Weak systems of determinacy and arithmetical quasi-inductive definitions

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(Paper & slides - of the same title - available at www.bris.ac.uk/˜mapdw)

Arithmetic operators

We consider *arithmetic* definitions of *operators*.

Definition 1 *(i) A set* $X \subseteq \mathbb{N}$ *is* arithmetic *if it can be defined over the structure of the natural numbers* $\mathbb{N} = \langle \mathbb{N}, +, \times, 0, ', \ldots, \rangle$ *(by a formula in an appropriate (first order) language).*
(ii) An operator $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ *is an* arithmetic operator *if "*$n \in \Gamma(X)$*" is an arithmetic relationship over*

$$\mathbb{N} = \langle \mathbb{N}, +, \times, 0, ', \ldots, X \rangle$$

(by a formula in an appropriate language with a symbol $\dot{X}$*).*

Thus:

$$\Gamma(X) = \{\ n \mid \langle \mathbb{N}, +, \times, 0, ', \ldots, X \rangle \models \varphi[n, X]\ \}.$$

Classifying Arithmetical sets

Definition 2 *(i) A set $X \subseteq \mathbb{N}$ is Σ^0_k if it can be defined over $\mathbb{N}$ by a formula*

$$\exists m_1 \forall m_2 \ldots \forall\backslash\exists m_k R(m_1, \ldots, m_k, n)$$

where R contains only bounded quantifiers (" is recursive").

Iterating along the ordinals

We shall be concerned with iteration of such operators along the *ordinals*:

Define the following iterates:

$$\Gamma_0(X) = X; \quad \Gamma_{\alpha+1}(X) = \Gamma(\Gamma_\alpha(X)); \quad \Gamma_\lambda(X) = ?$$

Monotone Operators

Definition 3 Γ *is* monotone *iff* $\quad \forall X \subseteq Y \quad \Gamma(X) \subseteq \Gamma(Y)$

In this case we set:

$$\Gamma_\lambda(X) = \bigcup_{\alpha<\lambda} \Gamma_\alpha(X)$$

Clearly Γ monotone implies there will be *fixed points*:

Definition 4 *We say that* Γ reaches a fixed point on X, *if there is* α *so that* $\Gamma_\alpha(X) = \Gamma_{\alpha+1}(X)$; *and if so we call* $\Gamma_\infty(X) = \Gamma_\alpha(X)$ *the* least fixed point (over X).

Clearly α will be countable.

Definition 5 *We say* X *is* arithmetically inductive *if for some monotone arithmetical operator* $\Gamma : \mathcal{P}(\mathbb{N} \times \mathbb{N}) \to \mathcal{P}(\mathbb{N} \times \mathbb{N})$ *there is* m_0 *so that*

$$X = \{n \mid (n, m_0) \in \Gamma_\infty(\emptyset)\}$$

A formula φ is X-positive if $\dot{X}$ is enclosed in an even number of $\neg$ symbols.

• *X-positive formulae define monotone inductive operators.*

This need not be restricted to the structure $\mathbb{N}$.

Example: Fix a recursive mapping

$$\ulcorner . \urcorner : Formulae \longrightarrow \mathbb{N}$$

(a "Gödel numbering") of formulae by numbers. The *truth set for arithmetic* T is:

$$T = \{\ulcorner A \urcorner \mid A \text{ a sentence in the language of arithmetic } \}.$$

Theorem 1 *The inductive sets over* $\mathbb{N}$ *are precisely the* Π^1_1 *sets.*

Arithmetical quasi-inductive definitions.

Let $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be an arithmetic operator. Define iterates as before except for limits λ:

$$\Gamma_\lambda(X) = \liminf_{\alpha \to \lambda} \Gamma_\alpha(X) = \cup_{\alpha < \lambda} \cap_{\lambda > \beta > \alpha} \Gamma_\beta(X).$$

Any such operator has a least countable $\zeta = \zeta(\Gamma)$ with $\Gamma_\zeta(\varnothing) = \Gamma_{On}(\varnothing)$.

