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RAMSEY-LIKE CARDINALS II

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Abstract. This paper continues the study of the *Ramsey-like* large cardinals introduced in [Git09] and [WS08]. Ramsey-like cardinals are defined by generalizing the "existence of elementary embeddings" characterization of Ramsey cardinals. A cardinal κ is Ramsey if and only if every subset of κ can be put into a κ-size transitive model of ZFC for which there exists a weakly amenable countably complete ultrafilter. Such ultrafilters are fully iterable and so it is natural to ask about large cardinal notions asserting the existence of ultrafilters allowing only α-many iterations for some countable ordinal α. Here we study such *α-iterable* cardinals. We show that the α-iterable cardinals form a strict hierarchy for $\alpha \leq \omega_1$ and they are downward absolute to L for $\alpha < \omega_1^L$.

We show that the strongly Ramsey and super Ramsey cardinals from [Git09] are downward absolute to the core model K. Finally, we use a forcing argument from a strongly Ramsey cardinal to separate the notions of Ramsey and virtually Ramsey cardinals. Welch and Sharpe introduced *virtually Ramsey* cardinals in [WS08] as an upper bound on the consistency strength of the Intermediate Chang's Conjecture.

1. Introduction

The definitions of measurable cardinals and stronger large cardinal notions follow the template of asserting the existence of elementary embeddings $j : V \to M$ from the universe of sets to a transitive subclass with that cardinal as the critical point. Many large cardinal notions below a measurable cardinal can be characterized by the existence of elementary embeddings as well. The definitions of these smaller large cardinals κ follow the template of asserting the existence of elementary embeddings $j : M \to N$ with critical point κ from a weak κ-model or κ-model M of set theory to a transitive set[1]. A *weak κ-model* M of set theory is a transitive set of size κ satisfying ZFC$^-$ (ZFC without the *Powerset Axiom*) and having $\kappa \in M$. If a weak κ-model M is additionally closed under $< \kappa$-sequences, that is $M^{<\kappa} \subseteq M$, it is called a *κ-model* of set theory. Having embeddings on κ-models is particularly important for forcing indestructibility arguments, where the techniques rely on $< \kappa$-closure. Weakly compact cardinals is one example of smaller large cardinals that are characterized by the existence of elementary embeddings. A cardinal κ is *weakly compact* if $\kappa^{<\kappa} = \kappa$ and every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ. Another example are the strongly unfoldable cardinals. A cardinal κ is *strongly unfoldable* if for every ordinal α, every $A \subseteq \kappa$ is contained a weak κ-model M for which there exists an elementary embedding $j : M \to N$ with critical point κ, $\alpha < j(\kappa)$, and $V_\alpha \subseteq N$.

An embedding $j : V \to M$ with critical point κ can be used to construct a κ-complete ultrafilter on the powerset of κ. These measures are *fully iterable*; they

[1] It will be assumed throughout the paper that, unless stated otherwise, all embeddings are elementary and between transitive structures.

allow iterating the ultrapower construction through all the ordinals. The iteration proceeds by taking ultrapowers by the image of the original ultrafilter at successor ordinal stages and direct limits at limit ordinal stages to obtain a directed system of elementary embeddings of *well-founded* models of length *Ord*. Kunen showed in [Kun70] that countable completeness is a sufficient condition for an ultrafilter to be fully iterable, that is, for every stage of the iteration to produce a well-founded model. Returning to smaller large cardinals, an embedding $j : M \to N$ with critical point κ can be used to construct an ultrafilter on the powerset of κ of M that is κ-complete *from the perspective* of M. These small measures are called *M-ultrafilters* because all their measure-like properties hold only from the perspective of M. It is natural to ask what kind of iterations can be obtained from M-ultrafilters. Here, there are immediate technical difficulties arising from the fact that an M-ultrafilter is, in most interesting cases, external to M. In order to start defining the iteration, the M-ultrafilter needs to have the additional property of being *weakly amenable*. The existence of weakly amenable ultrafilters with well-founded ultrapowers is equivalent to the existence of embeddings $j : M \to N$ where M and N have the same subsets of κ. We will call such embeddings *κ-powerset preserving*.

Gitman observed in [Git09] that weakly compact cardinals are not strong enough to imply the existence of κ-powerset preserving embeddings. She called a cardinal κ *weakly Ramsey* if every $A \subseteq \kappa$ is contained in a weak κ-model for which there exists a κ-powerset preserving embedding $j : M \to N$. Weakly Ramsey cardinals are consistency-wise above the completely ineffable cardinals and therefore much stronger than weakly compact cardinals. We associate iterating M-ultrafilters with Ramsey cardinals because Ramsey cardinals imply the existence of fully iterable M-ultrafilters. It was shown in [Dod82, Mit79] among others, that κ is Ramsey if and only if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a weakly amenable countably complete M-ultrafilter. The full iterability follows by Kunen's result. The α-iterable cardinals were introduced in [Git09] to fill the gap between the weakly Ramsey cardinals that assert the existence of M-ultrafilters that can potentially be iterated and Ramsey cardinals that assert the existence of fully iterable M-ultrafilters. A cardinal κ is *α-iterable* if every subset of κ is contained in a weak κ-model M for which there exists an M-ultrafilter allowing an iteration of length α. By a well-known result of Gaifman [Gai74], an ultrafilter that allows an iteration of length ω_1, is fully iterable. So it only makes sense to study the α-iterable cardinals for $\alpha \leq \omega_1$.

Welch and Sharpe showed in [WS08] that the ω_1-iterable cardinals are strictly weaker than Ramsey cardinals. In fact, they are weaker than ω_1-Erdős cardinals. In Section 3, we show that for $\alpha < \omega_1^L$, the α-iterable cardinals are downward absolute to L. In Section 4, we show that for $\alpha \leq \omega_1$, the α-iterable cardinals form a hierarchy of strength.

Gitman also introduced *strongly Ramsey* cardinals and *super Ramsey* cardinals by requiring the existence of κ-powerset preserving embeddings on κ-models instead of weak κ-models. The strongly Ramsey and super Ramsey cardinals fit in between Ramsey cardinals and measurable cardinals in strength. In Section 5, we show that these two large cardinal notions are downward absolute to the core model K. In Section 6, we use a forcing argument starting from a strongly Ramsey cardinal to separate the notions of virtually Ramsey and Ramsey cardinals. *Virtually Ramsey* cardinals were introduced by Welch and Sharpe in [WS08] as an upper bound on the consistency of the Intermediate Chang's Conjecture.

2. Preliminaries

In this section, we review facts about M-ultrafilters and formally define the α-iterable cardinals. We begin by giving the precise definition of an M-ultrafilter.

Definition 2.1. Suppose M is a transitive model of ZFC$^-$ and κ is a cardinal in M. Then $U \subseteq \mathcal{P}(\kappa) \cap M$ is an *M-ultrafilter* if $\langle M, \in, U\rangle \models$ "U is a normal ultrafilter".

Recall that an ultrafilter U on κ is *normal* if every regressive function $f : A \to \kappa$ with $A \in U$ is constant on some $B \in U$. Normality implies κ-completeness. The M-ultrafilters are normal *from the perspective of* M but most likely not even countably complete from outside. An ultrapower by an M-ultrafilter is not necessarily well-founded. A way to obtain M-ultrafilters with well-founded ultrapowers is from an elementary embedding $j : M \to N$.

Proposition 2.2. *Suppose M is a weak κ-model and $j : M \to N$ is an elementary embedding with critical point κ, then $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is an M-ultrafilter.*

In this case, we say that U is *generated by κ via j*.

To define α-iterable cardinals, we will need the corresponding key notion of *α-good* M-ultrafilters.

Definition 2.3. Suppose M is a weak κ-model and U is an M-ultrafilter on κ, then U is 0-*good* if the ultrapower of M by U is well-founded.

If M is a weak κ-model and $j : M \to N$ is an embedding with critical point κ, then the M-ultrafilter generated by κ via j is 0-good. This follows from the fact that the ultrapower of M by U embeds into N and hence must be well-founded.

