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Amplitude distribution of eigenfunctions in mixed systems

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Abstract

We study the amplitude distribution of irregular eigenfunctions in systems with mixed classical phase space. For an appropriately restricted random wave model, a theoretical prediction for the amplitude distribution is derived and a good agreement with numerical computations for the family of limaçon billiards is found. The natural extension of our result to more general systems, e.g. with a potential, is also discussed.

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1. Introduction

The semiclassical behaviour of the eigenfunctions of a quantum mechanical system strongly depends on the ergodic properties of the underlying classical system. The semiclassical eigenfunction hypotheses [1, 2] state that the Wigner function of a semiclassical eigenstate is concentrated on a region in phase space explored by a typical trajectory of the classical system. In integrable systems the phase space is foliated into invariant tori, and the Wigner functions of the quantum mechanical eigenfunctions tend to delta functions on these tori in the semiclassical limit [3]. On the other hand, in an ergodic system almost all trajectories cover the energy shell uniformly, and hence the Wigner functions of the eigenstates are expected to become a delta function on the energy shell. That this actually happens for an ergodic system for almost all eigenstates follows from the quantum ergodicity theorem, see [4–6] and [7, 8] for billiards (the relation of the quantum ergodicity theorem with the semiclassical behaviour of Wigner functions is explicitly derived for Hamiltonian systems in [9]). However, a generic system is neither integrable nor ergodic [10], but has a mixed phase space in which regular regions (e.g. islands around stable periodic orbits) and stochastic regions coexist. Whether these numerically observed stochastic regions are ergodic and of positive measure is an open question, see [11] for a review on the coexistence problem. The eigenfunctions in mixed systems are expected to be separated into regular and irregular eigenfunctions according to an early conjecture by Percival [12] which has been numerically confirmed for several systems (see e.g. [13–16]). In addition, at finite energies there is a small (semiclassically vanishing) fraction of 'hierarchical states' which are of intermediate nature, and localize in regions bounded by cantori [17].

Besides the localization properties of the Wigner function, the local amplitude fluctuations of the eigenfunctions also strongly depend on the classical system, as has been pointed out in [1, 18]. The basic idea is that an eigenfunction can be represented locally as a superposition of de Broglie waves with wavelength determined by the energy and momenta distributed according to the semiclassical limit of the Wigner function. In a chaotic system one therefore expects an isotropic distribution of the momenta. If one additionally assumes that the phases are randomly distributed, one obtains locally a Gaussian amplitude distribution of a typical eigenfunction in a quantum mechanical system with chaotic classical limit. For instance, in a chaotic billiard a Gaussian amplitude distribution is expected, and this has been confirmed by several numerical studies (see e.g. [19–24]). Predictions of the random wave model on the behaviour of the maxima of eigenfunctions have been derived and successfully tested in [22, 25]. In mixed systems the situation is more complicated; for some studies on matrix elements and eigenfunctions in this case, see, for example [26–28]. In contrast, in an integrable system the localization of the Wigner function on the invariant tori enforces a more coherent superposition of the de Broglie waves, leading to a regular structure of the eigenfunction [1].

Our aim is to determine the amplitude distribution for irregular states in systems with mixed classical dynamics. We assume that the motion on a stochastic region D in phase space is ergodic and that the statistical properties of eigenfunctions can be described by a random wave model restricted to D (see the following section for a precise definition). The derivation shows that locally the fluctuations are Gaussian with a position-dependent variance which is given by the classical probability density on position space defined by the ergodic density on D. Thus the resulting amplitude distribution may be significantly different from a Gaussian. In section 3 we compare the theoretical prediction of the restricted random wave model with numerical computations.

2. Amplitude distribution for the restricted random wave model

In this section we consider a restricted random wave model for the two-dimensional Euclidean quantum billiards in order to describe the statistical properties of irregular eigenfunctions in systems with a mixed classical phase space. The quantum mechanical system is defined by the Euclidean Laplacian on a compact domain $\Omega \subset \mathbb{R}^2$ with suitable boundary conditions on the boundary $\partial\Omega$. (Usually one chooses the Dirichlet conditions.) The quantum mechanical eigenvalue problem is given by

$$\Delta\psi_n(\boldsymbol{q}) = E_n\psi_n(\boldsymbol{q}) \qquad \text{with} \quad \psi_n(\boldsymbol{q}) = 0 \quad \text{for} \quad \boldsymbol{q} \in \partial\Omega \tag{1}$$

and we are interested in the behaviour of the eigenfunctions ψ_n in the semiclassical limit $E_n \to \infty$.

The corresponding classical system is given by a free particle moving along straight lines inside the billiard, making elastic reflections on the billiard boundary $\partial\Omega$. The phase space is $T^*\Omega = \mathbb{R}^2 \times \Omega$, and the Hamiltonian is $H(\boldsymbol{p}, \boldsymbol{q}) = |\boldsymbol{p}|^2$. Since the Hamiltonian is scaled we can restrict our attention to the equi-energyshell with energy $E = 1$,

$$S^*\Omega := \{(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^2 \times \Omega\,;\, |\boldsymbol{p}| = 1\}. \tag{2}$$

Introducing polar coordinates (r, ϕ) for the momentum $\boldsymbol{p}$, we can parametrize $S^*\Omega$ by $(\phi, \boldsymbol{q}) \in [0, 2\pi) \times \Omega$ where ϕ is the direction of the momentum. In these coordinates the Liouville measure on $S^*\Omega$ is given by

$$\mathrm{d}\mu = \mathrm{d}\phi\, \mathrm{d}^2 q \tag{3}$$

which is invariant under the Hamiltonian flow on $S^*\Omega$.