Definition 6 *$Y \subseteq \omega$ is* arithmetically quasi-inductive *if for some such Γ, Y is (1-1) reducible to* $\Gamma_{On}(\varnothing)$.

Examples

• H.Field *"Revenge-immune Solution to the Semantic Paradoxes"*, J.Philosophical Logic, April 2003.

• J.Hamkins & A.Lewis *"Infinite Time Turing Machines*, JSL 2002.

• H.Herzberger *"Notes on Naive Semantics"*

• S. Kreutzer *Partial fixed point logic on infinite structures*, Annual Conference of the European Association for Computer Science Logic (CSL), 2002, Lecture Notes in Computer Science, 2471.

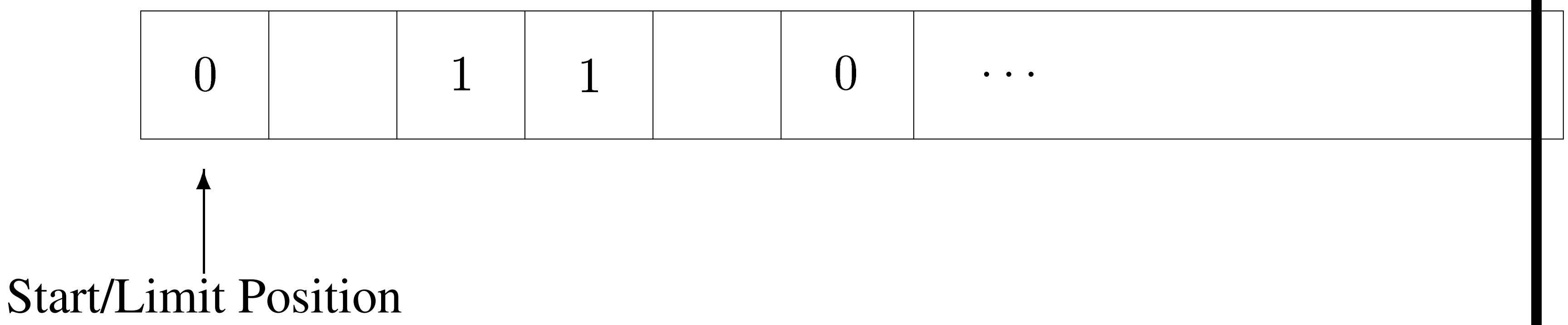
Example: Herzberger's Theory of Truth
(a slide for Philosophers)

Let $\mathcal{L}_{\dot{T}}$ be the language of arithmetic augmented with a new one-place predicate symbol.
Let $T_0 = \emptyset$;
$T_{\alpha+1} = \{\ulcorner A \urcorner \mid \langle \mathbb{N}, \ldots T_\alpha \rangle \models A\}$ where $A \in Sent(\mathcal{L}_{\dot{T}})$;
$T_\lambda = \liminf_{\alpha \to \lambda} \ T_\alpha$.

$T^* = \liminf_{\alpha \to On} T_\alpha$ is then the set of *stable truths of arithmetic*.

Example: Infinite Time Turing Machines

Hardware: just as for ordinary Turing Machines,



Software: just as for ordinary Turing Machines, but with the addition of an extra "limit state" q_L.

Action: *At times* $\nu = \beta + 1$: obey the usual T.M. rules of the program P_e.

At limit times ν: R/W Head returns to the Limit Position; enters state q_L. Cells get their limiting values or are blank otherwise.

Can ask the following concrete question:

Question: What is $\{e : P_e(0){\downarrow}\}$?

or, again:

Question: What is $\{y : P_e(0){\downarrow}y\ \}$?

Proposition 1 *(Burgess) (i) A set Y is arithmetical quasi-inductive iff $Y \in \Sigma_2(L_\zeta)$.*
(ii) ζ is the least Σ_2-extendible ordinal; that is the least ζ so that there is a $\Sigma > \zeta$ with $L_\zeta \prec_{\Sigma_2} L_\Sigma$.