To begin talking about iterability of M-ultrafilters, we need the following key definitions.

Definition 2.4. Suppose M is a weak κ-model and U is an M-ultrafilter on κ. Then U is *weakly amenable* if for every $A \subseteq \kappa \times \kappa$ in M, the set $\{\alpha \in \kappa \mid A_\alpha \in U\} \in M$.

Definition 2.5. An elementary embedding $j : M \to N$ is *κ-powerset preserving* if M and N have the same subsets of κ.

It turns out that the existence of weakly amenable M-ultrafilters on κ with well-founded ultrapowers is equivalent to the existence of κ-powerset preserving embeddings.

Proposition 2.6.

(1) *If $j : M \to N$ is the ultrapower by a weakly amenable M-ultrafilter on κ, then j is κ-powerset preserving.*

(2) *If $j : M \to N$ is a κ-powerset preserving embedding, then the M-ultrafilter $U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$ is weakly amenable.*

(see [Kan03], p. 246)

Definition 2.7. Suppose M is a weak κ-model and U is an M-ultrafilter on κ, then U is 1-*good* if it is weakly amenable and 0-good.

Lemma 2.8. *Suppose M is a weak κ-model, U is a 1-good M-ultrafilter on κ, and $j : M \to N$ is the ultrapower by U. Define*

$$j_U(U) = \{A \subseteq j_U(\kappa) \mid A = [f]_U \text{ and } \{\alpha \in \kappa \mid f(\alpha) \in U\} \in U\}.$$

Then $j_U(U)$ is well-defined and $j_U(U)$ is a weakly amenable N-ultrafilter on $j_U(\kappa)$ such that $A \in U$ implies $j_U(A) \in j_U(U)$. ([Kan03], p. 246)

Lemma 2.8 is essentially saying that we can take the ultrapower of the structure $\langle M, \in, U\rangle$ by the ultrafilter U and a partial Łoś Theorem still goes through due to the weak amenability of U. Previously, we only took ultrapowers of the structures $\langle M, \in\rangle$. The Łoś Theorem there relied on the fact that $M \models \text{ZFC}^-$, but the structure $\langle M, \in, U\rangle$ need not satisfy any substantial fragment of ZFC. The additional assumption of weak amenability is precisely what is required to carry out the argument. Weak amenability serves the basis of any fine structtural analysis of measures and extenders [Zem02]. The ultrafilter $j_U(U)$ is simply the relation corresponding to U in the ultrapower. It is not necessarily the case that the ultrapower by $j_U(U)$ is well-founded.

Suppose M is a weak κ-model and U_0 is a 1-good M-ultrafilter. Let $j_{U_0}(U_0) = U_1$ be the weakly amenable ultrafilter obtained as above for the ultrapower of M by U. If the ultrapower by U_1 happens to be well-founded, we will say that U_0 is 2-*good*. In this way, we can continue iterating the ultrapower construction so long as the ultrapowers are well-founded. For $\xi \leq \omega$, we will say that U is ξ-*good* if the first ξ-many ultrapowers are well-founded. Suppose next that the first ω-many ultrapowers are well-founded. We can form their direct limit and ask if that is well-founded as well. If the direct limit of the first ω-many iterates turns out to be well-founded, we will say that U is $\omega + 1$-*good*. In general, we define that for an ordinal α, an M-ultrafilter is α-*good*, if we can iterate the ultrapower construction for α-many steps.

Gaifman showed in [Gai74] that to be able to iterate the ultrapower construction through all the ordinals it suffices to know that we can iterate through all the countable ordinals.

Theorem 2.9. *An ω_1-good M-ultrafilter is α-good for every ordinal α.*

Thus, the study of α-good ultrafilters only makes sense for $\alpha \leq \omega_1$.

Definition 2.10. For $\alpha \leq \omega_1$, a cardinal κ is α-*iterable* if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an α-good M-ultrafilter.

A few easy observations about the definition are in order.

Remarks 2.11.

(1) If $\kappa^{<\kappa} = \kappa$, then κ is 0-iterable if and only if κ is weakly compact. Without that extra assumption, being 0-iterable is not necessarily a large cardinal notion. Hamkins showed in [Ham07] that it is consistent that ω_1 can be 0-iterable.

(2) Weakly Ramsey cardinals are exactly the 1-iterable cardinals. Unlike 0-iterability, 1-iterability implies inaccessibility.

(3) By our previous comments, Ramsey cardinals are ω_1-iterable.
(4) ω_1-iterable cardinals are *strongly unfoldable* in L (see [Vil98]).

3. α-iterable cardinals in L

In this section, we show that for $\alpha < \omega_1^L$, the α-iterable cardinals are downward absolute to L. This result is optimal since ω_1-iterable cardinals cannot exist in L.

Many of our arguments below will use the following two simple facts about weak κ-models.

Remarks 3.1.

(1) If M is a weak κ-model of height α, then $L^M = L_\alpha$. Note that $M \cap L$ can be a proper superset of L_α. That is, M might contain constructible elements that it does not realize are constructible.
(2) If $j : M \to N$ is an elementary embedding with critical point κ and X has size κ in M, then $j \upharpoonright X$ is an element of N. This follows since $j \upharpoonright X$ is definable from an enumeration of X in M together with $j(f)$, both of which are elements of N.

Next, we give a short argument why ω_1-iterable cardinals cannot exist in L.

Proposition 3.2. *If there is an ω_1-iterable cardinal, then $0^\#$ exists.*

Proof. Suppose κ is an ω_1-iterable cardinal. Fix a weak κ-model M and an ω_1-good M-ultrafilter U on κ. By Theorem 2.9, U is fully iterable. Let $j_\alpha : M_\alpha \to N_\alpha$ be the α^{th}-iterated ultrapower of U. Observe that $\mathcal{P}(\kappa) \cap M = \mathcal{P}(\kappa) \cap M_\alpha$ for all α. By remark 3.1 (1), $M_\alpha \cap L$ contains L_α. Thus, for a large enough α, we have $\mathcal{P}(\kappa)^L \subseteq L_\alpha \subseteq M_\alpha$. It follows that $L_{\kappa^+} \subseteq M$. Thus, j restricts to an embedding on L_{κ^+} and hence $0^\#$ exists. □

We will first show that if $0^\#$ exists, then for $\alpha < \omega_1^L$, the Silver indiscernibles are α-iterable in L. We will modify this argument to show that for $\alpha < \omega_1^L$, the α-iterable cardinals are downward absolute to L. We begin with the case of 1-iterable cardinals. We will make use of a standard lemma below (see [BJW82] for a proof).

Lemma 3.3. *If $0^\#$ exists and κ is a Silver indiscernible, then $\text{cf}^V((\kappa^+)^L) = \omega$.*

Theorem 3.4. *If $0^\#$ exists, then the Silver indiscernibles are 1-iterable in L.*

Proof. Let $I = \{i_\xi \mid \xi \in Ord\}$ be the Silver indiscernibles enumerated in increasing order. Fix $\kappa \in I$ and let $\lambda = (\kappa^+)^L$. Define $j : I \to I$ by $j(i_\xi) = i_\xi$ for all $i_\xi < \kappa$ and $j(i_\xi) = i_{\xi+1}$ for all $i_\xi \geq \kappa$ in I. The map j extends, via the Skolem functions, to an elementary embedding $j : L \to L$ with critical point κ. Restrict j to $j : L_\lambda \to L_{j(\lambda)}$. Since $\lambda = (\kappa^+)^L$, j is clearly κ-powerset preserving. Let U be the weakly amenable L_λ-ultrafilter generated by κ via j as in Proposition 2.2. Since every $\alpha < \lambda$ has size κ in L_λ, by weak amenability, $U \cap L_\alpha$ is an element of L_λ. Construct, using Lemma 3.3, a sequence $\langle \lambda_i : i \in \omega \rangle$ cofinal in λ, such that each $L_{\lambda_i} \prec L_\lambda$, and $U \cap L_{\lambda_i}, L_{\lambda_i} \in L_{\lambda_{i+1}}$. Let j_i be the restriction of j to L_{λ_i}. Each $j_i : L_{\lambda_i} \to L_{j(\lambda_i)}$ is an element of L by remark 3.1 (2), since it has size κ in L_λ. These observations motivate the construction below.