Now let $D \subset S^*\Omega$ be an open domain which is invariant under the classical flow, and on which the flow is chaotic. The existence of such a domain where the flow is, for instance, ergodic, is an open problem. But numerically one observes invariant domains on which the flow is at least irregular in the sense that most orbits are unstable, and regular islands inside this domain are very small. The uncertainty principle implies a finite quantum mechanical resolution of phase space quantities at finite energies. Therefore at finite energies the small islands of such an irregular domain are not resolved by the quantum system.

So we expect, in the spirit of [1], that the statistical properties of irregular eigenfunctions associated with D can be described by those of a superposition of plane waves with wave vectors of the same lengths and directions distributed uniformly on D. Furthermore if we assume random phases, we arrive at the following restricted random wave model for real valued functions, which is a superposition of plane waves of the form

$$\psi_{\mathrm{RRWM},D}(\boldsymbol{q}) = \sqrt{\frac{4\pi}{\mathrm{vol}(D)N}} \sum_{n=1}^{N} \chi_D(\widehat{\boldsymbol{k}}_n, \boldsymbol{q}) \cos(\boldsymbol{k}_n \cdot \boldsymbol{q} + \varepsilon_n). \tag{4}$$

Here $\chi_D(\cdot)$ is the characteristic function of D, the phases ε_n are independent random variables equidistributed on $[0, 2\pi]$, and the momenta $\boldsymbol{k}_n \in \mathbb{R}^2$ are independent random variables which are equidistributed on the circle of radius $\sqrt{E}$. So the characteristic function $\chi_D(\cdot)$ ensures the localization on D. Furthermore, it is natural to take $N \sim \sqrt{E}$, the scaling of the number of line segments of a typical Heisenberg-length orbit. The volume of D measured with the Liouville measure (3) is denoted by vol(D). With this choice of normalization the expectation value of the norm $\|\psi_{\mathrm{RRWM},D}\|$ is 1.

Let us first consider the value distribution $P_{\boldsymbol{q}}(\psi)$ of $\psi_{\mathrm{RRWM},D}(\boldsymbol{q})$ at a given point $\boldsymbol{q} \in \Omega$. Our restricted random wave model (4) is a sum of identical independent random variables which have zero mean and whose variance is given by

$$\sigma^2(\boldsymbol{q}) = \mathbb{E}\left(\frac{4\pi}{\mathrm{vol}(D)}(\chi_D(\hat{\boldsymbol{k}}_n, \boldsymbol{q}) \cos(\boldsymbol{k}_n \cdot \boldsymbol{q} + \varepsilon_n))^2\right) = \frac{1}{\mathrm{vol}(D)} \int_0^{2\pi} \chi_D(\boldsymbol{e}(\phi), \boldsymbol{q})\, \mathrm{d}\phi \tag{5}$$

where $\boldsymbol{e}(\phi) := (\cos(\phi), \sin(\phi))$ denotes the unit vector in the ϕ-direction. So by the central limit theorem we obtain for $E \to \infty$, i.e. $N \to \infty$, a Gaussian distribution of $\psi_{\mathrm{RRWM},D}(\boldsymbol{q})$ at $\boldsymbol{q}$,

$$P_{\boldsymbol{q}}(\psi) \longrightarrow \sqrt{\frac{1}{2\pi\sigma^2(\boldsymbol{q})}} \exp\left(-\frac{\psi^2}{2\sigma^2(\boldsymbol{q})}\right) \tag{6}$$

with variance given by (5). If the classical dynamics on D is ergodic, then the variance $\sigma^2(\boldsymbol{q})$ is exactly the probability density of finding the particle at the point $\boldsymbol{q} \in \Omega$ if it moves on a generic trajectory in D. So $\sigma^2(\boldsymbol{q})$ is the classical probability density in position space.

By integrating equation (6) over Ω we obtain the complete amplitude distribution as a mean over a family of Gaussians with variances given by (5),

$$P_{\mathrm{RRWM},D}(\psi) = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega P_{\boldsymbol{q}}(\psi)\, \mathrm{d}^2 q \tag{7}$$

$$= \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega \sqrt{\frac{1}{2\pi\sigma^2(\boldsymbol{q})}} \exp\left(-\frac{1}{2\sigma^2(\boldsymbol{q})}\psi^2\right) \mathrm{d}^2 q. \tag{8}$$

So the amplitude distribution is completely determined by the classical probabality density (5), and it will be typically non-Gaussian if $\sigma^2(\boldsymbol{q})$ is not constant.

The moments of the distribution (8) can be computed directly and turn out to be proportional to the moments of the classical probability density $\sigma^2(\boldsymbol{q})$,

$$\int \psi^{2k} P_{\mathrm{RRWM},D}(\psi)\,\mathrm{d}\psi = \rho_{2k} \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega [\sigma^2(\boldsymbol{q})]^k \,\mathrm{d}q \tag{9}$$

where the factor $\rho_{2k} = \frac{(2k)!}{k!2^k}$ denotes the $2k$th moment of a Gaussian. The odd moments are of course zero. Note that the second moment is always $1/\mathrm{vol}(\Omega)$, due to the normalization of ψ.

