Algorithmic Theories of Truth and Turing Machines with an Infinite Amount of Time

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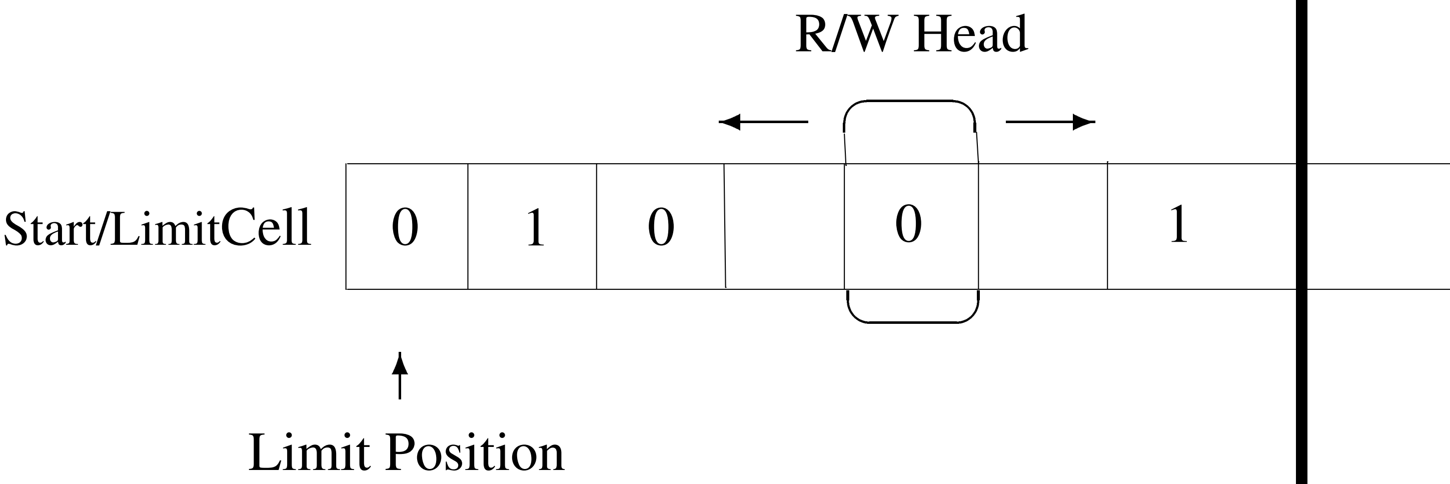
Infinite Time Turing Machines

Hardware: just as for ordinary Turing Machines an infinite tape (to the right). One tape position is designated in advance, as the "Limit Position".



Software: just as for ordinary Turing Machines, but with the addition of an extra "limit state" q_L.

Let $\langle P_n \mid n < \omega \rangle$ enumerate all programs.

Action: *At times* $\nu = \beta + 1$: obey the usual T.M. rules of the program P_n.

At limit times ν: R/W Head returns to the Limit Position;
Cell values $C_i(\nu) \simeq$ eventual value as $\tau \to \nu$ (if such exists): otherwise a B.

- $\varphi_n(x)(x \in {}^{\omega}\{0, 1, B\})$ denotes the result of running P_n on input string x.

- $\forall n, x \exists \alpha < \omega_1$ so that the behaviour of $\varphi_n(x)$ has "settled" by stage α: either it has entered a permanent looping cycle or it has halted: $\varphi_n(x){\downarrow}y$ for some $y \in {}^{\omega}3$.

• Decidable properties: Arithmetic; "$x \in WO$"; (thus) any Π^1_1 predicate.

Note: "$\varphi_p(x){\downarrow}y$ (in α steps)", "$\varphi_p(x){\uparrow}$" are both Δ^1_2 properties (in a code for α).

Kripkean Theories of Truth

An operator j_{vF} or j_K that assigns to each sound pair $A = (A^+, A^-)$ an extension/anti-extension of $\dot{T}$ over $\langle \mathbb{N}, A \rangle$ a superset pair $j(A) = B = (B^+, B^-) \supseteq A$

• These are monotone inductive definitions and hence result in fixed points.

