Squares below a Jónsson cardinal

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Aim: to discuss some situations where, using Covering Lemmas and condensation arguments, we can abstract from inner models knowledge about combinatorial principles in V.

We shall see that in certain situations, such an inner model can correctly calculate the successor cardinal function on ‘‘thick’’ sets of regular cardinals.

We have already seen:

$\vec{C} = \langle C_\alpha | \alpha \in \mathrm{Sing} \rangle$ is a global $\Box$-sequence iff

(i) $C_\alpha \subseteq \alpha$ is unbounded in α, and closed below α;
(ii) $\mathrm{ot}(C_\alpha) < \alpha$;
(iii) $\beta \in (C_\alpha)^* \longrightarrow C_\beta = \beta \cap C_\alpha$.

If $\vec{C}$ is the sequence we have been looking at, as defined in L, the only thing stopping it being a global $\Box$-sequence in V, is that "$\alpha \in \mathrm{Sing}$".

Jensen's L-Covering Lemma (a consequence thereof)

CL(L):
If there is no non-trivial elementary embedding $\pi: L \longrightarrow_e L$:
then for any γ

a) γ singular $\Longrightarrow (\gamma \text{ singular})_L$;

b) $(\lambda = \gamma^+)_L \wedge \lambda \geq \omega_2 \longrightarrow \mathrm{cf}(\lambda) \geq |\lambda|$.

- Consequently if L-Covering holds, our L-defined sequence is good in V too.

- The hypothesis about the non-existence of such embeddings is abbreviated as "$\neg O^{\#}$".

This scenario could be played out with other models W if we knew:
(a) the model W had a similar fine structural feel as L, and so we could run a proof of Global $\Box$ there;
(b) $\mathrm{CL}(W)$ held.

For many models (a) is still possible, but (b) gets lost on the way up through the hierarchy of inner models. Nevertheless a sufficient amount of Covering remains in many situations - a ‘‘weak’’ covering lemma holds. In particular if we look only at $\Box_\kappa$-sequences, we can still say significant things about the ordinal combinatorics of V.

For example:

Theorem 1. $(\kappa^+)^L = \kappa^+ \Longrightarrow \Box_\kappa$ *holds (in V).*

In particular for many models W the antecedent holds for κ a
(a) singular cardinal; or a
(b) weakly compact cardinal.

We shall look at a class of situations that enable us to have the antecedent holding at other inaccessible cardinals than weakly compacts. All that is needed is to have a sufficently vital failure of the condensation lemma for the underlying hierarchy of the model W.

Definition 2.
(a) A cardinal κ *is* δ-Jónsson *for* $\delta \leq \kappa$ *a cardinal,*

if for any $A \subseteq H_{_\kappa}$, if $\mathfrak{A} = \langle H_\kappa, \in, \lhd_\kappa, A, \ldots \rangle$ *then*

$\exists\ \mathfrak{B} \prec \mathfrak{A}$ *with* $|\mathfrak{B}|_\cdot \cap \kappa \neq \kappa$*, but with order type* $|\mathfrak{B}|_\cdot \cap \kappa = \delta$*.*

(b) κ *is* Jónsson *if it is* δ-*Jónsson;*

Fact 1 (KUNEN)
$\mathrm{Con}(\mathrm{ZFC} + \exists\kappa\,\text{Jónsson}) \Longleftrightarrow \mathrm{Con}(\mathrm{ZFC} + \exists\kappa\,\text{Ramsey})$;

Fact 2 (TRYBA, WOODIN) For no regular κ is κ^+ is Jónsson.

Fact 3 κ regular and Jónsson $\not\Longrightarrow \kappa$ strongly inaccessible. ($2^{\aleph_0} > \kappa$ is possible.)

Suppose κ is Jónsson. Consider $\mathfrak{A} = \langle H_\kappa, \in, \lhd_\kappa, L_\kappa, \ldots \rangle$.
Then there is $\mathfrak{B} = \langle X, \in, \lhd_\kappa \cap X^2, L_\kappa \cap X, \ldots \rangle \prec \mathfrak{A}$ with $|X| = \kappa$, but $X \cap \kappa \neq \kappa$.

Let $\pi\colon \langle H, \in, \bar{\lhd}, L_\kappa, \ldots \rangle \longleftrightarrow X$ be the transitive collapse of X; then we have that $\pi \restriction L_\kappa\colon L_\kappa \longrightarrow_e L_\kappa$ is elementary and non-trivial. This is enough to show that $O^\#$ exists (we extend or ''lift'' the embedding $\pi \restriction L_\kappa$ to a $\tilde{\pi}\colon L \longrightarrow_e L$). This shows that $\mathrm{CL}(L)$ must fail.

