boldsymbol{K}$ be a κ-structure, and let

$$M =_{\mathrm{df}} \boldsymbol{K}|(\kappa^+)^{\boldsymbol{K}} \frown \langle \mathcal{A} \rangle.$$

Using lemma 6.23, let C be an ω_1-club in κ such that for all $\alpha \in C$, there is $Y \prec M$ such that $Y \cap \omega_1 \in \omega_1$, $\sup(Y \cap \lambda) = \alpha$, and $\mathrm{otp}(Y \cap \kappa) = \omega_1$. Then there is $C' \in \boldsymbol{V}$, $C' \subseteq C$ such that C' is $(>\omega)$-closed, using the fact that $\mathbb{P}$ is ω-closed and κ-c.c. However for any $\alpha \in C' \cap \mathrm{Cof}_{>\omega}$, Y as above gives rise to an elementary embedding $j : \overline{M} \to M$ (the inverse of the transitive collapse). By the argument of Theorem 6.17 there is a countable N_0 iterating past M and ultimately again a set of good indiscernibles for $\mathcal{A}$, cofinal in α. The result now follows by the Jensen Indiscernibles Lemma. □

Another strategy for better understanding the consistency strength of the existence of a virtually Ramsey cardinal would be to attempt to force a model of ZFC $+(\exists\kappa)(\kappa$ *virtually Ramsey*$)$ from a model of ZFC plus some better understood property.

Note, also, that the forcing destroys weak compactness of κ, if κ is weakly compact. Lemmas 6.21 and 4.32 suggest that this would necessarily be the case if we hope to force a virtually Ramsey cardinal from a greatly Erdős cardinal.

It is consistently possible to separate virtual Ramseyness from Ramseyness, but to date only by starting with a slightly stronger notion:

Theorem 6.27. (Gitman-Hamkins-Welch, [16]) *If κ is a strongly Ramsey cardinal, then there is a forcing partial order $\mathbb{P}$, so that after forcing with $\mathbb{P}$ κ remains virtually Ramsey, but not weakly compact.*

However the notion of κ being strongly Ramsey implies that κ is a limit of completely Ramsey cardinals, (for which, see [15]), so this is not quite good enough. We should like to start with a simple Ramsey cardinal but obtain the same conclusion (or know the reason why not).

Our final result suggests that such an approach might have potential to be successful for a greatly Erdős cardinal if the right iterated forcing were chosen. It shows that it is possible to shoot a single club and retain the greatly Erdős property.

Lemma 6.28. *Assume $\boldsymbol{V} = \boldsymbol{K}$. Suppose κ is greatly Erdős, $\mathcal{F}$ witnesses this, and $X \in \mathcal{F}$. Let $\mathbb{P}$ be the set of closed bounded subsets of X, ordered by end-extension. Then in the forcing extension by $\mathbb{P}$, κ remains greatly Erdős and X contains a club.*

Remark 6.29. *It is possible to prove other results similar to Lemma 6.28. For example, if $\mathbb{P}_X$ is Mitchell's forcing for shooting a club through a set $X \subseteq \kappa$ ([21]), and $X \in \mathcal{F}$, where $\mathcal{F}$ witnesses that κ is greatly Erdős, then it may be shown that $\mathbb{P}_X$ forces $\kappa = \omega_2$ and Chang's Conjecture.*

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