Proposition 2 *There is a recursive operator* Γ *with* $\zeta(\Gamma) = \zeta$.

Some quasi-inductive definitions may reach a fixed point.

Definition 3 We say that Γ *reaches a fixed point on* X, if there is α so that $\Gamma_\alpha(X) = \Gamma_{\alpha+1}(X)$; and if so we call $\Gamma_\alpha(X)$ the *fixed point*.

Definition 4 A *repeat pair* $(\zeta(\Gamma, X), \Sigma(\Gamma, X))$ is a pair of ordinals so that $\Gamma_\zeta(X) = \Gamma_\Sigma(X)$. If an arithmetic operator Γ acting on X has a repeat pair, we say that Γ *converges* (with input X).

Definition 5 **AQI** is the sentence: "For every arithmetic operator Γ, for every $X \subseteq \omega$, there is a wellordering W with a repeat pair $(\zeta(\Gamma, X), \Sigma(\Gamma, X))$ in $\mathrm{Field}(W)$". If an arithmetic operator Γ acting on X has a repeat pair, we say that Γ *converges* (with input X).

A certain amount of set theory (or analysis) is needed to show that every operator converges.

Question: How strong in terms of analysis is **AQI**?

One answer using Prop. 1, this is thus:

Lemma 6 $\mathrm{KP} \vdash \mathbf{AQI} \longleftrightarrow \forall X \subseteq \omega\ \exists \xi, \sigma (L_\xi[X] \prec_{\Sigma_2} L_\sigma[X])$.

Classifying Projective sets

Definition 7 *(i) A set* $X \subseteq \mathbb{N}$ *is* Σ^1_k *if it can be defined over* $\mathbb{N}$ *by a formula*

$$\exists f_1 \forall f_2 \ldots \forall \backslash \exists f_k R(f_1, \ldots, f_k, n)$$

where f_i *are function quantifiers in* $\mathbb{N}^{\mathbb{N}}$ *and* R *contains only bounded quantifiers.*

(ii) X *is* Π^1_k *if* $\overline{X}$ *is* Σ^1_k.

A slide for Proof Theorists

Recall: The strength of Π^1_2-CA_0 by work of RATHJEN is tied up with analysing chains of finite length $L_{\kappa_n} \prec_{\Sigma_1} L_{\kappa_{n+1}}$.

• *The first level L_α which is an infinite tower of such models, is the first whose reals form a β-model of* Π^1_2-CA_0.
Equivalently , it is the least α so that $L_\alpha \models \Sigma_1$-Separation.

The same occurs for Π^1_3-CA_0:

•*The first L_γ whose reals form a β-model of* Π^1_3-CA_0 *is the union of an infinite tower of models $L_{\zeta_n} \prec_{\Sigma_2} L_{\zeta_{n+1}}$.*
Equivalently, it is the least α so that $L_\gamma \models \Sigma_2$-Separation.

Moral: if anyone is to analyse Π^1_3-CA_0, they will have to analyse **AQI** along the way.

Games of perfect information

Given $A \subseteq \mathbb{N}^{\mathbb{N}}$, players I,II play G_A picking $n_k \in \mathbb{N}$:

$$\begin{array}{lllll} I & n_0 & n_2 & n_4 \ldots & \\ II & & n_1 & n_3 & n_5 \ldots \end{array}$$

• I *wins* iff $x = (n_0, n_1, \ldots, n_k, \ldots) \in A$.

• A is *determined* if either I or II has a winning strategy. A class Σ of sets is determined ("$Det(\Sigma)$") if G_A is determined for all $A \in \Sigma$.

• A strategy is essentially a map $\sigma : Fin(\mathbb{N}) \to Fin(\mathbb{N})$, and thus can itself be considered as a subset of $\mathbb{N}$.

• We call a game G_A a Σ-*game*, if $A \in \Sigma$.