If the system is ergodic one has $\sigma^2(\boldsymbol{q}) = \frac{1}{\mathrm{vol}(\Omega)}$ and we get the classical result that $P_{\mathrm{RRWM},D}(\psi)$ is Gaussian with variance $\sigma^2 = \frac{1}{\mathrm{vol}(\Omega)}$. However, if $\sigma^2(\boldsymbol{q})$ depends on $\boldsymbol{q}$ then the corresponding distribution can show deviations from the Gaussian distribution. In particular, if $\sigma^2(\boldsymbol{q}) = 0$ for some region $\Omega' \subset \Omega$, we get a contribution $\frac{\mathrm{vol}(\Omega')}{\mathrm{vol}(\Omega)}\delta(\psi)$ to the corresponding distribution of $P_{\mathrm{RRWM},D}(\psi)$ as the integrand in (7) tends to a δ distribution as $\sigma^2(\boldsymbol{q}) \to 0$.

Finally, we would like to point out that the main ingredient in formula (7) is the assumption that the local amplitude distribution of an irregular eigenfunction around a point $\boldsymbol{q}$ in position space is Gaussian, with a variance given by the classical probability density in position space $\sigma^2(\boldsymbol{q})$, defined by the projection of the invariant measure on D in the position space. Clearly this assumption is not restricted to billiards, but is expected to be true for arbitrary quantum mechanical systems for which the underlying classical system contains chaotic components in phase space. So formula (7) is expected to be valid in far more general situations, with $\sigma^2(\boldsymbol{q})$ denoting the classical probability density defined by the ergodic measure on the chaotic component.

3. Comparison with irregular eigenfunctions

We now compare the predictions of the restricted random wave model with the results for some numerically computed eigenfunctions. As systems to study the amplitude distribution of irregular states in mixed systems, we have chosen the family of limaçon billiards introduced by Robnik [29, 30] with boundary given in polar coordinates by $\rho(\varphi) = 1 + \varepsilon\cos(\varphi)$, $\varphi \in [-\pi, \pi]$, with $\varepsilon \in [0, 1]$ being the system parameter. We consider the case $\varepsilon = 0.3$, for which the billiard has a phase space of mixed type [29], see figure 1. In [31] examples of eigenstates far into the semiclassical regime have been studied in this system and, in particular, the amplitude distribution has been studied numerically, but no analytical predictions have been made.

First we have to determine the classical position space probability density $\sigma^2(\boldsymbol{q})$ of the ergodic measure on the invariant domain D. The normalized ergodic measure on D is given by

$$\mathrm{d}\mu_D(\phi, \boldsymbol{q}) = \frac{1}{\mathrm{vol}(D)} \chi_D(\boldsymbol{e}(\phi), \boldsymbol{q})\,\mathrm{d}\phi\,\mathrm{d}^2 q$$

so we can express the variance $\sigma^2(\boldsymbol{q})$ as a mean value

$$\sigma^2(\boldsymbol{q}) = \int_{S^*\Omega} \delta(\boldsymbol{q} - \boldsymbol{q}')\,\mathrm{d}\mu_D(\phi', \boldsymbol{q}'). \tag{10}$$

As the motion on D is assumed to be ergodic, in order to determine $\sigma^2(\boldsymbol{q})$ we could replace the integral over $S^*\Omega$ by a time average over a typical trajectory of D and the δ function by a smoothed δ function, e.g. a narrow Gaussian. However, as we will see below, the eigenfunctions turn out not to be concentrated on the whole chaotic component, but rather on a subset which is almost invariant in the sense that it is bounded by partial barriers in phase

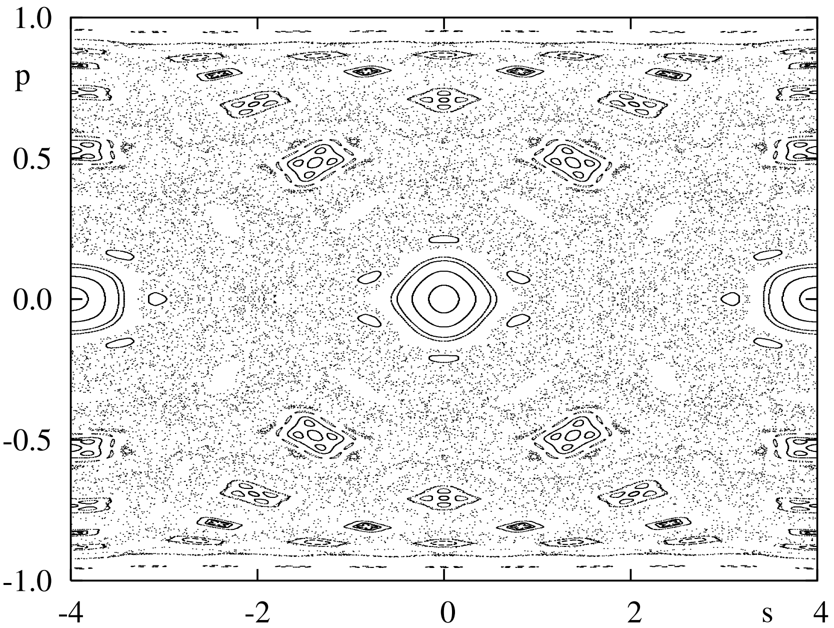

Figure 1. Plot of several stable and irregular orbits in the Poincaré section $\mathcal{P}$ of the limaçon billiard for $\varepsilon = 0.3$. Here $\mathcal{P}$ is parametrized by the (rescaled) arclength coordinate $s \in [-4, 4]$ along the billiard boundary and $p \in [-1, 1]$ which is the projection of the unit velocity vector on the tangent at the point s after the reflection.

space. Since at finite energies quantum mechanics has only a finite resolution in phase space, these partial barriers appear like real barriers. But since any classical trajectory will pass such a barrier after a certain time, the time average is not suitable for the determination of $\sigma^2(\boldsymbol{q})$ in such a situation.