• j_{vF} is a "supervaluation" operator

• j_{vF} is an operator derived from Kleene's 3 valued logics.

Pro: Elegance, simplicity, metatheoretically undemanding.

Contra: For j_K: $\vdash \varphi \not\Rightarrow \ulcorner \varphi \urcorner \in A$ for a fixed point A. j_K, j_{vF} both allow $T(\ulcorner \varphi \urcorner) \vee \neg T(\ulcorner \varphi \urcorner)$ to hold at A, without either of $T(\ulcorner \varphi \urcorner) \in A^+$ or $T(\ulcorner \varphi \urcorner) \in A^-$

Circular definitions

Let $\mathcal{L}$ be a first order language, and let $\mathcal{L}^+$ be its extension by a possibly infinite set of new predicate symbols $\dot{G}_n(x_1, \ldots x_n)$. For each $\dot{G}$ there is a definition from the set of definitions $\mathcal{D}$ of the form

$$(1) \quad \dot{G}_n(x_1, \ldots x_n) =_{df} A_G(x_1, \ldots x_n).$$

If we specialise $\mathcal{M}$ to $\mathbb{N}$, and have a single definition arising from a fixed first order formula $\varphi(v_0, \dot{X})$, we set:

$$X_{a+1} = \delta_\varphi(X_\alpha) =_{df} \{n | \langle \mathbb{N}, \ldots, X_\alpha \rangle \models \varphi(n, \dot{X})\}$$

For the Tarskian operator δ_τ in the language $\mathcal{L}_{\dot{T}}$, we would write:

$$X_{a+1} = \delta_\varphi(X_\alpha) =_{df} \{\ulcorner \sigma \urcorner | \langle \mathbb{N}, \ldots, X_\alpha \rangle \models \sigma[\dot{T}])\}$$

Coherency for Bootstrapping Policies

Such policies should have the *Coherency* features that

(i) if for all sufficiently large ordinals $\alpha < \lambda$ x falls into (falls out of) the definiendum G at stage α (we say that it is "stably t (f) at the limit") then it should fall in (out resp.) at stage λ.

(ii) Γ then assigns truth values to the other $G(x)$ not covered by (i) in some other way.

We can then view Γ as acting on sequences of limit ordinal length

Revision Sequences

Definition 1 *A* revision sequence based on δ, Γ *is then* $\vec{s} = \langle X_\alpha \mid \alpha < \infty \rangle$ *where at each stage the operators* δ *or* Γ *have been applied.*

- For successor $\alpha = \gamma + 1 : X_\alpha = \delta(X_\gamma)$;
- For limit $\lambda : \Gamma(\langle X_\alpha \mid \alpha < \lambda \rangle) = X_\lambda$.

Possible bootstrapping policies:
Herzberger Rule Γ_0: $X_\lambda(G(x))$ is always f unless forced to be otherwise. A "$\liminf$" rule.
Gupta Rule Γ_G: for those values not determined by stability, we set $X_\lambda(G(x)) = X_0(G(x))$.
Belnap Rule Γ_∞: any X_λ is allowed consistent with the Coherency requirement.

The Gupta-Belnap Theory of Truth

The Tarskian "revision operator" $\tau = \delta_{\tau,\mathcal{M}}$,

Definition 2 *(Belnap-Gupta) The* categorical truth set *using S_Γ, over arithmetic $V_\mathbb{N}$ are those sentences of $\mathcal{L}_{\dot{T}}$ that are* stably true *in all revision sequences using δ_τ and Γ: $V_\mathbb{N} =$*

$$\bigcap\{\, s^+_{<\infty} \mid \vec{s} = \langle X_\alpha \mid \alpha < \infty\rangle \text{ is a revision sequence based on } \delta_\tau, \Gamma \,\}.$$

where

$$s^+_{<\infty} =_{df} \{k = \ulcorner\sigma\urcorner \mid \exists\alpha\forall\beta > \alpha \quad k \in X_\beta\}$$

Pro: Solves quite a few of the "difficulties" mentioned for the Kripkean theory.