To show that κ is 1-iterable in L, for every $A \subseteq \kappa$ in L, we need to construct in L a weak κ-model M containing A and a 1-good M-ultrafilter on κ. Fix $A \subseteq \kappa$ in L. Define in L, the tree T of finite sequences of the form

$$s = \langle h_0 : L_{\gamma_0} \to L_{\delta_0}, \ldots, h_n : L_{\gamma_n} \to L_{\delta_n} \rangle$$

ordered by extension and satisfying the properties:

(1) $A \in L_{\gamma_0} \models \text{ZFC}^-$,
(2) $h_i : L_{\gamma_i} \to L_{\delta_i}$ is an elementary embedding with critical point κ,
(3) $\delta_i < j(\lambda)$.

Let W_i be the L_{γ_i}-ultrafilter generated by κ via h_i. Then:

(4) for $i < j \leq n$, we have $L_{\gamma_i}, W_i \in L_{\gamma_j}$, $L_{\gamma_i} \prec L_{\gamma_j}$, $L_{\delta_i} \prec L_{\delta_j}$, and h_j extends h_i.

We view the sequences s as better and better approximations to the embedding we are trying to build.

Consider the sequences

$$s_n = \langle j_0 : L_{\lambda_0} \to L_{j(\lambda_0)}, \ldots, j_n : L_{\lambda_n} \to L_{j(\lambda_n)} \rangle.$$

Clearly each s_n is an element of T and $\langle s_n : n \in \omega \rangle$ is a branch through T in V. Hence the tree T is ill-founded. By absoluteness, it follows that T is ill-founded in L as well. Let $\{h_i : L_{\gamma_i} \to L_{\delta_i} : i \in \omega\}$ be a branch of T in L and W_i be the L_{γ_i}-ultrafilters as above. Let

$$h = \cup_{i\in\omega} h_i,\ L_\gamma = \cup_{i\in\omega} L_{\gamma_i},\ L_\delta = \cup_{i\in\omega} L_{\delta_i},\ \text{and } W = \cup_{i\in\omega} W_i.$$

It is clear that $h : L_\gamma \to L_\delta$ is an elementary embedding with critical point κ and W is a weakly amenable L_γ-ultrafilter generated by κ via h. Since the ultrapower of L_γ by W is a factor embedding of h, it must be well-founded.

We have now found a weak κ-model L_γ containing A for which there exists a weakly amenable L_γ-ultrafilter on κ with a well-founded ultrapower. This completes the proof that κ is 1-iterable. □

The next lemma will allow us to modify the proof of Theorem 3.4 to show that the 1-iterable cardinals are downward absolute to L.

Lemma 3.5. *If κ is a 1-iterable cardinal, then every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists a 1-iterable M-ultrafilter U satisfying the conditions:*

(1) $M = \cup_{n\in\omega} M_n$,
(2) $M_n, M_n \cap U \in M_{n+1}$,
(3) *for* $i < j$, *we have* $M_i \prec M_j$,
(4) $M \models$ *"I am H_{κ^+}" (every set has transitive closure of size at most κ).*

Proof. Fix $A \subseteq \kappa$. Since κ is 1-iterable, there is a weak κ-model M' containing A and a 1-good M'-ultrafilter U' on κ. Let $h : M' \to N'$ be the ultrapower embedding by U'. We can assume without loss of generality that $M' \models$ "I am H_{κ^+}" by taking $\{B \in M' \mid M' \models \text{transitive closure of } B \text{ has size } \kappa\}$ instead of M' and restricting the embedding accordingly. It then follows that $M' = H^{N'}_{\kappa^+}$. In N', let M_0 be a transitive elementary submodel of $H_{\kappa^+} = M'$ of size κ containing A. Since U' is weakly amenable, it follows that $U_0 = M_0 \cap U'$ is an element of $H_{\kappa^+} = M'$. Let M_1 be a transitive elementary submodel of H_{κ^+} of size κ containing M_0 and U_0. Similarly let M_{n+1} be a transitive elementary submodel of H_{κ^+} of size κ containing

M_n and U_n. Let $M = \cup_{n\in\omega} M_n$ and $U = \cup_{n\in\omega} U_n$. Clearly U is a weakly amenable M-ultrafilter and the ultrapower of M by U embeds into N and hence is well-founded. □

Theorem 3.6. *If κ is 1-iterable, then κ is 1-iterable in L.*

Proof. Fix $A \subseteq \kappa$ in L. Choose a weak κ-model M of *least* height α satisfying the requirements that $A \in M$, $M \models A \in L$, and there is a 1-good M-ultrafilter U on κ. We can assume that M satisfies the conclusion of Lemma 3.5 since the construction in the proof can only decrease the height. Let $j : M \to N$ be the ultrapower embedding by U, then j restricts to $j : L_\alpha \to L_\beta$ where β is the height of N. Observe that $(\kappa^+)^{L_\beta} \le \alpha$ since if $\gamma < \beta$ of size κ in L_β, then $\gamma \in M$. First, we argue that $W = U \cap L_\alpha$ is a weakly amenable L_α-ultrafilter. It suffices to show that $U \cap L_\xi \in L_\alpha$ for every ξ of size κ in L_α. Since L_ξ has size κ in L_α, we have $j \restriction L_\xi \in L_\beta$ and thus $U \cap L_\xi \in L_\beta$. Since $U \cap L_\xi$ has size at most κ in L_β, it must be in $L_{(\kappa^+)^{L_\beta}} \subseteq L_\alpha$. Thus, U is a weakly amenable L_α-ultrafilter. Next, we argue that $L_\alpha \models$ "I am H_{κ^+}". If this were not true, then there would be $\xi < \alpha$ such that $\xi = (\kappa^+)^{L_\alpha}$. But, then by the same argument as above, we can restrict the ultrafilter to L_ξ and have it remain weakly amenable. This would contradict the assumption that the initial model M was chosen of least height since we could have taken L_ξ instead.

Therefore the embedding $j : L_\alpha \to L_\beta$ has exactly the same properties as the embedding with the Silver indiscernible as the critical point. Namely, $\mathrm{cf}^V(\alpha) = \omega$, $L_\alpha \models$ "I am H_{κ^+}", and the ultrafilter generated by κ via j is weakly amenable. So we can proceed as in the proof of Theorem 3.4 to show that the critical point κ is 1-iterable in L. □

The next lemma is a simple observation that will prove key to generalizing the arguments above for α-iterable cardinals. Let us say that

$$\{j_{\gamma\xi} : M_\gamma \to M_\xi \mid \xi < \gamma < \alpha\}$$

is a *good commuting system of elementary embeddings of length α* if:

(1) for all $\xi_0 < \xi_1 < \xi_2 < \alpha$, $j_{\xi_1\xi_2} \circ j_{\xi_0\xi_1} = j_{\xi_0\xi_2}$.

Let κ_ξ be the critical point of $j_{\xi\xi+1}$ and let U_ξ be the M_ξ-ultrafilter generated by κ_ξ via $j_{\xi\xi+1}$, then:

(2) for all $\xi < \beta < \alpha$, if $A \in M_\xi$ and $A \subseteq U_\xi$, then $j_{\xi\beta}(A) \subseteq U_\beta$.

Remarks 3.7. The directed system of embedding resulting from the iterated ultrapowers construction is a good commuting system of elementary embeddings.

The next lemma shows that existence of good commuting systems of elementary embeddings of length α is basically equivalent to existence of α-good ultafilters.