For a more direct approach to determine $\sigma^2(\boldsymbol{q})$ we use the Poincaré section $\mathcal{P} = \{(s, p);\, s \in [-4, 4],\, p \in [-1, 1]\}$, which is parametrized by the (rescaled) arclength coordinate s (corresponding to $\varphi \in [-\pi, \pi]$) along the boundary $\partial\Omega$ and the projection p of the unit velocity vector on the tangent at the point s after the reflection. Let $\mathcal{D} \subset \mathcal{P}$ be the projection of the region D in the energy shell $S^*\Omega := \{(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^2 \times \Omega;\, \|\boldsymbol{p}\| = 1\}$ on the Poincaré section. This projection is defined as follows: for a point $(\boldsymbol{e}(\phi), \boldsymbol{q}) \in D$ we can associate the trajectory which passes through $\boldsymbol{q}$ in direction $\boldsymbol{e}(\phi)$, then $s(\phi, \boldsymbol{q})$ is defined as the first intersection with the boundary $\partial\Omega$ when traversing the trajectory backwards from $\boldsymbol{q}$ and $p(\phi, \boldsymbol{q}) := \boldsymbol{e}(\phi)\boldsymbol{T}(s(\phi, \boldsymbol{q}))$ which is the projection of the unit velocity vector $\boldsymbol{e}(\phi)$ on the unit tangent vector $\boldsymbol{T}(s(\phi, \boldsymbol{q}))$ to $\partial\Omega$ at $s(\phi, \boldsymbol{q})$.

For a given point $\boldsymbol{q}$ we therefore get a curve parametrized by ϕ

$$(p(\phi, \boldsymbol{q}), s(\phi, \boldsymbol{q})) \in \mathcal{P}. \tag{11}$$

Since $\chi_D(\boldsymbol{e}(\phi), \boldsymbol{q}) = \chi_{\mathcal{D}}(p(\phi, \boldsymbol{q}), s(\phi, \boldsymbol{q}))$, we get

$$\sigma^2(\boldsymbol{q}) = \frac{1}{\mathrm{vol}(D)} \int_0^{2\pi} \chi_{\mathcal{D}}(p(\phi, \boldsymbol{q}), s(\phi, \boldsymbol{q}))\,\mathrm{d}\phi \tag{12}$$

and therefore we have to determine the fraction of the angular interval(s) for which the curve (11) is in $\mathcal{D}$. That is, one has to determine the angles $\phi_i^{\mathrm{entry}}(\boldsymbol{q})$ and $\phi_i^{\mathrm{exit}}(\boldsymbol{q})$ where the curve (11) enters or leaves the region $\mathcal{D}$, i.e. the intersection points of (11) with the boundary of $\mathcal{D}$. In terms of these angles we obtain

$$\sigma^2(\boldsymbol{q}) = \frac{1}{\mathrm{vol}(D)} \sum_i \phi_i^{\mathrm{exit}}(\boldsymbol{q}) - \phi_i^{\mathrm{entry}}(\boldsymbol{q}) \tag{13}$$

which is proportional to the fraction of directions in the ergodic component visible from the point $\boldsymbol{q}$.

With this classical probability density $\sigma^2(\boldsymbol{q})$, one can compute the corresponding amplitude distribution via equation (8). If $\sigma^2(\boldsymbol{q}) = 0$ for some region, then the local amplitude distribution (6) becomes a delta function, and it is necessary to consider for a concrete comparison a binned distribution,

$$P_{\mathrm{binned}}(\psi, \Delta\psi) := \frac{1}{\Delta\psi} \int_{\psi-\Delta\psi/2}^{\psi+\Delta\psi/2} P(\psi')\, \mathrm{d}\psi' \tag{14}$$

$$= \frac{1}{2|\Omega|} \int_\Omega \left[\mathrm{erf}\left(\frac{\psi + \Delta\psi/2}{\sqrt{2\sigma^2(\boldsymbol{q})}} \right) - \mathrm{erf}\left(\frac{\psi - \Delta\psi/2}{\sqrt{2\sigma^2(\boldsymbol{q})}} \right) \right] \mathrm{d}^2 q. \tag{15}$$

We now use a Husimi Poincaré section representation of the eigenstate (see e.g. [32, 33]) to determine the boundary of the relevant component $\mathcal{D}$ by a spline approximation. The Poincaré Husimi representation of an eigenfunction ψ_n in a billiard is defined by projecting the normal derivative $u_n(s)$ of an eigenfunction $\psi_n(\boldsymbol{q})$ at the boundary onto a coherent state on the boundary. The coherent states, semiclassically centred in $(s, p) \in \mathcal{P}$, are defined as

$$c_{(s,p),k}(s') := \left(\frac{k}{\sigma\pi} \right)^{1/4} \sum_{m=-\infty}^{\infty} \exp(\mathrm{i}pk(s' - mL - s)) \exp\left(-\frac{k}{2\sigma}(s' - mL - s)^2 \right) \tag{16}$$

where $s' \in [-4, 4]$, $\sigma > 0$ and $L = 8$ is the total (rescaled) length of the boundary. This definition is just a periodized version of the standard coherent states. The Poincaré Husimi function of a state ψ_n with normal derivative $u_n(s)$ is then defined as

$$H_n(s, p) = \frac{k_n}{2\pi} \frac{1}{\int_{-4}^{4} |u_n(s)|^2\, \mathrm{d}s} \left| \int_{-4}^{4} c^*_{(s,p),k_n}(s')\, u_n(s')\, \mathrm{d}s' \right|^2 \tag{17}$$

with $k_n = \sqrt{E_n}$; the prefactor ensures the normalization $\iint H_n(s, p) \mathrm{d}p\, \mathrm{d}s = 1$.