Contra: Complicated; metatheoretically demanding.

Kripkean theory of truth $\longrightarrow$ Monotone inductive definition $\longrightarrow$

an example of

Analysis of Π^1_1 & positive elementary definitions $\longleftrightarrow$ The next admissible set

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Gupta-Belnap theory of truth $\longrightarrow$ Revision Theoretic definability scheme $\longrightarrow$

an example of

? $\longleftrightarrow$??

Validity in S_Γ

Definition 3 *Let $\mathcal{L}$ be any first order language, and $\mathcal{L}^+ \supseteq \mathcal{L}$ contain predicate letters for new definienda $G \in \mathcal{D}$. Let $\Gamma = \Gamma_\infty$.*

(i) A is valid on $\mathcal{D}$ in M in the system S_Γ *($M \models_{\mathcal{D}} A$) iff for all initial hypotheses h, and all revision sequences $\vec{s}$ with $X_0 = h$, then A is stably true in ∞. That is for all sufficiently large α $\langle M, X_\alpha \rangle \models A$.*

(ii) A is valid on $\mathcal{D}$ in S_Γ ($\models^{\mathcal{D}}_{\Gamma_\infty}$) iff for all models M of $\mathcal{L}$, $M \models_{\mathcal{D}} A$.

Definability in S_Γ

Definition 4 *Let $\mathcal{L}$, $\mathcal{L}^+ \supseteq \mathcal{L}$, $\Gamma = \Gamma_\infty$ be as above.*

(i) A formula $A(v_0)$ of $\mathcal{L}^+$ (weakly) defines *a set $Y \subseteq |M|$ in S_Γ if: $x \in Y$ iff for all initial hypotheses h, and all revision sequences $\vec{s}$ with $X_0 = h$, then $A(x)$ is stably true in ∞. That is for all sufficiently large α $\langle M, X_\alpha \rangle \models A$.*

(ii) If additionally $x \notin Y$ iff .... stably false ..... then we say A strongly defines Y.

Theorem 1 *The stable truth set* $V_{\mathbb{N}}$ *for arithmetic is definable over* $\mathbb{N}$ *(*via *any limit rule* Γ *that is* $\Delta_1(HC, \in)$ *definable.) The sets of integers so definable are the* Π^1_2 *sets.*

Corollary 2 *The strongly definable reals are precisely the* Δ^1_2 *reals.*

Let $\mathcal{M} = \langle |\mathcal{M}|, R_1, \ldots \rangle$ be a very weakly acceptable countable structure (*i.e.* has a RT-strongly definable pairing function).

Definition 5 *(i) Let* $\mathbb{S}_{\mathcal{M}} = L_{\sigma_{\mathcal{M}}}(\mathcal{M})$ *be the least level of the Gödel hierarchy built over* $\mathcal{M}$ *(with elements of* $|\mathcal{M}|$ *as* urelemente*); with* $\sigma_{\mathcal{M}}$ *the least* $\mathcal{M}$-stable ordinal*; where:*
(ii) $\sigma_{\mathcal{M}}$ *is the least* σ *so that* $L_\sigma(\mathcal{M})$ *is a* Σ_1 *substructure of the universe.*

$\mathbb{S}_{\mathcal{M}}$ is thus the "next stable" set over $\mathcal{M}$.

Theorem 3 *The strongly definable subsets of a countable v. weakly acceptable $\mathcal{M}$ are those of the first stable set over $\mathcal{M}$, that is $\mathbb{S}_{\mathcal{M}}$.*

Paraphrasing by Levy-Schoenfield, we have the characterisation:

Corollary 4 *Let $\mathcal{M}$ be as above. The RT-definable subsets of $|\mathcal{M}|$ are those $\Pi^1_2(\mathcal{M})$, and the strongly definable are those $\Delta^1_2(\mathcal{M})$.*

Fully Varied Theory of truth

Definition 6 *A revison sequence* $\vec{s} = \langle X_\alpha \mid \alpha < \infty \rangle$ *is* fully varied (fv) *if for all functions* X_∞ *that could be coherently applied at* ∞*, there are cofinally many* $\lambda < \infty$ *when* $X_\lambda = X_\infty$.

