Stable *versus* ultimate truths in Field's theory of truth[a]

Philip Welch

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[a] H. Field, *"A revenge-immune solution to the Semantic Paradoxes"*, J.Phil.Logic, April. 2003

Theories of Truth

(I) Field proposes an extension of the Kripkean theory of truth to allow a conditional operator $\longrightarrow$ between sentences in a language $\mathcal{L}_{Tr}$ extending, e.g. that of arithmetic augmented by a predicate symbol for truth $\dot{T}r$. Let this latter language with $\longrightarrow$ be $\mathcal{L}^{+}_{Tr}$

(II) As for Kripke, $\dot{T}r$ is partially interpreted, with semantic values $0, \frac{1}{2}, 1$

(III) The ultimate theory allows for the full inter-substitutivity of A with $Tr(\ulcorner A \urcorner)$. More particularly he gets (defining $\longleftrightarrow$ as $\leftarrow \wedge \rightarrow$) all instances of the Tarskian truth schema

$$\dot{T}r(A) \longleftrightarrow A \text{ for sentences } A \text{ of } \mathcal{L}^{+}_{Tr}.$$

The Construction over $\mathbb{N}$

At the beginning of any stage β, $|Tr(A)|_\beta$ is set to $= 1/2$ for any A of $\mathcal{L}^+_{Tr}$.

(1) Stage 0: Set $|A \to B|_0 = 1/2$ for all A, B of $\mathcal{L}^+_{Tr}$. Do the Kripkean construction using Strong Kleene with these assigned values to reach a fixed point. Call the resulting truth assignments $|A|^+_0$.

(2) Stage $\alpha + 1$: Do the Kripkean construction after assigning:

$$|A \longrightarrow B|_{\alpha+1} = \begin{cases} 1 & : \quad \text{iff } |A|^+_\alpha \leq |B|^+_\alpha \\ 0 & : \quad \text{iff } |A|^+_\alpha > |B|^+_\alpha \\ 1/2 & : \quad \text{otherwise.} \end{cases}$$

(3) Limit Stage λ: Do the Kripkean construction after assigning:

$$|A \longrightarrow B|_\lambda = \begin{cases} 1 & : \quad \text{iff } \exists\beta < \lambda \forall\alpha \in (\beta, \lambda) |A|^+_\alpha \leq |B|^+_\alpha \\ 0 & : \quad \text{iff } \exists\beta < \lambda \forall\alpha \in (\beta, \lambda) |A|^+_\alpha > |B|^+_\alpha \\ 1/2 & : \quad \text{otherwise} \end{cases}$$

Ultimate Truths

For any A of $\mathcal{L}^{+}_{Tr}$: $||A|| =_{df} \begin{cases} 1 & : \quad \text{iff } \exists\beta\forall\alpha > \beta |A|^{+}_{\alpha} = 1 \\ 0 & : \quad \text{iff } \exists\beta\forall\alpha > \beta |A|^{+}_{\alpha} = 0 \\ 1/2 & : \quad \text{otherwise} \end{cases}$

$U =_{df} \{A : ||A|| = 1\}$.

Field proves the existence of *acceptable* ordinals, *i.e.* , an ordinal Δ so that $\forall A \in \mathcal{L}^{+}_{Tr} \quad |A|_{\Delta} = ||A||$.

Let Δ denote the least such.

Field further:

• defines a notion of validity for $\longrightarrow$, $\models_{LCC}$ and partially axiomatises this to give a deductive system $\vdash_{LCC}$.

• proves a result on the substitutivity of truth:

Lemma: For any sentences A, B of $\mathcal{L}^{+}_{Tr}$, if B^* results fromB by replacing one or more occurrences of A by $Tr(A)$ then $||B|| = ||B^*||$.

• claims that his notion of "truth-in-a-model" is *revenge-immune*; that is, no "Strengthened Liar Paradox" (or *hyper-paradox*) is possible with this system.

The “revenge immunity” claim

• Field asserts that his notion of “truth-in-a-model” is *revenge-immune*; that is, no “Strengthened Liar Paradox” (or *hyper-paradox*) is possible with this system.

Determinately true operators

Field defines:

- DA abbreviates $(\top \longrightarrow A) \wedge A$ " A is *determinately true*".

Note $||DA|| = 1 \Longleftrightarrow ||A|| = 1$. For a (usual) Liar sentence ("$L_0 \Leftrightarrow \neg Tr(L_0)$") we have that although $||L_0|| = 1/2$, we have that $||\neg DL_0|| = 1$. Etc etc.