An example is shown in figure 2. In (*a*) a high-lying eigenfunction ($E = 1002\,754.70\ldots$, approximately the 130 568th state of odd symmetry) in the limaçon billiard with $\varepsilon = 0.3$ is shown as density plot (black corresponding to high intensity of $|\psi|^2$). In (*b*) the corresponding Husimi representation on the Poincaré section is shown. The boundary of the irregular region $\mathcal{D}$ is described by a cubic spline which is shown as a full curve. With these boundary curves we can use (13) to compute $\sigma^2(\boldsymbol{q})$, which is shown in figure 2(*c*). Finally, in figure 2(*d*) the comparison of the amplitude distribution of ψ with the prediction of the restricted random wave model is given. Clearly, $P(\psi)$ is non-Gaussian, and the agreement is very good. Table 1 lists the first moments and also a very good agreement of the results using (9) and the moments of ψ is found. Both the resulting amplitude distribution $P_{\mathrm{RRWM},D}$ and the moments turn out to be quite robust with respect to small changes of the selection of $\mathcal{D}$. Note that we have rescaled $\sigma^2(\boldsymbol{q})$ such that the variance of the distributions is 1.

Another example is shown in figure 3. The eigenfunction ($E = 1003\,030.75\ldots$, approximately the 130 607th state of odd symmetry) plotted in (*a*) has a quite large region in the centre where it is almost vanishing. So from this alone the amplitude distribution is expected to show a very clear deviation from the normal distribution. Using the same procedure as in the previous case, we determine $\mathcal{D}$, compute $\sigma^2(\boldsymbol{q})$ and then $P_{\mathrm{RRWM}}(\psi)$. The comparison