One may show:

Theorem 5 *Let* $V^+ = V^+_{\mathbb{N}}$ *be the stable truth set over arithmetic in this revision scheme. Then* V^+ *is (i) RT-definable over* $\mathbb{N}$*; and is (ii) complete* Π^1_3.

Pro: Correctly classifies sets of sentences of the Gupta-Puzzle type; ensures certain RT-definitions converge to their ostensible values (McGee).

Contra: Membership questions about the categorical truth set of arithmetic are now independent of ZFC

Let

$$T = \{e \in \mathbb{N} \mid \forall x \in 3^\omega \exists y \in 3^\omega \ \varphi_e(x) \downarrow y\}$$

- T is complete Π^1_2.

Theorem 6 *Let the universal machine $\varphi_U(0)$ be started on a zero tape: the ordinal ζ at which the machine starts to loop and the ordinal Σ of its periodicity is precisely those ordinals associated by Herzberger to his revision sequence $\vec{h} = \langle h_\alpha \mid \alpha < \infty \rangle$ starting with an empty hypothesis $h_0 = \emptyset$, and using the Herzberger limit rule Γ_0.*

Realistic and Algorithmic Theories

Definition 7 *A revision sequence $\vec{s}$ is* realistically varied *if for all limit $\lambda < \infty$ s_λ is chosen as a coherent extension of s in the following fashion: let $s = (s^+_{<\lambda}, s^-_{<\lambda})$ be the local pair of stabilities at λ.*
(i) Either s_λ is recursive in s or in some s_α for an $\alpha < \lambda$, and s_λ has not been used as a bootstrapping policy cofinally in λ;
(ii) Or, if at stage λ there is no s_λ that satisfies clause (i), then s_λ may be chosen arbitrarily.

Maxim: "Use the simple ones first"

Algorithmic variance

Let us suppose that in the theory of realistic variance the choices of limit extensions s_λ have been done in some fashion that shows that s_λ has been chosen in some reasonably uniform manner in λ from the preceding sequence. Let us say, being generous, that s_λ is $\Delta_1(\vec{s} \upharpoonright \lambda)$ definable uniformly in λ in some weak set theory, say KP, Kripke-Platek. (The Γ_G and Γ_H both conform to this, but we are adding to these the requirement of realistic variance.) Call such a sequence a *(generalised) algorithmically varied* sequence.

Maxim: “Use some algorithm to choose which limit rule you use”

Theorem 7 *Let* $X_\infty = (X^+_\infty, X^-_\infty)$ *arise from an algorithmically varied revision sequence. Then:*
(i) X_∞ *is a complete arithmetical quasi-inductive set.*
(ii) X_∞ *is a Kripkean fixed point using the supervaluation scheme.*

Challenge Problem Find a set of sentences B that is intuitively of a certain category under some starting hypotheses, but, for example, that is badly classified as "sometimes unstable", according to realistic or algorithmic variance.

Contra: difficult to describe.
Pro: Metatheoretically far simpler than G-B RT; sets high the hurdle of finding sets of sentences ill-treated by these schemes

Definition 8 *Let $\vec{s} = \langle s_\alpha | \alpha < \infty \rangle$ be an algorithmically varied revision sequence. Let the* stabilization ordinal*, $\sigma(\vec{s})$, be the least σ so that $\forall \alpha \geq \sigma \exists \beta > \alpha \; s_\alpha = s_\beta$.*

Theorem 8 *Under the hypotheses of the last theorem $\sigma(\vec{s})$ is (equivalently) at most:*
(i) The least ordinal ζ so that L_ζ has a transitive Σ_2 end extension;
(ii) The supremum of the infinite time turing machine "eventually writable" ordinals;
(iii) The starting point of the "Herzberger Grand Loop" based on initial hypothesis $h = 0$.
(iv) The closure ordinal of arithmetical quasi-inductive definitions.

