On specialisations of the Jónsson property and sequences of mutually stationary sets.

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The basic concept

Let $\langle \kappa_\alpha | \alpha < \delta \rangle$ for $\delta < \kappa_0$ be an increasing sequence of uncountable regular cardinals. (Often $\delta = \omega$, $\kappa_n = \aleph_n$.) Set $\kappa = \kappa_\delta =_{df} \sup \kappa_\alpha$. We consider stationary sets in $[\kappa]^\gamma$ for some $\gamma \leq \kappa_0$.

Theorem 1 *(Liu '93) Let* $f : \mathbb{N} \to 2$ *and suppose*
$\forall n, m \quad f(n) = f(m) = 0 \longrightarrow |n - m| \geq 4$. *Then*

$Con(ZFC + GCH + \exists j : V_\lambda \longrightarrow_e V_\lambda) \Rightarrow Con(ZFC + GCH +$
"$S =_{df} \{X \in [\aleph_\omega]^{\omega_1} \,|\, \forall m > 1\, cf(\sup(X \cap \omega_m)) = \omega_{f(m)}\}$ *is stationary.")*

The latter conclusion is equivalent to saying " *For all algebras* $\mathfrak{A}$ *on* $\aleph_\omega$*,*
$\exists \mathfrak{B} \prec \mathfrak{A}$ *with* $|\mathfrak{B}| = \omega_1 \wedge \forall m > 1\, cf(\sup(\mathfrak{B} \cap \omega_m) = \omega_{f(m)}$."

Theorem 2 *(Liu-Shelah '96) Let* $A \subseteq \omega$, $A = \{m_n \mid n < \omega\}$, *with* $1 < m_n < m_n + 1 < m_{n+1}$. *Then* $Con(ZFC + \exists^\infty$ *measurable cardinals*$) \Rightarrow Con(ZFC+$ *"*$S =_{df}$

$$\left\{ X \in [\aleph_\omega]^{\omega_1} \;\middle|\; \forall m > 1 \; cf(\sup(X \cap \omega_m)) = \begin{array}{ll} \omega & : \textit{if } m \in A \\ \omega_1 & : \textit{if } m \notin A \end{array} \right\}$$

is stationary.")

Note:

- in Theorems 1,2 f (or A) is chosen first, and then assumptions are fitted up accordingly.
- gaps are required in the patterns of cofinalities.

Theorem 3 *(Magidor)* $ZFC \vdash$ *"If* $A \subseteq \omega$ *is infinite/coinfinite and* $S =_{df}$

$$\left\{ X \in [\aleph_\omega]^{\omega_1} \;\middle|\; \forall m > 1 \; cf(\sup(X \cap \omega_m)) = \begin{array}{ll} \omega & : \textit{if } m \in A \\ \omega_1 & : \textit{if } m \notin A \end{array} \right\}$$

is stationary, then there exists an inner model with infinitely many measurable cardinals."

Theorem 4 *Let* $A \subseteq \omega$, $A = \{m_n \mid n < \omega\}$, *with* $1 < m_n < m_{n+1}$. *Suppose that both the sets* $\{n|m_n + 1 = m_{n+1}\}$, $\{n|m_n + 2 < m_{n+1}\}$ *are infinite.*

Now if the same S *is stationary, then there is an inner model with a strong cardinal.*

Proof: Covering arguments and linear iterations.

If we raise the cofinalities in Magidor's Theorem 3:

Theorem 5 $ZFC \vdash$ *"If $A \subseteq \omega$ is infinite/coinfinite and*
$S =_{df}$

$$\left\{ X \in [\aleph_\omega]^{\omega_2} \;\middle|\; \forall m > 2 \;\; cf(\sup(X \cap \omega_m)) = \begin{array}{l} \omega_1 : \textit{ if } m \in A \\ \omega_2 : \textit{ if } m \notin A \end{array} \right\}$$

is stationary, then there exists an inner model with infinitely many measurable cardinals κ with $o(\kappa) = \omega_1$."

Keeping the cofinalities constant

joint with Peter Koepke - Bonn

Let $\langle\kappa_\alpha|\alpha<\delta\rangle$ for $\delta<\kappa_0$ be an ascending sequence of uncountable regular cardinals. Let $\sup\kappa_\alpha =_{df} \kappa_\delta$.

Definition 1 *Let $\mathcal{S}=\langle S_\alpha|\alpha<\delta\rangle$ be a sequence of stationary sets, with $S_\alpha\subseteq\kappa_\alpha\cap cof(\lambda)$ ($\lambda<\kappa_0$).*

Then $\mathcal{S}$ is mutually stationary *if $S =_{df}$*

$$\{[\kappa]^\lambda \mid \sup(X\cap\kappa_\alpha)\in S_\alpha\}$$

is stationary.