- The same argument works to show that for any $B \subset \gamma < \kappa$ $B^\#$ exists.

If now we try to replace L with some other more general model W, that under an assumption analogous to "$\neg O^{\#}$", is "rigid", i.e. there is no non-trivial $\sigma: W \longrightarrow_e W$, we would conclude in the above scenario that if:

$$\pi: \langle H, \in, \bar{\lhd}, \overline{W}, \ldots\rangle \Longleftrightarrow$$
$$\Longleftrightarrow \langle X, \in, \lhd_\kappa \cap X^2, W \restriction \kappa \cap X, \ldots\rangle \prec \langle H_\kappa, \in, \lhd_\kappa, W \restriction \kappa, \ldots\rangle$$

that $W \restriction \kappa \neq \overline{W}$ i.e. there must be a failure of condensation.

Fine Structural models

Fine structure is used to define a class of models W so that under certain anti-large cardinal assumptions:

(A) W is rigid: there is no non-trivial $\sigma : W \longrightarrow_e W$;
(B) $\mathrm{WCL}(W)$: $(\lambda = \gamma^+)_W \wedge \lambda \geq \omega_2 \longrightarrow \mathrm{cf}(\lambda) \geq |\lambda|$.

In the sequel we shall assume initially that the anti-large cardinal assumption is: ‘‘there is no inner model with a measurable cardinal’’, and that W will be the Dodd-Jensen Core model that we shall hear about tomorrow, K_{DJ}.

Mice: the components of such models.

A typical component of such a model is a structure:

$$M = \langle J_\beta^{E^M}, E^M, E_\beta^M \rangle,$$

with an E_β^M, when not empty, a measure on $\mathcal{P}(\kappa) \cap M$, where κ is a measurable cardinal in the sense of M. E^M codes the sequence of previous filters E_α^M for $\alpha < \beta$.

We may take iterated ultrapowers of such an $M = M_0$, and E^M_β has been chosen so that all such ultrapowers are wellfounded.

$M_0 = \langle J^{E^{M_0}}_{\beta_0}, E^{M_0}, E^{M_0}_{\beta_0}\rangle \longrightarrow \mathrm{Ult}(M_0, E^{M_0}_{\beta_0}) \cong M_1 = \langle J^{E^{M_1}}_{\beta_1}, E^{M_1}, E^{M_1}_{\beta_1}\rangle \longrightarrow \ldots$

with $\pi_{\alpha\beta}$: $M_\alpha \longrightarrow M_\beta$ the iteration maps.

We may define an inner model $W_M = \bigcup_{\alpha \in \mathrm{On}} H^{M_\alpha}_{\kappa_\alpha}$.

The Dodd-Jensen K_{DJ}

A proper class inner model of ‘‘$\mathrm{ZFC} + V = L[\vec{E}]$’’ is obtained, where $\vec{E}$ is a predicate coding such a suitable sequence of filters. Under the above assumptions we have:

(I) $W = K_{\mathrm{DJ}} \models \mathrm{ZFC} + V = L[\vec{E}]$;
(II) Every $L[\vec{E}]$-like proper inner model of W can be seen to be ‘‘generated’’ by iterating some M in this way. In particular:
(III) If $\pi\colon W' \longrightarrow_e W$ then some ‘‘least’’ $M = M(W)$ has been omitted from W', and it is this M that will generate W' in the above fashion.

Theorem 3. *Assume W satisfies (A),(B) above. Let κ be Jónsson. Then $E = \{\gamma < \kappa \,|\, \gamma \text{ regular} \wedge \gamma^+ = \gamma^{+W}\}$ is stationary in κ.*

Proof Let $D \subseteq \kappa$ be a closed and unbounded set; consider then π: $\langle H, \in, \bar{\lhd}, \overline{W}, \overline{D}, \ldots\rangle \cong \langle X, \in, \lhd_\kappa \cap X^2, W \upharpoonright \kappa \cap X, D \cap X, \ldots\rangle \prec \langle H_\kappa, \in, \lhd_\kappa, W \upharpoonright \kappa, D, \ldots\rangle$
where $X \cap \kappa \neq \kappa$.

(1) $\overline{W} \neq W \upharpoonright \kappa$.

To show this one has to argue that $\pi \upharpoonright \overline{W}$: $\overline{W} \longrightarrow W \upharpoonright \kappa$ can be "lifted" or extended to an embedding $\widetilde{\pi}$: $W \longrightarrow W$ contradicting rigidity. We have not shown how to do this, but it can be done.