Lemma 3.8. *Suppose $\{j_{\gamma\xi} : M_\gamma \to M_\xi : \xi < \gamma < \alpha\}$ is a good commuting system of elementary embeddings of length α. Suppose further that $m_0 = \cup_{i\in\omega} m_0^{(i)}$ is a transitive elementary submodel of M_0 such that $m_0^{(i)} \prec m_0^{(i+1)}$ and $U_0 \cap m_0^{(i)}$, $m_0^{(i)} \in m_0^{(i+1)}$. Then $u_0 = m_0 \cap U_0$ is an α-good m_0-ultrafilter.*

Proof. Let $\{h_{\gamma\xi} : m_\gamma \to m_\xi : \xi < \gamma < \alpha\}$ be the not necessarily well-founded directed system of embeddings obtained by iterating u_0 and let u_ξ be the ξ^{th}-iterate of u_0. Define $u_0^{(i)} = u_0 \cap m_0^{(i)}$ and $u_\xi^{(i)} = h_{0\xi}(u_0^{(i)})$. It is easy to see that

$h_{\beta\xi}(u_\beta^{(i)}) = u_\xi^{(i)}$ and $u_\xi = \cup_{i\in\omega} u_\xi^{(i)}$. To show that each m_ξ is well-founded, we will argue that we can define elementary embeddings $\pi_\xi : m_\xi \to M_\xi$. More specifically we will construct the following commutative diagram:

$$\begin{array}{ccccccccccc}
M_0 & \xrightarrow{j_{01}} & M_1 & \xrightarrow{j_{12}} & M_2 & \xrightarrow{j_{23}} & \dots & \xrightarrow{j_{\xi\xi+1}} & M_{\xi+1} & \xrightarrow{j_{\xi+1\xi+2}} & \dots \\
\uparrow \pi_0 & & \uparrow \pi_1 & & \uparrow \pi_2 & & \dots & & \uparrow \pi_\xi & & \\
m_0 & \xrightarrow{h_{01}} & m_1 & \xrightarrow{h_{12}} & m_2 & \xrightarrow{h_{23}} & \dots & \xrightarrow{h_{\xi\xi+1}} & m_{\xi+1} & \xrightarrow{h_{\xi+1\xi+2}} & \dots
\end{array}$$

where

(1) $\pi_{\xi+1}([f]_{u_\xi}) = j_{\xi\xi+1}(\pi_\xi(f))(\kappa_\xi)$,
(2) if λ is a limit ordinal and t is a thread in the direct limit m_λ with domain $[\beta, \lambda)$, then $\pi_\lambda(t) = j_{\beta\lambda}(\pi_\beta(t(\beta)))$,
(3) $\pi_\xi(u_\xi^{(i)}) \subseteq U_\xi$.

We will argue that the π_ξ exist by induction on ξ. Let π_0 be the identity map. Suppose inductively that π_ξ has the desired properties. Define $\pi_{\xi+1}$ as in (1) above. Since $\pi_\xi(u_\xi^{(i)}) \subseteq U_\xi$ by the inductive assumption, it follows that $\pi_{\xi+1}$ is a well-defined elementary embedding. The commutativity of the diagram is also clear. It remains to verify that $\pi_{\xi+1}(u_{\xi+1}^{(i)}) \subseteq U_{\xi+1}$. Recall that

$$u_{\xi+1}^{(i)} = h_{\xi\xi+1}(u_\xi^{(i)}) = [c_{u_\xi^{(i)}}]_{u_\xi}.$$

Let $\pi_\xi(u_\xi^{(i)}) = v$. Then by inductive assumption, $v \subseteq U_\xi$. Thus,

$$\pi_{\xi+1}(u_{\xi+1}^{(i)}) = j_{\xi\xi+1}(c_v)(\kappa_\xi) = c_{j_{\xi\xi+1}(v)}(\kappa_\xi) = j_{\xi\xi+1}(v).$$

By hypothesis, $j_{\xi\xi+1}(v) \subseteq U_{\xi+1}$. This completes the inductive step. The limit case also follows easily. □

Below, we give another useful example of a good commuting system of elementary embeddings.

Lemma 3.9. *Suppose L (or L_α) is the Skolem closure of a collection of order indiscernibles I and let i_ξ be the ξ^{th} element of I. Fix δ below the order type of I and consider the system of elementary embeddings $j_{\alpha\beta}$ defined by $j_{\alpha\beta}(i_\xi) = i_\xi$ for $\xi < \delta + \alpha$ and $j_{\alpha\beta}(i_\xi) = i_{\xi+\gamma}$ where $\alpha + \gamma = \beta$. Then this is a good commuting system of elementary embeddings.*

The proof is a straightforward application of indiscernibility. Thus, if κ is a Silver indiscernible, then there is a good commuting system of elementary embeddings of length *Ord* with the first embedding having critical point κ.

Now we can generalize Lemma 3.5 to the case of α-iterable cardinals.

Lemma 3.10. *If κ is an α-iterable cardinal, then every $A \subseteq \kappa$ is contained in a weak κ-model M for which there exists an α-good M-ultrafilter U satisfying the conditions:*

(1) $M = \cup_{n\in\omega} M_n$,
(2) $M_n, M_n \cap U \in M_{n+1}$,
(3) *for* $i < j$, *we have* $M_i \prec M_j$,
(4) $M \models$ *"I am H_{κ^+}".*

Proof. Start with any weak κ-model M' and an α-good M'-ultrafilter U' on κ. Use proof of Lemma 3.5 to find a transitive elementary submodel M of M' satisfying the requirements and use Lemma 3.8 to argue that $U = M \cap U'$ is α-good. □

We are now ready to show that if $0^{\#}$ exists, for $\alpha < \omega_1^L$, the Silver indiscernibles are α-iterable in L. It will follows using the same techniques that for $\alpha < \omega_1^L$, the α-iterable cardinals are downward absolute to L.

Theorem 3.11. *If $0^{\#}$ exists, then for $\alpha < \omega_1^L$, the Silver indiscernibles are α-iterable in L.*

Proof. Fix a Silver indiscernible κ_0 and let $\lambda_0 = (k_0^+)^L$. Let U_0 be an ω_1-good L_{λ_0}-ultrafilter on κ_0 that exists by Lemma 3.8 combined with Lemma 3.9. Let

$$\{j_{\gamma\xi} : L_{\lambda_\gamma} \to L_{\lambda_\xi} : \xi < \lambda < \alpha\}$$

be the good commuting system of elementary embeddings obtained from the first α-steps of the iteration. Let κ_ξ be the critical point of $j_{\xi\xi+1}$ and let U_ξ be the ξ^{th}-iterate of U_0. As before, we find a cofinal sequence $\langle \lambda_0^{(i)} : i \in \omega\rangle$ in λ_0 such that $L_{\lambda_0^{(i)}} \prec L_{\lambda_0}$ and $U_0 \cap L_{\lambda_0^{(i)}}, L_{\lambda_0^{(i)}} \in L_{\lambda_0^{(i+1)}}$. Let $U_0^{(i)} = U_0 \cap L_{\lambda_0^{(i)}}$, $L_{\lambda_\xi^{(i)}} = j_{0\xi}(L_{\lambda_0^{(i)}})$, and $U_\xi^{(i)} = j_{0\xi}(U_0^{(i)})$. Finally, let $j_{\gamma\xi}^{(i)}$ be the restriction of $j_{\gamma\xi}$ to $L_{\lambda_\xi^{(i)}}$. Observe that each $j_{\gamma\xi}^{(i)}$ is an element of L by remark 3.1 (2). As before, we will use these sequences to show that the tree we construct below is ill-founded.

To show that κ is α-iterable in L, for every $A \subseteq \kappa$ in L, we need to construct in L a weak κ-model M containing A and an α-good M-ultrafilter on κ. Fix $A \subseteq \kappa$ in L. Also, fix in L, a bijection $\rho : [\alpha]^2 \to \omega$.