Theorem 2 *(Gale-Stewart). Any open game (= Σ^0_1-game) is determined.*

Theorem 3 *(Martin) ZFC proves that all Borel games are determined.*

Theorem 4 *(Friedman) Weaker theories (*e.g.* of analysis) cannot prove Det(Borel).*

Determinacy and Inductive Definitions

Theorem 5 *Any Σ^0_1-game has a monotone Π^1_1-inductive strategy (if I wins), in fact a strategy $\sigma \in L_{\omega_1^{ck}}$; if II wins, then a strategy can be found* $\Pi_1(L_{\omega_1^{ck}})$.

Theorem 6 *(Solovay) Any Σ^0_2-game has a monotone Σ^1_1- inductive strategy (if I wins),* i.e. *a strategy $\sigma \in L_{\sigma^1_1}$; if II wins, then a strategy can be found in $L_{(\sigma^1_1)^+}$.*

A slide for Reverse Mathematicians

Theorem 7 *(Steel) Over RCA_0 the following are equivalent:*
(i) arithmetical transfinite recursion;
(ii)$Det(\Sigma^0_1)$*;*
(iii)$Det(\Delta^0_1)$

Theorem 8 *(Tanaka) Over RCA_0 the following are equivalent:*
(i) Σ^1_1-MI*;*
(ii)$Det(\Sigma^0_2)$

$\boxed{\textbf{The strength of AQI}}$

In descending order of strength are:

$\mathbf{\Pi^1_3}$-$\mathbf{CA_0}$ $>>$ $\mathbf{\Delta^1_3}$-$\mathbf{CA_0} + \mathbf{\Sigma^0_3}$-Determinacy $>>$

$>>$ $\mathbf{\Delta^1_3}$-$\mathbf{CA_0}$+**AQI** $>>$ $\mathbf{\Delta^1_3}$- $\mathbf{CA_0}$.

Slides for Set Theorists

More precisely, we locate winning strategies for various Σ^0_3-games in the L-hierarchy in order to prove the following:

Theorem 9 $\mathrm{KP} \vdash \Sigma_2\text{-Separation} \rightarrow \exists$ *a* β*-model of*

$$\boldsymbol{\Delta^1_3}\textbf{-CA}_\mathbf{0} + \boldsymbol{\Sigma^0_3}\text{-Determinacy}\ .$$

Alternatively: $\boldsymbol{\Pi^1_3}\textbf{-CA}_\mathbf{0} \vdash$ there is a β-model of $\boldsymbol{\Delta^1_3}\textbf{-CA}_\mathbf{0} + \boldsymbol{\Sigma^0_3}\text{-Determinacy}$. The implication is not reversible.

Theorem 10

$\mathrm{KP} + \Delta_2\text{-Comprehension} + \Sigma_2\text{-Collection} + \mathbf{AQI} \nvdash \boldsymbol{\Sigma^0_3}\text{-Determinacy}$.

Alternatively:

Theorem 11 $\mathbf{\Delta^1_3}$-$\mathbf{CA_0} + \mathbf{AQI} \nvdash \mathbf{\Sigma^0_3}$-Determinacy .

Proof: Let δ be least so that $L_\delta \models$ KP $+\mathbf{AQI}$. We show
$L_\delta \not\models \mathbf{\Sigma^0_3}$-Determinacy .
Let $\Psi = \Sigma_1$-$Th(L_\delta)$. For $\psi \in \Psi$, let α_ψ be least so that

$L_\alpha \models KP + \psi$. One shows that for each such ψ there is a Π^0_3 game without a winning strategy in L_{α_ψ}. Fix such a $\psi \in \Psi$.

I plays $m_{0,} m_{1,} \ldots, m_i$ $x = (m_{0,} m_{1,} \ldots, m_i, \ldots)$

II plays $n_{0,} n_{1,} \ldots, n_i$ $y = (n_{0,} n_{1,} \ldots, n_i, \ldots)$

Rules for I.

Let T be the theory KP $+V = L + \psi$. x must be a code for the complete $\Sigma_1 \cup \Pi_1$-theory of an ω-model of $T+$"there is no set model of T".