Granting (1) then by the same considerations as (III) above, some least component M has been ‘‘omitted’’, and in fact this is a component of size $< \kappa$. M then generates $\bar{W}$ in the sense above:

(2) $\overline{W} = \bigcup_{\alpha<\kappa} H^{M_\alpha}_{\kappa_\alpha}$.

We set $C_0 =_{\mathrm{df}} \{\kappa_\alpha | \alpha < \kappa\}$. Let $C =_{\mathrm{df}} \{\alpha = \kappa_\alpha | \alpha \in C_0\}$, be its set of fixed points. Then $C \cap \bar{D}$ is still cub in κ new.

Let κ^+_α = $(\kappa^+_\alpha)^{M_\alpha}$ if such exists,
$= (\mathrm{On})^{M_\alpha}$ if not.

The fact we need, and can be shown easily using elementary ultrapower theory is:

(3) $\pi_{\alpha\beta}$ “ κ_α^+ is cofinal in κ_β^+. Hence $\forall\alpha<\kappa\ \mathrm{cf}(\kappa_\alpha^+)=\mathrm{cf}(\kappa_0^+)$.

A final further Fact is needed:

(4) (SHELAH) κ Jónsson $\implies$ κ is weakly Mahlo.

So pick $\delta<\kappa,$ with δ regular, $\delta\in C\cap\overline{D}$, and $\delta>|M|$. Then

(5) $H\vDash\delta$ regular.

Suppose now $\pi(\delta) \notin E$. Then

(6) $\pi(\delta)^+ > (\pi(\delta)^+)^W$; hence $H \models$ "$\delta^+ > \delta^{+\overline{W}}$".

But
(7) $H \models$ "$\mathrm{cf}(\delta^{+\overline{W}}) = \delta$".

But this is a contradiction: $\delta > |M| \geq \mathrm{cf}(\kappa_0^+) = \mathrm{cf}(\kappa_\delta^+) = \mathrm{cf}(\delta^{+\overline{W}})$!.
Hence $\pi(\delta) \in E$. QED

Similar considerations give:

Theorem 4. *Suppose κ is ω_3-Jónsson. Then*
$B = \{\gamma < \kappa \mid \gamma \text{ regular} \wedge \gamma^+ = \gamma^{+W}\}$ *is stationary in κ.*

What are the issues about generalising such arguments?

Issue 1: We assumed that W and so $\overline{W}$, contained no measurable cardinals of their own.

We shall be forced to ‘‘compare’’ $\overline{W}$ with M in a coiteration, with possible ultrapowers of $\overline{W}$ to be considered.

Issue 2: We tacitly assumed that there was some small (i.e. of cardinality < κ or $< \omega_3$) structure M that would iterate up through the ordinals of $\overline{W}$, leaving $\overline{W}$ behind. This means that in general the argument that such a set C of a points that get iterated up to κ exists is much harder.

Generalisations

Theorem 5. *Suppose there is no inner model with a Woodin cardinal. Assume that there is a measurable cardinal Ω; let $W = K^{\mathrm{Steel}}$. Let $\kappa < \Omega$ be Jónsson.*
Then $B = \{\gamma < \kappa \mid \gamma \text{ regular} \wedge \gamma^+ = \gamma^{+W}\}$ is stationary in κ.

Theorem 6. *Assume that there is no inner model with a measurable of Mitchell order ω_1; let W be the appropriate $L[\vec{E}]$-model. Suppose κ is ω_3-Jónsson. Then*
$B = \{\gamma < \kappa \mid \gamma \text{ regular} \wedge \gamma^+ = \gamma^{+W}\}$ is stationary in κ.

Further results

Theorem 7. *Assume* $\neg O^{\mathbb{P}}$ *(that is no* $\#$ *for an inner model of a strong cardinal). Let* κ *be a regular Jónsson cardinal.*
Then $\forall A \subseteq \kappa\ A^{\#}$ *exists.*

Open Question: Assume there is no inner model of a Woodin cardinal. Let κ be a regular Jónsson cardinal. Does the conclusion of the last theorem hold?

Theorem 8. *Under the assumptions of Theorem 6 weak covering holds at* κ*:* $\kappa^{+} = \kappa^{+W}$.

Full proofs of the above can be found in ''*Some remarks on the maximality of core models*'' at the above website, or in Proc. Logic Coll.98 Praha, ASL Lecture Notes in Logic, vol.13.