Define in L, the tree T of finite tuples of sequences $t = \langle s_0, \ldots, s_{n-1}\rangle$ where each s_i is a sequence of length n consisting of approximations to the elementary embedding from stage ξ to stage β where $\rho(\xi, \beta) = i$. That is

$$s_i = \langle h_{\xi\beta}^{(0)} : L_{\gamma_\xi^{(0)}} \to L_{\gamma_\beta^{(0)}}, \ldots, h_{\xi\beta}^{(n-1)} : L_{\gamma_\xi^{(n-1)}} \to L_{\gamma_\beta^{(n-1)}}\rangle$$

Note that if t is an n-tuple, then we require all sequences in the tuple to have length n. We define $t \leq t'$ whenever the length of t' is greater than or equal to the length of t and the i^{th} coordinate of t' extends the i^{th} coordinate of t. They are required to satisfy the properties:

(1) $A \in L_{\gamma_0^{(0)}} \models \text{ZFC}^-$
(2) $h_{\xi\beta}^{(i)} : L_{\gamma_\xi^{(i)}} \to L_{\gamma_\beta^{(i)}}$ are commuting elementary embeddings,
(3) $\gamma_\xi^{(i)} < \lambda_\alpha = \cup_{\xi<\alpha}\lambda_\xi$.

Let κ_ξ be the critical points of $h_{\xi\xi+1}$ and let $W_\xi^{(i)}$ be the $L_\xi^{(i)}$-ultrafilters generated by κ_ξ via $h_{\xi\xi+1}^{(i)}$, then:

(4) for $i < j \leq n$, $L_{\gamma_\xi^{(i)}}, W_\xi^{(i)} \in L_{\gamma_\xi^{(j)}}$, $L_{\gamma_\xi^{(i)}} \prec L_{\gamma_\xi^{(j)}}$ and $h_{\xi\beta}^{(j)}$ extends $h_{\xi\beta}^{(i)}$,
(5) $h_{\xi\beta}^{(i)}(W_\xi^{(i)}) = W_\beta^{(i)}$.

As before, we argue using the iterated ultrapowers of U_0, that T is ill-founded in V and hence in L. Unioning up the branch in L, we obtain a good commuting system of elementary embeddings of length α and therefore an α-good ultrafilter for the first model in the system by Lemma 3.8. □

Theorem 3.12. *If κ is α-iterable, then κ is α-iterable in L.*

Proof. Use the proof of Theorem 3.6 together with Lemma 3.8. □

4. The hierarchy of α-iterable cardinals

In this section, we show using the techniques developed in the previous section, that for $\alpha \leq \omega_1$, the α-iterable cardinals form a hierarchy of strength. We also make some observations about the relationship between the α-iterable cardinals and the α-Erdős cardinals.

Theorem 4.1. *If κ is an α-iterable cardinal, then for $\xi < \alpha$, the cardinal κ is a limit of ξ-iterable cardinals.*

Proof. Suppose κ is an α-iterable cardinal. Choose a weak κ-model M_0 containing V_κ as an element for which there exists an α-good M_0-ultrafilter, satisfying the conclusion of Lemma 3.10. Let $j_\xi : M_\xi \to M_{\xi+1}$ be the ξ^{th} step of the iteration by U_0. Suppose, first, that $\alpha = \beta + 1$ is a successor ordinal. In this case, the iteration will have a final model namely M_α. It suffices to argue that κ is β-iterable in M_α. To see this, suppose that κ is β-iterable in M_α, then κ is β-iterable in M_1 as well. But then M_1 satisfies that there is a β-iterable cardinal below $j_0(\kappa)$ and hence, by elementarity, M_0 satisfies that κ is a limit of β-iterable cardinals. But since $V_\kappa \in M_0$, the model must be correct about this assertion. Now we exactly follow the argument that Silver indiscernibles are β-iterable in L, with M_α in the place of L. Let $M_0 = \cup_{i \in \omega} M_0^{(i)}$ where the $M_0^{(i)}$ have the properties of the conclusion of Lemma 3.10 and let $M_\xi^{(i)} = j_{0\xi}(M_0^{(i)})$ for $\xi \leq \alpha$. Observe that for $\xi < \gamma \leq \beta$, the restrictions $j_{\xi\gamma}^{(i)} : M_\xi^{(i)} \to M_\gamma^{(i)}$ are all elements of M_β by remark 3.1 (2). Moreover, M_β is a set in M_α and hence the ordinals of M_β are bounded in M_α by some ordinal δ. This suffices to run same tree building argument. The bound δ is needed to insure that the tree is a set. Next, suppose, that α is a limit ordinal. Fix $\xi < \alpha$, then κ is $\xi + 1$-iterable. So by the inductive assumption, κ is a limit of ξ-iterable cardinals. □

It is not completely clear what is the exact relationship between the α-iterable cardinals and the α-Erdős cardinals for $\alpha \leq \omega_1$. Sharpe and Welch showed that:

Theorem 4.2. *An ω_1-Erdős is a limit of ω_1-iterable cardinals.*

It is also easy to see that:

Theorem 4.3. *If κ is a γ-Erdős cardinal for some $\gamma < \omega_1$, then there is a countable ordinal α and a real r such that $L_\alpha[r]$ is a model of* ZFC *having a class of cardinals that are δ-iterable for every $\delta < \gamma$.*

Proof. Suppose κ is a γ-Erdős cardinal, then there is a set $I = \{i_\xi \mid \xi \in \gamma\}$ of good indiscernibles for $L_\kappa[r]$ (see Section 6 for definition) where r codes the fact that γ is a countable ordinal. Let $L_\alpha[r]$ be the collapse of the Skolem closure of I in $L_\kappa[r]$, then it is the Skolem closure of some collection $K = \{k_\xi \mid \xi \in \gamma\}$ of indiscernibles. The indiscernibles k_ξ are unbounded in α. Since each $L_{i_\xi}[r] \prec L_\kappa[r]$, we have $L_\beta[r] \prec L_\kappa[r]$ where β is the sup of I. It follows that the Skolem closure of I is contained in $L_\beta[r]$ and hence the k_ξ are unbounded in $L_\alpha[r]$. By Lemma 3.9, for every indiscernible k_ξ, there is a good commuting system of elementary embeddings of length γ on L_α with the first embedding having critical point κ_ξ.

Thus, by Lemma 3.8, there is a γ-good ultrafilter for every $L_{(k_\xi^+)^{L_\alpha[r]}}$. Since α is countable, it is clear that $(k_\xi^+)^{L_\alpha[r]}$ has countable cofinality. Now we can use the argument in the proof of Theorem 3.11 to show that each k_ξ is δ-iterable in $L_\alpha[r]$ for $\delta < \gamma$. Notice that we cannot use these techniques to make the argument that k_ξ are γ-iterable since the tree of embedding approximations must be a set in $L_\alpha[r]$. □

Question 4.4. For $\alpha \leq \omega_1$, what is the relationship between the α-iterable cardinals hierarchy and the α-Erdős cardinals hierarchy?

5. Ramsey-like cardinals and downward absoluteness to to K

In this section, we show that the strongly Ramsey and super Ramsey cardinals defined in [Git09] are downward absolute to the core model K.

Definition 5.1. A cardinal κ is *strongly Ramsey* if every $A \subseteq \kappa$ is contained in a κ-model M for which there exists a κ-powerset preserving embedding $j : M \to N$.

Definition 5.2. A cardinal κ is *super Ramsey* if every $A \subseteq \kappa$ is contained in a κ-model $M \prec H_{\kappa^+}$ for which there exists a κ-powerset preserving embedding $j : M \to N$.

Strongly Ramsey cardinals are limits of *completely Ramsey* cardinals that top Feng's Π_α-Ramsey hierarchy [Fen90]. They are Ramsey, but not necessarily completely Ramsey. They were introduced with the motivation of using them for indestructibility arguments involving Ramsey cardinals. Such an application is made in Section 6. Super Ramsey cardinals are limits of strongly Ramsey cardinals and have the advantage that the embedding is on a κ-model is that is stationarily correct. Note that we can restate the definition of strongly Ramsey and super Ramsey cardinals in terms of the existence of weakly amenable M-ultrafilters. Since we require the embedding to be on a κ-model, such an ultrafilter is automatically countably complete and therefore has a well-founded ultrapower.