(Equivalently if for all algebras $\mathfrak{A}$ on κ_δ, there exists a sub-algebra $\mathfrak{B}\prec\mathfrak{A}$ with $\forall\alpha<\delta: \quad \kappa_\alpha\in\mathfrak{B}\longrightarrow\sup(|\mathfrak{B}|\cap\kappa_\alpha)\in S_\alpha.$)

Definition 2 *Let* $MS(\langle \kappa_a \rangle_{\alpha<\delta}, \lambda)$ *mean:*
For all sequences $\mathcal{S} = \langle S_\alpha | \alpha < \delta \rangle, S_\alpha \subseteq \kappa_\alpha \cap cof(\lambda), S_\alpha$ *stationary,* $\mathcal{S}$ *is mutually stationary.*

- $MS(\langle \kappa_a \rangle_{\alpha<\delta}, \omega)$ for any sequence $\langle \kappa_a \rangle_{\alpha<\delta}$.

Theorem 6 *(Cummings-Foreman-Magidor) If each* κ_α *is measurable, then any such* $\mathcal{S}$ *is mutually stationary.*

Theorem 7 *(Foreman-Magidor) Assume* $V = L$*. Then*
$\forall k > 0 \quad \neg MS(\langle \aleph_n \rangle_{n<\omega}, \omega_k)$

Remark: Their argument can be seen to be valid for any fine-structural inner model of the form $L[\vec{E}]$ (*i.e.* extender models, core models ...)

Theorem 8 *(Cummings-Foreman-Magidor)*
$Con(ZFC+$*"there exists a measurable cardinal"*
$\Longrightarrow Con(ZFC + \exists\langle\kappa_n|n<\omega\rangle(\forall\lambda<\kappa_0\ MS(\langle\kappa_n\rangle_{n<\omega},\lambda))$

Theorem 9 *(Koepke-W)*
$Con(ZFC+$*"there exists a measurable cardinal"*
$\Longleftarrow Con(ZFC + \exists\langle\kappa_n|n<\omega\rangle(\ MS(\langle\kappa_n\rangle_{n<\omega},\omega_1))$

Theorem 10 *(Koepke-W)* $ZFC + 2^{\aleph_m} < \aleph_\omega + MS(\langle\aleph_n\rangle_{n<\omega},\omega_{m+1}) \vdash$ *"There exists an inner model with a measurable κ with $o(\kappa)=\omega_{m+1}$".*

Theorem 11 *(Koepke-W)* $ZFC + MS(\langle\aleph_n\rangle_{n<\omega},\omega_1) \vdash$ *"There exists an inner model with a measurable κ which is a limit of measurables $\lambda<\kappa$".*

Obtaining strength

Theorem 12 *(F-M) Assume* $V = L$. *Fix* $k > 0$. *If each* $S_n \subseteq \omega_n \cap cof(\omega_k) \cap defcof^L(h_n)$ *and* $\langle h_n \rangle$ *is not eventually constant, then* $\mathcal{S} = \langle S_n \rangle$ *is not mutually stationary.*

Here "$defcof^L(h)$" is the class

$$\{\alpha \in On | \alpha \text{ is singular in } L, \text{but is first } \Sigma_h \text{ definably collapsed.}\}$$

We associate $ot(C_\alpha)$, where $\langle C_\alpha | \alpha \in On \rangle$ is the canonically defined *global* $\Box$-*sequence* in L.

$\langle C_\alpha | \alpha$ singular limit ordinal$\rangle$ satisfies:

(i) $C_\alpha \subseteq \alpha$ is closed and unbounded in α;
(ii) $ot(C_\alpha) < \alpha$;
(iii) $\eta \in (C_\alpha)^* \longrightarrow \eta$ is singular & $C_\eta = C_\alpha \cap \eta$.

• If we now assume $\neg O^{\#}$ we have Jensen Covering over L and for $\omega_2 < \alpha$ we have α singular $\longrightarrow (\alpha \text{ singular})^L$.

• Now let $S_n = \{\alpha < \omega_n \cap cof(\omega_1) \mid ot(C_\alpha) \geq \omega_{n-1}\}$. For $n > 0$ each S_n is stationary (using coherence - and absoluteness - properties of the Global $\Box$ sequence *defined in* L).

• Let $\mathfrak{A} = \langle H_{\omega_{\omega+1}}, \in, < \rangle$; suppose $\mathcal{S} = \langle S_n \rangle_n$ is mutually stationary, for a contradiction. Let $\mathfrak{B} \prec \mathfrak{A}$ with $\chi_n =_{df} \sup(\omega_n \cap \mathfrak{B}) \in S_n$.

• If $\pi : \overline{H} \leftrightarrow \mathfrak{B}$ is the transitive collapse, and $L_\gamma = (L)^{\overline{H}}$ then "lift-up" arguments (from the canonical singularising structure for L_γ to those of each of the L_{χ_n}) show that then there is an ordinal $\overline{\beta} < \omega_\omega$ that bounds the order types of all the C_{χ_n}, contrary to the definition of $\mathfrak{B}$. Thus $\mathcal{S}$ is not mutually stationary. Hence $MS(\langle \aleph_n \rangle, \omega_1) \Rightarrow O^{\#}$.