Further:

- $D^{n+1}A =_{df} D(D^n A); D^\omega A, \ldots, D^\gamma A \quad (\gamma < \lambda_0)$,
for some (unspecified) $\lambda_0 < \omega_1^{ck}$.

- *Lemma* $\lambda_0 < \Delta$.

Analysis of Field's system

Theorem 1 Stable *v.* Ultimate Truth *U is recursively isomorphic to*
(i) the set of stable truths in Herzberger's revision theory of truth, starting with a null hypothesis;
(ii) the complete Σ_2-Theory(L_Δ).

Sketch of b): $\underline{U \leq_1 \widetilde{O}} \equiv \Sigma_2$-Th$(L_\Delta)$.
Run Field's construction inside L_Δ:
"$A \in U$" $\Longleftrightarrow$ $(L_\Delta \models$ "$\exists\alpha\forall\beta > \alpha.\beta...A....$")

$\underline{U \geq_1 \widetilde{O}}$.We use the following result:

Fact (Kripke) *Let $C \subseteq \mathbb{N}$. The least fixed point, C^+, using Strong Kleene 3-valued logic over $\langle \mathbb{N}, +, \times, \ldots, C\rangle$ is $\Pi_1^{1,C}$-complete.*

Hence $C^+ \equiv_1$ the complete Σ_1-Th$(\langle L_{\omega^C_{1ck}}[C], \in, C\rangle)$.

Let $\langle \omega_\alpha \,|\, 1 \leq \alpha < \infty\rangle$ enumerate in ascending order
$ADM^* =_{df} \{\beta > \omega \,|\, \beta$ is admissible or a limit of such $\}$.

In essence Field starts out defining $C_0 = \varnothing$ and at the first stage constructs

- $C_0^+ \sim (\{A : |A|_0^+ = 1\}, \{A : |A|_0^+ = 0\})$
$$\equiv \text{complete } \Pi_1^{1,C_0} \text{ set} \equiv \text{complete } \Sigma_1\text{-Th}(\langle L_{\omega_{1ck}}, \in, C_0\rangle).$$

- C_1 then codes up C_0^+ through
$$\{\top \to A : |A|_0^+ = 1\} \cup \{A \to \bot : |A|_0^+ = 0\}.$$

- $C_\alpha^+ \sim (\{A : |A|_a^+ = 1\}, \{A : |A|_a^+ = 0\})$
$$\equiv \text{complete } \Pi_1^{1,C_\alpha} \text{ set} \equiv \text{complete } \Sigma_1\text{-Th}(\langle L_{\omega_{\alpha+1}}[C_\alpha], \in, C_\alpha\rangle).$$

- $C_{\alpha+1}$ then codes up C_α^+ through
$$\{\top \to A : |A|_\alpha^+ = 1\} \cup \{A \to \bot : |A|_\alpha^+ = 0\}.$$

- $C_\lambda = \liminf_{\alpha\to\lambda} C_\alpha^+$.

Lemma 1 *If* $L_{\omega_\lambda} \models \Sigma_1$*-Separation, then* $C_\lambda \equiv_1$ *pure* Σ_2*-Th(*L_{ω_λ}*). These reductions are uniform and independent of* λ*.*

Proof: For $\underline{(\leq_1)}$, just note that $C_\lambda \in \Sigma_2(L_{\omega_\lambda})$.
$\underline{(\geq_1)}$: Suppose $L_{\omega_\lambda} \models \exists n \forall m \psi(n, m) \equiv \sigma$. Then there is n_0, and hence $\alpha < \lambda$ so that:

$L_{\omega_{\alpha+1}} \models \forall m \psi(n_0, m) \equiv \tau(n_0)$ So:

$L_{\omega_{\alpha+1}} \models$ "$L_{\omega_\alpha} \models \tau(n_0)$" This latter quoted statement is a Σ_1 statement about the last admissible set. Call it $T(n_0)$.

Let $G : \Sigma_1$-Th($\langle L_{\omega_{\beta+1}}[C_\beta], C_\beta \rangle$) $\longrightarrow C_{\beta+1}$ (uniformly in β).

As $L_{\omega_\lambda} \models \sigma$, $G(T(n_0)) \in C_\lambda$.

For the converse, if $G(T(n_0)) \in C_\lambda$, we have to use Σ_1-Separation to ensure that $L_{\omega_\lambda} \models$ the corresponding σ.