As a representative *core model* K here we take that constructed using extender sequences which are *non-overlapping* (see [Zem02]). In such a model a strong cardinal may exist but not a sharp for such. The argument does not depend on any particular fine structural considerations, simply the definability of K up to κ^+ in any H_{κ^+} with applications of the Weak Covering Lemma (*cf.* [Zem02]).

Proposition 5.3. *If κ is strongly Ramsey, then κ is strongly Ramsey in K.*

Proof. Let κ be strongly Ramsey and fix $A \subseteq \kappa$ in K. Choose a weak κ-model M containing A such that $M \models A \in K$ for which there exists a weakly amenable M-ultrafilter U on κ. To see that we can choose such M, note that $A \in P = \langle J_\alpha^{E^K}, \in, E^K \rangle$ for some $\alpha < \kappa^+$ where E^K is the extender sequence from which K is constructed. We may assume that a code for P is definable over V as a subset of κ. Hence we may assume that the M witnessing strong Ramseyness has this code, and so P, as an element. Note that with $P \in M$, K_κ is an initial segment of K^M. Finally observe that $V_\kappa \in M$ since M is a κ-model. Let $\overline{\kappa} = (\kappa^+)^{K_M}$ and set $\overline{K} = K^M_{\overline{\kappa}}$.

First, we argue that $U \cap \overline{K} \in K$.

Case (1) $\overline{\kappa} < (\kappa^+)^M$

We argue that $\mathrm{cf}^V(\overline{\kappa}) = \kappa$. Note that we may iterate the measure U on M throughout all the ordinals leaving behind a ZFC model $W = W_M = \bigcup_{\alpha\in Ord} H^{M_\alpha}_{\kappa_\alpha}$ where M_α is the α^{th}-iterate of M. Then $\overline{\kappa} = (\kappa^+)^{K_W}$ (by weak amenability of U). By the Weak Covering Lemma applied inside W_M the cofinality of $\overline{\kappa}$ is greater than or equal to κ. This suffices to see the same is true in V.

However the assumption that $\overline{\kappa} < (\kappa^+)^M$ implies that $\mathcal{P}(\kappa)\cap\overline{K}$ has size κ in M, and so by weak amenability we have that $U\cap\overline{K}\in M$. It follows that $U\cap\overline{K}\in W_M$ and in W_M we can form the well-founded ultrapower $\mathrm{Ult}(K_W, U\cap\overline{K})$. However this would imply that we had a non-trivial embedding of K_W to itself. The only way this can happen in W_M is for $U\cap\overline{K}$ to be a measure on the K-extender sequence defined in W_M, namely E^{K_M}. We conclude that $U\cap\overline{K}\in K_M$ and so $U\cap\overline{K}\in K$.

Case (2) $\overline{\kappa} = (\kappa^+)^M$

As above $U\cap\overline{K}$ is a countably complete ultrafilter, which is weakly amenable to $\overline{K}$. Further by a standard comparison argument we have that $\overline{K} = \langle J^{E^{K_M}}_{\overline{\kappa}}, \in, E^{K_M}\rangle$ is an initial segment of K. However then $\langle\overline{K}, U\cap\overline{K}\rangle$ is a mouse and again a standard argument shows that $U\cap\overline{K}$ is a filter on the the K-sequence E^K.

This completes the first part of the argument. It remains to show that $\overline{K}^{<\kappa}\subseteq\overline{K}$ in K. Fix $\alpha<\kappa$ and $f:\alpha\to\overline{K}$ in K. Without loss of generality we shall assume that f is 1-1. Since M is a κ-model, we have $f\in M$. Suppose $f(\gamma)\in\overline{K}_{\delta(\gamma)}$ for some $\delta(\gamma)<\overline{\kappa}$. Let $\delta = \sup_{\gamma<\alpha}\delta(\gamma)$. Then $\delta<\overline{\kappa}$. Let $G\in\overline{K}$ be such that $G:\kappa\to\overline{K}_\delta$ is a bijection. Then if $C = G^{-1}\text{``}\mathrm{ran}(f)$, then C is a bounded subset of κ with $C\in K$. However $C\in V_\kappa\to C\in K^M$. As both f, G are (1-1) there is a permutation $\pi : C\longrightarrow C$ with $f = G\circ\pi$. Again $\pi\in V_\kappa\cap K\cap M$. Hence $f\in\overline{K}$.

Thus, the embedding $k:\overline{K}\to\mathrm{Ult}(\overline{K},\, U\cap\bar{K})$ witnesses the strong Ramseyness of κ in K. □

Proposition 5.4. *If κ is super Ramsey, then κ is super Ramsey in K.*

Proof. Note that if $M\prec H_{\kappa^+}$ then $K^M\prec(K)^{H_{\kappa^+}} = K_{\kappa^+}$. By the Weak Covering Lemma, $\kappa^+ = (\kappa^+)^K$ since κ is weakly compact, and Weak Covering implies that successors of such are correctly computed by K. As above if U is a weakly amenable M-ultrafilter, then this restricts to be weakly amenable to $\overline{K} = K^M$, with $U\cap\overline{K}\in K$. This suffices. □

6. Virtually Ramsey cardinals

In [WS08], Sharpe and Welch defined a new large cardinal notion, the *virtually Ramsey* cardinals. Virtually Ramsey cardinals are defined by an apparently weaker statement about the existence of good indiscernibles than Ramsey cardinals. The definition was motivated by the conditions needed to get an upper bound on the consistency strength of the Intermediate Chang's Conjecture. In this section, we separate the notions of Ramsey and virtually Ramsey cardinals using an old forcing argument of Kunen's showing how to destroy and then resurrect a weakly compact cardinal [Kun78].

Definition 6.1. Suppose κ is a cardinal and $A\subseteq\kappa$. Then $I\subseteq\kappa$ is a set of *good indiscernibles* for $\langle L_\kappa[A], A\rangle$ if for all $\gamma\in I$:

1. $\langle L_\gamma[A], A\rangle\prec\langle L_\kappa[A], A\rangle$.
2. $I-\gamma$ is a set of indiscernibles for $\langle L_\kappa[A], A, \xi\rangle_{\xi\in\gamma}$.

Theorem 6.2. *A cardinal κ is Ramsey if and only if for every $A \subseteq \kappa$, the structure $\langle L_\kappa[A], A\rangle$ has a collection of good indiscernibles of size κ.*

See [Dod82] for details on good indiscernibles and proof of above theorem. In general, for $A \subseteq \kappa$, let $\mathscr{I}_A = \{\alpha \in \kappa \mid$ there is an unbounded collection $I_\alpha \subseteq \alpha$ of good indiscernibles for $\langle L_\kappa[A], A\rangle\}$.

Definition 6.3. A cardinal κ is *virtually Ramsey* if for every $A \subseteq \kappa$, the set $\mathscr{I}_A$ contains a club.

There is no obvious reason to suppose that the good indiscernibles below each of the ordinals in $\mathscr{I}_A$ can be glued together into a set of good indiscernibles of size κ, suggesting that virtually Ramsey cardinals are not necessarily Ramsey. First, we make some easy observations about virtually Ramsey cardinals.

Proposition 6.4. *Ramsey cardinals are virtually Ramsey.*

Proof. Suppose κ is a Ramsey cardinal. Fix $A \subseteq \kappa$, then there is a set I of size κ of good indiscernibles for the structure $\langle L_\kappa[A], A\rangle$. Clearly the club of all limit points of I is contained in $\mathscr{I}_A$. This verifies that κ is virtually Ramsey. □

The next proposition confirms that virtually Ramsey cardinals is a large cardinal notion.