Approximate Rules for II. In effect she tries to build an $r_{n+1} <_M r_n$-descending chain through the ordinals that I defines using Σ_1-terms. At any move she may "pass" by playing $n_k = 0$, or she can pick out one of the terms *I has previously played as defining an ordinal*, which is $<_M$ below her previous choice.

II wins if both players obey their rules, and infinitely often II does not pass.

Lemma 7 *I has a winning strategy.*

(Play $A_{\alpha_\psi} =_{df}$ the true Σ_1 theory of L_{α_ψ}).

Lemma 8 *Let σ be any winning strategy for I in G_ψ. Then $A_{\alpha_\psi} \leq_T \sigma$.*

As $A =_{df} A_{\alpha_\psi} \notin L_{\alpha_\psi}$ this implies $\sigma \notin L_{\alpha_\psi}$ for any such σ as we required.

The key lemma says the *only* way I can win is by playing the truth.

Lemma 9 *Let σ be any strategy for I. If at any stage I asserts "$m_k = \tau \underset{\notin}{\overset{\in}{}} A$", whilst in fact $\tau \underset{\in}{\overset{\notin}{}} A$ then II can win.*

We have to describe how II can win in this latter case. This amounts to showing how, under our "smallness" assumption about δ, the L-hierarchy up to δ is sufficiently simple, and consequently for any code x of an ω-model of the theory of one of its levels L_α,, if the model is illfounded, then *recursively in x and A_α the standard Σ_1 code of L_α* we can pick out a descending chain through the ordinals named by Σ_1-Skolem terms in that theory x.

Suppose I plays an untruth. Then II knows M will be illfounded.

Suppose I says "τ_0 *is a* Σ_1 *truth of* M *as first witnessed at stage* $a_0 = t_l$" (where $\langle t_i | i < \omega \rangle$ enumerates all Σ_1-terms), and II sees $\tau_0 \notin A$.

Then II could put $r_0 = t_l$ at the head of her putative descending chain. She then waits until I says "τ_1 *is a* Σ_1 *truth of* M *as first witnessed at stage* $a_1 = t_p \wedge t_p < t_l$ ", and II sees $\tau_1 \notin A$.
II could then set $r_1 = t_p < r_0$.

However this will not work: I could force II down into a stretch of the M-ordinals of the form $[b, b + \beta)$ where
$\beta = \beta_M =_{df} On^M \cap WFP(M)$. II is stuck: there may be no new Σ_1-M-truths becoming true in the interval (β, b).

Suppose I plays an untruth. Then II knows M will be illfounded.
Suppose II somehow knew what β_M was going to be.

She waits for an $r_0 = t_l$ as before. For $\gamma \in On$ let $T^\gamma =_{df} \Sigma_2\text{-}Th(L_\gamma)$.

Now she waits until I says: "$\tau_1 \in T^{r_0}$" but II sees that $\tau_1 \notin T^\beta$.

Claim: (For some such t_l) There is such a τ_1. (Using leastness of δ.)

She continues to wait until I says: "t_l is the least ordinal so that $\forall c \in (t_l, r_0] L_c \models \tau_1$"

She then sets $r_1 = t_l < r_0$.

She continues to wait until I says: "t_p is the least ordinal so that $\forall c \in (t_p, r_1] L_c \models \tau_2$ where $\tau_2 \in T^{r_1}$" but II knows that $\tau_2 \notin T^\beta$.

She then sets $r_2 = t_p < r_1$. Etc.
This tactic can be made to work.

Let $\langle \beta_i | i < \omega \rangle$ enumerate the admissibles $\leq \alpha_\psi$.

The correct *Rules for II* involve expressing $\mathbb{N} = \oplus_i A_i$ with each A_i infinite, and in rounds n_i she plays *with the working assumption* that $\beta_M = \beta_i$, and building putative chains $r^i_{n+1} <^M r^i_n$.

II then wins iff

"$\exists i(\{n_k | k \in A_i \wedge n_k \neq 0\}$ is infinite)."

This is a Σ^0_3 condition for II.