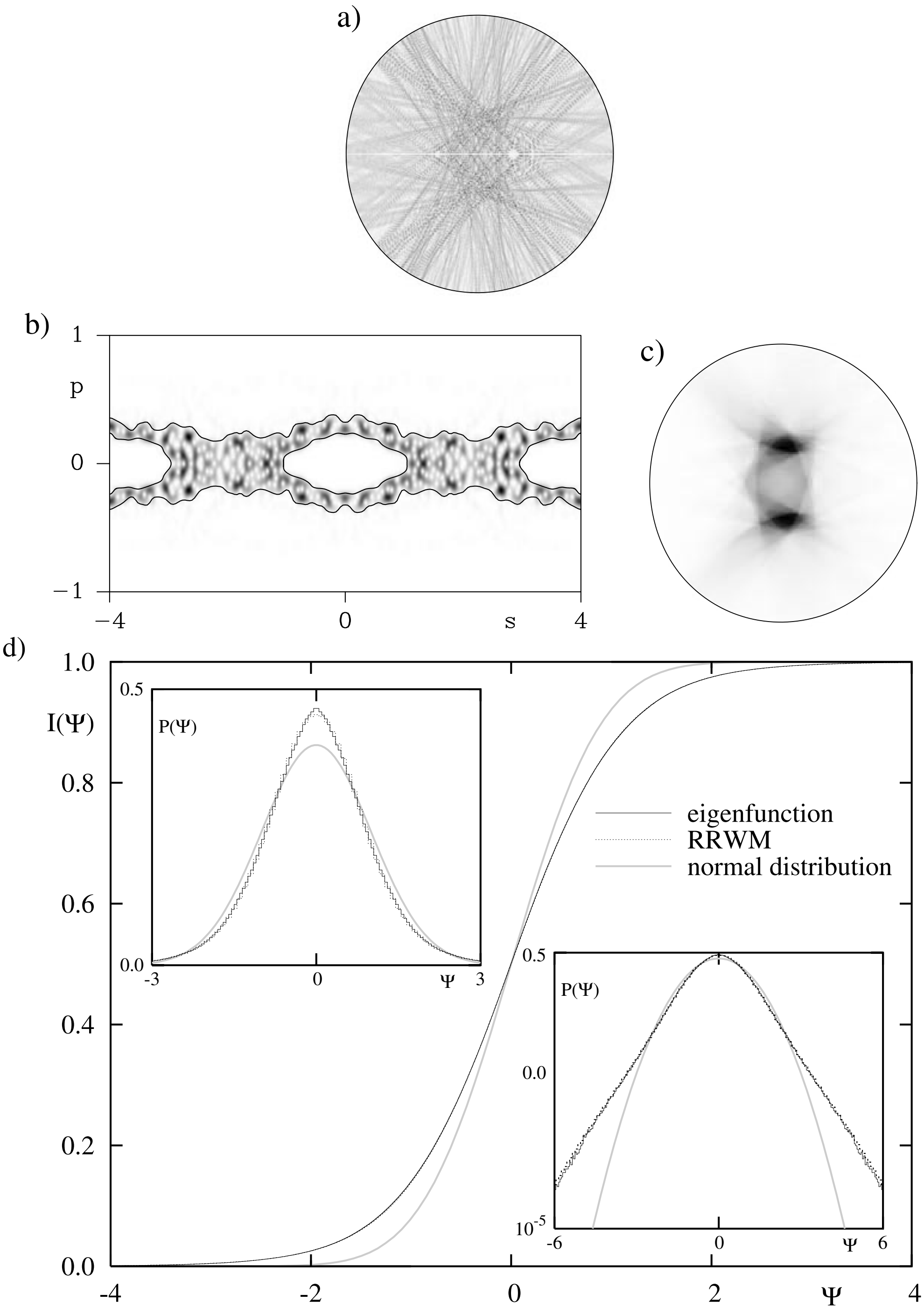

Figure 2. In (*a*) a high-lying eigenfunction ($E = 1002\,754.70\ldots$, approximately the 130 568th state) in the limaçon is shown as a grey scale plot (black corresponding to high intensity). In (*b*) the corresponding Husimi function on the Poincaré section is shown together with the boundary (full curves) of the region on which the eigenfunction is concentrated. In (*c*) a density plot of $\sigma^2(\boldsymbol{q})$, computed via equation (13), is shown. In (*d*) the cumulative amplitude distribution of the eigenfunction is compared with the prediction of the RRWM; on this scale no differences are visible. The left inset shows $P(\psi)$, and for the right inset a logarithmic vertical scale is used to emphasize the tails of the distribution. For comparison the normal distribution is shown as grey curve.

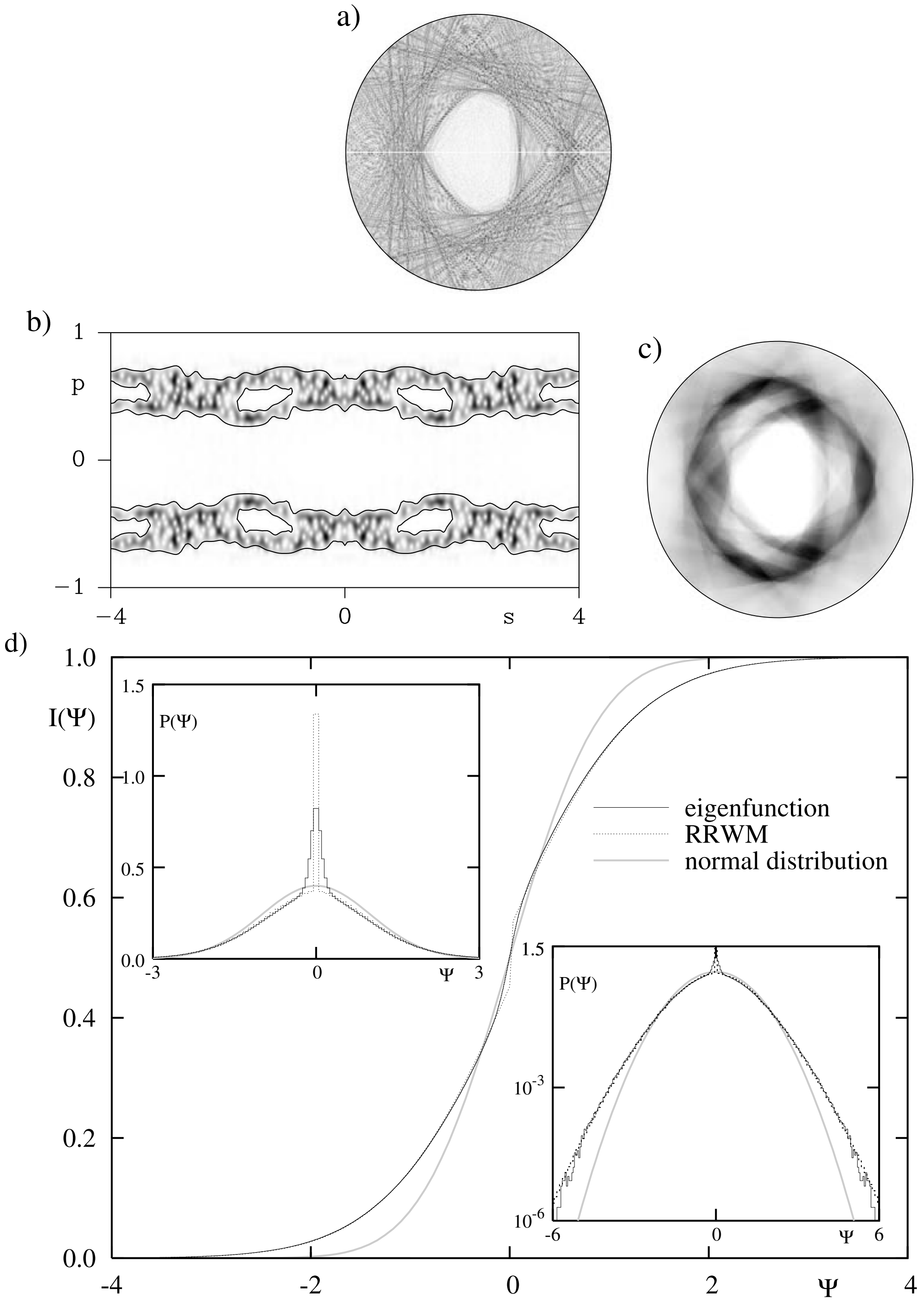

Figure 3. The same plots as in the previous figure are shown for another high-lying eigenfunction ($E = 1003\,030.75\ldots$, approximately the 130 607th state). In this case there is a deviation of the amplitude distribution of the eigenfunction from the prediction of the restricted random wave model around $\psi = 0$. This is because $\sigma^2(\boldsymbol{q}) = 0$ in the central region, whereas the eigenfunction does not vanish there (see the text for further discussion). For the tails of the distribution, the agreement of the two distributions is again very good.