Proposition 6.5. *Virtually Ramsey cardinals are Mahlo.*

Proof. Suppose κ is virtually Ramsey. First we show that κ is inaccessible. If κ is not regular, then there is a cofinal $f : \alpha \to \kappa$. Consider $L_\kappa[f]$ and let $C \subseteq \mathscr{I}_f$ be a club. Choose any $\beta_0 \in C$ above α and choose ξ_0 such that $f(\xi_0) > \beta_0$. Next, choose $\beta_1 \in C$ above $f(\xi_0)$ and ξ_1 such that $f(\xi_1) > \beta_1$. In this way, define a sequence $\{\beta_n : n \in \omega\}$ of elements of C. Let β be the limit of the sequence, then $\beta \in C$. Note that the image of f is unbounded in β. But this violates indiscernibility since every $f(\xi)$ should be either below the least indiscernible in β or above β. Thus, κ is regular. If κ is not a strong limit, then there is $\alpha < \kappa$ such that $2^\alpha \geq \kappa$. Let $\vec{A} = \langle A_\xi : \xi < \kappa\rangle$ be a sequence of distinct subsets of α and consider $L_\kappa[\vec{A}]$. Let $\beta > \alpha$ such that there is an unbounded $I_\beta \subseteq \beta$ of good indiscernibles. Choose any $\gamma_0 \neq \gamma_1$ in I_β above α. Then A_{γ_0} and A_{γ_1} are distinct subsets of α. Without loss of generality, assume there is ξ such that $\xi \in A_{\gamma_0}$ and $\xi \notin A_{\gamma_1}$. But since γ_0 and γ_1 are above ξ, this contradicts indiscerniblity. Thus, κ is a strong limit. To show that κ is Mahlo, we need to argue that every club in κ contains a regular cardinal. Fix a club C in κ. Let $\vec{f} = \langle f_\xi \mid \xi < \kappa\rangle$ where $f_\xi : \alpha \to \xi$ witnesses that $\mathrm{cf}(\xi) = \alpha$. We will work with the structure $L_\kappa[\vec{f}, C]$. Suppose i is a good indiscernible for $L_\kappa[\vec{f}, C]$, then $L_i[\vec{f}, C] \prec L_\kappa[\vec{f}, C]$, and hence C is unbounded in i. It follows that all good indiscernibles are elements of C. So it will suffice to see that good indiscernibles are regular cardinals. Let $\xi \in \mathscr{I}_{\vec{f}, C}$ and X be the Skolem closure of I_ξ in $L_\xi[\vec{f}, C] \prec L_\kappa[\vec{f}, C]$. Let M be the transitive collapse of X, then M is the Skolem closure of indiscernibles I'_ξ corresponding to I_ξ via the collapse. If $i \in I_\xi$ is a good indiscernible and i' its under the collapse, define an elementary embedding $j : M \to N$ with critical point i'. Observe that i' is regular in M since if $f : \alpha \to i'$ is cofinal for $\alpha < i'$, then so is $j(f) : \alpha \to j(i')$, which is impossible. Since the collapse is an isomorphism, it follows that i is regular in X, and hence in

$L_\kappa[\vec{f}, C]$. But by our choice of $\vec{f}$, $L_\kappa[\vec{f}, C]$ is correct about this. This completes the argument that κ is Mahlo. □

Next, we give a sufficient condition needed to glue the good indiscernibles below the ordinals in $\mathscr{I}_A$ into a set of good indiscernibles of size κ.

Proposition 6.6. *If a cardinal is virtually Ramsey and weakly compact, then it is Ramsey.*

Proof. Suppose κ is virtually Ramsey and weakly compact. We will argue that we can glue together the good indiscernibles coming from the different ordinals of the club contained in $\mathscr{I}_A$ into a set of good indiscernibles of size κ. Fix $A \subseteq \kappa$ and let C be a club contained in $\mathscr{I}_A$. Fix any weak κ-model M containing A and C as elements. By weak compactness, there exists an embedding $j : M \to N$ with critical point κ. Observe that $M \models C \subseteq \mathscr{I}_A^M$, where $\mathscr{I}_A^M$ is the set $\mathscr{I}_A$ defined from the perspective of M. By elementarity $N \models j(C) \subseteq \mathscr{I}_{j(A)}^N$. Since $\kappa \in j(C)$, it follows that κ contains unboundedly many good indiscernibles for $\langle L_{j(\kappa)}[j(A)], j(A)\rangle$. But since $j(A) \cap \kappa = A$, it is easy to see that these are good indiscernibles for $\langle L_\kappa[A], A\rangle$ as well. This completes the proof that κ is Ramsey. □

Our strategy to separate virtually Ramsey and Ramsey cardinals will be to start with a Ramsey cardinal and force to destroy its weak compactness while preserving virtual Ramseyness. Although ideally we would like to start with a Ramsey cardinal, we will have to start with a strongly Ramsey cardinal instead. The reason being that strongly Ramsey cardinals have embeddings on sets with $< \kappa$-closure that is required for indestructibility techniques. The argument below was worked out jointly with Joel David Hamkins and we would like to thank him for his contribution.

The forcing we use is Kunen's well-known forcing from [Kun78] to destroy and then resurrect weak compactness. The next lemma is a key observation in the argument.

Lemma 6.7. *If $\mathbb{P}$ is a $< \kappa$-distributive, stationary preserving forcing, $G \subseteq \mathbb{P}$ is V-generic and κ is Ramsey in $V[G]$, then κ is virtually Ramsey in V.*

Proof. Fix $A \subseteq \kappa$. Since $\mathbb{P}$ is $< \kappa$-distributive, it cannot add any new sets of good indiscernibles to ordinals $\alpha < \kappa$. It follows that $\mathscr{I}_A = \mathscr{I}_A^{V[G]}$. If $\mathscr{I}_A$ does not contain a club in V, then the complement $\overline{\mathscr{I}}_A$ is stationary in V. Since $\mathbb{P}$ is stationary preserving, $\overline{\mathscr{I}}_A$ remains stationary in $V[G]$. This is clearly a contradiction since κ is Ramsey in $V[G]$ and hence $\mathscr{I}_A$ contains a club. □

First, we define a forcing $\mathbb{Q}$ to add a Souslin tree T together with a group of automorphisms $\mathscr{G}$ that acts *transitively* on T. A group of automorphisms $\mathscr{G}$ of a tree T is said to act transitively if for every a and b on the same level of T, there is $\pi \in \mathscr{G}$ with $\pi(a) = b$. The elements of $\mathbb{Q}$ will be pairs (t, f) where t is a *normal* $\alpha + 1$-tree for some $\alpha < \kappa$ such that $Aut(t)$ acts transitively and $f : \lambda \underset{\text{onto}}{\overset{1-1}{\to}} Aut(t)$ is some enumeration of $Aut(t)$. We have $(t_1, f_1) \leq (t_0, f_0)$ when

(1) t_1 end-extends t_0,
(2) for all $\xi \in Dom(f_0)$, $f_1(\xi)$ extends $f_0(\xi)$.

To show that the generic κ-tree T added by $\mathbb{Q}$ is Souslin, we need to argue that every maximal antichain of T is bounded. In the usual forcing to add a Souslin

tree, the conditions are normal $\alpha+1$-trees and the argument is made by proving the Sealing Lemma. The Sealing Lemma states that if a condition forces that $\dot{A}$ is a name for a maximal antichain, then there is a stronger condition forcing that it is bounded. The argument for the Sealing Lemma goes as follows:
Suppose $t_0 \Vdash \dot{A}$ is a maximal antichain of $\dot{T}$. Choose $t_1 \leq t_0$ such that for every $s \in t_0$, there is $a_s \in t_1$ compatible with s and $t_1 \Vdash a_s \in \dot{A}$. Build a sequence $\cdots \leq t_n \leq \cdots \leq t_1 \leq t_0$ such that for every $s \in t_n$, there is $a_s \in t_{n+1}$ compatible with s and $t_{n+1} \Vdash a_s \in \dot{A}$. Let t be the union of t_n and build the top level of t by adding a branch through every pair s and a_s. Since every new branch passes through an element of $\dot{A}$, this *seals* the antichain. We will carry out a similar argument with the forcing $\mathbb{Q}$, but it will be complicated by the fact that whenever we add a node on top of a branch B, we need to add nodes on top of branches $f(\xi)``B$. While $B$ passes through an element of $\dot{A}$, there is no reason why $f(\xi)``B$ should. In fact, since the automorphism groups act transitively, it will suffice to add a single carefully chosen branch to the limit tree of the conditions and take the limit level to be all the images of the branch under the automorphism group on the second coordinate. Thus, we need to build our sequence of conditions such that the limit of the trees on the sequence has a branch all of whose images under the automorphism group go through elements of the antichain.