Strong Theories of Formal Truth

• Let VF be Cantini's theory of formal truth corresponding to Kripke's theory of truth using j_{vF}.

• We may make an extension of VF by definitions, adding a predicate C for the "computably decided" sentences over arithmetic. Essentially C is interpreted to consist of those sentences which are "hereditarily algorithmically decided" - essentially these form an "inner model" of Kripke-Feferman.

One may then add further axioms stating that there are towers of sets of sentences in the class C which form an increasing chains of KF models.

• This also gives an axiom system to distinguish between supervaluation fixed points, and stable truth

The strength of the machines

There is the *reducibility relation* associated to reals or sets of reals: $x \leq_\infty y, A \leq_\infty B$:

$$x \leq_\infty y \longleftrightarrow \exists n \varphi_n(x){\downarrow}y$$

$$A \leq_\infty B \longleftrightarrow \exists n \forall x \varphi_n^B(x){\downarrow}1 \text{ if } x \in A, \ {\downarrow}0 \text{ if } x \notin A$$

• For $x \in 2^\omega$ let λ^x be the least ordinal that a machine can write on input x. Then:

(i) $x \leq_\infty y \longleftrightarrow x \in L_{\lambda^y}[y]$;
(ii) the assignment $x \to \lambda^x$ satisfies a Spector Criterion.

• *This lets us develop a degree theory on reals that is somewhere intermediate between Δ^1_1-degrees (or* hyperdegrees*) and Δ^1_2-degrees.*

$O^{\blacktriangledown} =_{df} \{(n,x) \in \omega \times 2^\omega \mid \varphi_n(x)\downarrow\}$ ("Strong Jump").

Def $\mathbf{\Gamma} =_{df}$
$\{A \subseteq 2^\omega \mid A$ is semi-decidable from a real $\} =$
$\{A \subseteq 2^\omega \mid \exists r \in 2^\omega \exists n \forall x \quad x \in A \longleftrightarrow \varphi_n^r(x)\downarrow 1\}$.
$\mathbf{\Delta} = \mathbf{\Gamma} \cap \neg\mathbf{\Gamma}$.

Fact (Hamkins-Lewis) $\mathbf{\Delta} \subseteq \mathbf{\Delta^1_2}$ is a σ-algebra, closed under Suslin's operation $\mathcal{A}$. $\mathbf{\Gamma}$ forms a Spector pointclass.

Let $A \leq_{r,\infty} B$ denote A is infinite time computable from a real and B.

It follows from the above that the sets of reals semi-decidable in $x \in 2^\omega$ are those A expressible as follows: let $\psi_0(v_0, v_1)$ be a Σ_1 formula, then let

$$y \in A \Longleftrightarrow L_{\lambda^{\langle x,y \rangle}}[x, y] \models \psi_0(x, y).$$

A is *semi-decidable in* B *(*$A, B \subseteq 2^\omega$*) and* $x \in 2^\omega$ can be seen to be given similarly as:

$$y \in A \Longleftrightarrow L_{\lambda^{\langle x,y,B \rangle}}[x, y, B] \models \psi_0(x, y, B).$$

Post's problem: are there
1) $A \subseteq 2^\omega$, $A \in \Gamma$, A not decidable but $0^\blacktriangledown \not\leq_{r,\infty} A$?
2) $A, B \in$ $A \not\leq_{r,\infty} B \wedge B \not\leq_{r,\infty} A$?

• In L, the constructible universe, there are A, B as in 2).

• $Det(\text{Boolean}(\Gamma)) \Rightarrow$ any $A \in \Gamma \backslash \Gamma \cap \neg\Gamma$ satisfies $0^\blacktriangledown \equiv_{r,\infty} A$.

• $Det(\text{Boolean}(\Gamma)) \Rightarrow$ there are inner models of large cardinals.

Conclusion: the pointclass of semi-/co-semidecidable sets sits high up within the Wadge hierarchy inside Δ^1_2.