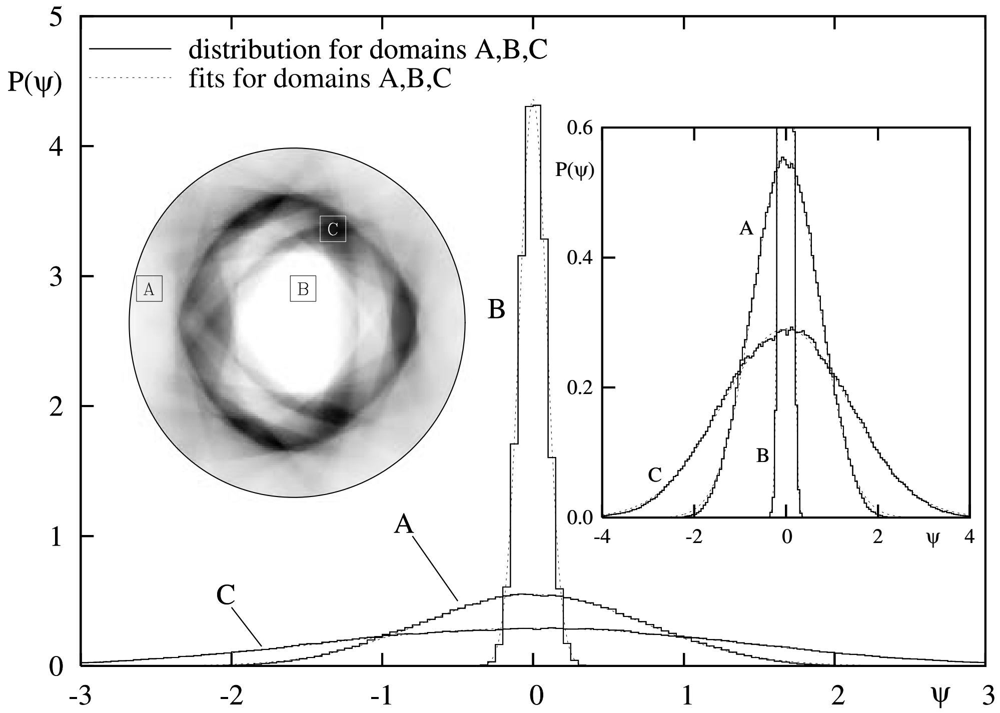

Figure 4. For the three domains indicated in the inset, the local amplitude distribution is shown (for the same state as in figure 3). The dotted curves are Gaussian fits and the agreement of the position-dependent variance for regions A and C is very good with the theoretical prediction (equation (5)). The non-zero width of the distribution for the region B corresponds to the widening of the δ-contribution (see figure 3).

Table 1. Comparison of the even moments for the distributions of the eigenfunction and the RRWM (equation (9)). The last column lists, for comparison, the moments of the normal distribution.

	Example 1, figure 2		Example 2, figure 3		
Moment	Eigenfunction	RRWM	Eigenfunction	RRWM	Normal distribution
4	4.39	4.46	3.85	3.75	3
6	45.1	47.6	26.9	25.8	15
8	819	899	269	269	105
10	2199	2501	3774	3841	945

of the prediction with $P(\psi)$ is shown in figure 3(d). The strongest deviation occurs for $\psi \approx 0$. The peak of $P_{\text{RRWM}}(\psi)$ at $\psi = 0$ is due to the fact that $\sigma^2(\boldsymbol{q}) = 0$ for the region in the centre of the billiard. The eigenfunction, however, is not exactly zero, but shows a decay in that region and thus still fluctuates there. This causes a broadening of the δ-contribution, which is clearly visible in the plot of $P(\psi)$ in figure 3(d). For $|\psi| > 0.25$ this region is not relevant anymore, and the agreement of $P(\psi)$ and $P_{\text{RRWM}}(\psi)$ is very good. In the right inset to figure 3(d) the distribution is shown with a logarithmic vertical scale to illustrate the agreement of the distributions even in the tails.

The moments, computed via equation (9), are listed in table 1. The agreement of the moments of the eigenfunction with the prediction of the restricted random wave model is quite good. All moments of the two examples are larger than those of a Gaussian, corresponding to the larger tails. Compared to the moments of the restricted random wave model, those of the eigenfunctions tend to be smaller, in particular, for the larger moments. This is reasonable, as an actual eigenfunction is always bounded, which reduces higher moments compared to the result of equation (9).

Furthermore, we have tested our basic assumption (6), that the local value distribution of a sufficiently high-lying eigenfunction is Gaussian with a variance given by the local classical probability density associated with D, more directly. To this end we have computed the value distribution of the eigenfunction in figure 3 for three small regions on which $\sigma^2(\boldsymbol{q})$ is almost invariant, and we therefore expect a Gaussian. The results are shown in figure 4, and a good agreement with the prediction (6) is found. Since many fewer wavelengths are contained in these small domains than those in Ω, the statistics is of course not as good as that for the full system, but the results give strong support for a local Gaussian behaviour. The variances for the two domains A and C coincide with the expected classical one $\sigma^2(\boldsymbol{q})$. But for domain B the observed variance is larger than $\sigma^2(\boldsymbol{q}) = 0$. This corresponds to the widening of the delta peak in figure 3, and is due to the fact that the eigenfunction cannot become exactly zero on some open set at finite energies, but instead fluctuates around zero.