Lemma 6.8 (Sealing Lemma)**.** *Suppose p is a condition in $\mathbb{Q}$, $\dot{T}$ is the canonical $\mathbb{Q}$-name for the generic κ-tree added by $\mathbb{Q}$, and $p \Vdash \dot{A}$ is a maximal antichain of $\dot{T}$. Then there is $q \leq p$ forcing that $\dot{A}$ is bounded.*

Proof. Fix $p \Vdash \dot{A}$ is a maximal antichain of $\dot{T}$. Let $p = (t_0, f_0)$ with t_0 of height $\alpha+1$ and $f : \lambda_0 \underset{\text{onto}}{\overset{1-1}{\to}} Aut(t_0)$. Choose some $M \prec H_{\kappa^+}$ of size $< \kappa$ containing $\mathbb{Q}$, p, and $\dot{A}$ with the additional property that $Ord^M \cap \kappa = \beta$ is an initial segment of κ. We will work inside M to build a condition (t, f) with t of height $\beta+1$ strengthening (t_0, f_0) and sealing $\dot{A}$. We will need a bookkeeping function $\varphi : \kappa \underset{\text{onto}}{\to} \kappa$ with the property that every ξ appears in the range cofinally often. Notice that by elementarity, M contains some such function φ. Working *entirely inside* M, we carry out the following construction for κ many steps. By going to a stronger condition, we can assume without loss of generality that there is $a \in t_0$ such that $(t_0, f_0) \Vdash a \in \dot{A}$. Let B_0 be any branch through a in t_0. Let a_0 be the top node of B_0. The node a_0 begins the branch we are trying to construct. Let (t_1, f_1) be a condition in $\mathbb{Q}$ strengthening (t_0, f_0) and having the property that for every $s \in t_0$, there is $a_s \in t_1$ compatible with s such that $(t_1, f_1) \Vdash a_s \in \dot{A}$. Consult the bookkeeping function $\varphi(1) = \gamma$. This will determine how a_0 gets extended. If $\gamma \geq \lambda_0$, let a_1 be the node on the top level of t_1 extending a_0. Otherwise, consider $f_0(\gamma)$ and $f_0(\gamma)(a_0) = s$. Let s' be on the top level of t_1 above s and a_s. Finally, let $f_1(\gamma)^{-1}(s') = a_1$. This has the effect that no matter how we extend $f_0(\gamma)$, the image under it of the branch we are building will pass through $\dot{A}$. At successor stages $\sigma+1$, we will extend the condition (t_σ, f_σ) to a condition $(t_{\sigma+1}, f_{\sigma+1})$ having the property that for every $s \in t_\sigma$, there is $a_s \in t_{\sigma+1}$ compatible with s such that $(t_{\sigma+1}, f_{\sigma+1}) \Vdash a_s \in \dot{A}$. Next, we will consult the bookkeeping function $\varphi(\sigma) = \gamma$ and let it decide as above how a_σ gets chosen. At limit stages λ, we will let t_λ be the union of t_ξ for $\xi < \lambda$ and f_λ be the coordinate-wise union of f_ξ. Now use the branch through a_ξ to define a limit level for t_λ, thereby extending to $(t_{\lambda+1}, f_{\lambda+1})$. From the perspective of M, we are carrying out this construction for κ many steps, but really we are only carrying it out for β many steps. In V, we build (t, f) by unioning the sequence and adding a limit level using the branch of the a_ξ. It should be clear that (t, f) forces that $\dot{A}$ is bounded. □

Corollary 6.9. *The generic κ-tree added by $\mathbb{Q}$ is Souslin.*

In the generic extension by $\mathbb{Q}$, the Souslin tree T it adds can be viewed as a poset. Next, we will argue that forcing with $\mathbb{Q} * \dot{T}$ is forcing equivalent $Add(\kappa, 1)$, where $Add(\kappa, 1)$ is the forcing to add a Cohen subset to κ. Since every $< \kappa$-closed poset of size κ is forcing equivalent to $Add(\kappa, 1)$, it suffices argue that $\mathbb{Q} * \dot{T}$ has a dense subset that is $< \kappa$-closed.

Lemma 6.10. *The forcing $\mathbb{Q} * \dot{T}$ has a dense subset that is $< \kappa$-closed.*

Proof. Conditions in $\mathbb{Q} * \dot{T}$ are triples $(t, f, \dot{a})$ where t is an $\alpha + 1$-tree, f is an enumeration of the automorphism group of t, and $\dot{a}$ is a name for an element of $\dot{T}$. We will argue that conditions of the form (t, f, a) where a is on the top level of t form a dense $< \kappa$-closed subset of $\mathbb{Q} * \dot{T}$. Start with any condition $(t_0, f_0, \dot{b}_0)$ and strengthen (t_0, f_0) to a condition (t_1, f_1) deciding that $\dot{b}$ is $b \in t_1$. Now we have $(t_0, f_0, \dot{b}) \geq (t_1, f_1, b) \geq (t_1, f_1, a)$ where a is above b on the top level of t_1. Thus the subset is dense. Suppose $\gamma < \kappa$ and we have a descending γ-sequence $(t_0, f_0, a_0) \geq (t_1, f_1, a_1) \geq \ldots \geq (t_\xi, f_\xi, a_\xi) \geq \ldots$. To find a condition that is above the sequence, we take unions of the first two coordinates and make the limit level of the tree in the first coordinate consist of images of the branch through $\langle a_\xi \mid \xi < \gamma \rangle$ under the automorphisms in the second coordinate. □

Let $\mathbb{P}_\kappa$ be the Easton support iteration which adds a Cohen subset to every inaccessible cardinal below κ. We will force with the iteration $\mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{T}$. This is equivalent to forcing with $\mathbb{P}_\kappa * Add(\kappa, 1)$. The forcing argument will rely crucially on the following standard theorem about preservation of strong Ramsey cardinals after forcing.

Theorem 6.11. *If κ is strongly Ramsey in V, then it remains strongly Ramsey after forcing with $\mathbb{P}_\kappa * Add(\kappa, 1)$.*

The proof uses standard techniques for lifting embeddings and will appear in [GJ09].

Now we have all the machinery necessary to produce a model where κ is virtually Ramsey but not weakly compact.

Theorem 6.12. *If κ is a strongly Ramsey cardinal, then there is a forcing extension, in which κ is virtually Ramsey, but not weakly compact.*

Proof. Let κ be a strongly Ramsey cardinal and $G * T * B \subseteq \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{T}$ be V-generic. Since $\mathbb{Q}$ adds a Souslin tree, κ is not weakly compact in the intermediate extension $V[G][T]$. But by Theorem 6.11, the strong Ramseyness of κ is resurrected in $V[G][T][B]$. Recall that forcing with a Souslin tree is $< \kappa$-distributive. The Souslin tree forcing is also κ-cc and hence stationary preserving. So by Lemma 6.7, we conclude that κ remains virtually Ramsey in $V[G][T]$. Thus, in $V[G][T]$, the cardinal κ is virtually Ramsey but not weakly compact, and hence not Ramsey. □

We showed starting from a strongly Ramsey cardinal that it is possible to have virtually Ramsey cardinals that are not Ramsey, thus separating the two notions. The following is still an open question.

Question 6.13. Are virtually Ramsey cardinals strictly weaker than Ramsey cardinals?

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