4. Summary

In this paper we have extended the random wave model for eigenfunctions from the case of chaotic systems to the case of irregular eigenfunctions in systems with mixed phase space. Our main result is one particular prediction of this model, namely, the amplitude distribution (7) of irregular eigenfunctions. Numerical tests have been performed for two high-lying eigenfunctions of the limaçon billiard with $\varepsilon = 0.3$, and impressive agreement, even in the tails of the distribution, with the theoretical prediction was found.

The physical picture underlying our analysis is that the local hyperbolicity in the irregular part of the phase space forces the eigenfunctions localizing on this part of phase space to behave locally like a Gaussian random function with a variance given by the classical probability density in position space defined by the uniform measure on the irregular component. By taking the mean over all these local Gaussians with varying variance, it gives our result for the global amplitude distribution. We have tested this intuitive picture by computing local amplitude distributions. The agreement of these with the Gaussian prediction is very good, giving further strong support to the picture of local Gaussian fluctuations with variance determined by the underlying classical system. A further natural question relates to the correlations of such eigenfunctions between different points in position space; this topic is addressed in [34].

We should point out that in view of the complicated structure of the phase space of a mixed system, it is quite surprising that our simple model fits so well. The only additional ingredient which appeared in the numerical tests was that the relevant irregular domains in phase space are only slighthy invariant, even for very high-lying eigenfunctions. A detailed understanding of these findings poses an important challenge for future research.

Although we have restricted our study to the Euclidean billiards, the general picture of local Gaussian fluctuations is of course not limited to these special types of systems. We therefore expect our results to be valid for irregular eigenfunctions in arbitrary systems (e.g. systems with potential), with $\sigma^2(\boldsymbol{q})$ defined as the projection of the ergodic measure on the irregular component to the position space.

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References

[1] Berry M V 1977 Regular and irregular semiclassical wavefunctions *J. Phys. A: Math Gen.* **10** 2083

[2] Voros A 1979 Semi-classical ergodicity of quantum eigenstates in the Wigner representation *Stochastic Behavior in Classical and Quantum Hamiltonian Systems (Lecture Notes in Physics no 93)* (Berlin: Springer) pp 326–33

[3] Berry M V 1977 Semi-classical mechanics in phase space: a study of Wigner's function *Phil. Trans. R. Soc.* A **287** 237

[4] Shnirelman A I 1974 Ergodic properties of eigenfunctions *Usp. Math. Nauk* **29** 181 (in Russian)

[5] Zelditch S 1987 Uniform distribution of eigenfunctions on compact hyperbolic surfaces *Duke. Math. J.* **55** 919

[6] Colin de Verdière Y 1985 Ergodicité et fonctions propres du laplacien *Commun. Math. Phys.* **102** 497 (in French)

[7] Gérard P and Leichtnam E 1993 Ergodic properties of eigenfunctions for the Dirichlet problem *Duke Math. J.* **71** 559

[8] Zelditch S and Zworski M 1996 Ergodicity of eigenfunctions for ergodic billiards *Commun. Math. Phys.* **175** 673

[9] Bäcker A, Schubert R and Stifter P 1998 Rate of quantum ergodicity in Euclidean billiards *Phys. Rev.* E **57** 5425
Bäcker A, Schubert R and Stifter P 1998 *Phys. Rev.* E **58** 5192 (erratum)

[10] Markus L and Meyer K 1974 *Generic Hamiltonian Dynamical Systems are Neither Integrable nor Chaotic (Mem. Am. Math. Soc. no 114)* (Providence, RI: American Mathematical Society)

[11] Strelcyn J-M 1991 The 'coexistence problem' for conservative dynamical systems: a review *Coll. Math.* **62** 331

[12] Percival I C 1973 Regular and irregular spectra *J. Phys. B: At. Mol. Phys.* **6** L229

[13] Bohigas O, Tomsovic S and Ullmo D 1990 Dynamical quasidegeneracies and separation of regular and irregular quantum levels *Phys. Rev. Lett.* **64** 1479

[14] Prosen T and Robnik M 1993 Survey of the eigenfunctions of a billiard system between integrability and chaos *J. Phys. A: Math. Gen.* **26** 5365

[15] Li B and Robnik M 1995 Separating the regular and irregular energy levels and their statistics in a Hamiltonian system with mixed classical dynamics *J. Phys. A: Math. Gen.* **28** 4843

[16] Carlo G, Vergini E and Fendrik A 1998 Numerical verification of Percival's conjecture in a quantum billiard *Phys. Rev.* E **57** 5397

[17] Ketzmerick R, Hufnagel L, Steinbach F and Weiss M 2000 New class of eigenstates in generic Hamiltonian systems *Phys. Rev. Lett.* **85** 1214

[18] Berry M V 1983 Semiclassical mechanics of regular and irregular motion *Comportement Chaotique des Systèmes Déterministes—Chaotic Behaviour of Deterministic Systems* ed G Iooss, R H G Hellemann and R Stora (Amsterdam: North-Holland) pp 171–271

[19] Shapiro M and Goelman G 1984 Onset of chaos in an isolated energy eigenstate *Phys. Rev. Lett.* **53** 1714

[20] McDonald S W and Kaufmann A N 1988 Wave chaos in the stadium: statistical properties of short-wave solutions of the Helmholtz equation *Phys. Rev.* A **37** 3067

[21] Aurich R and Steiner F 1991 Exact theory for the quantum eigenstates of a strongly chaotic system *Physica* D **48** 445

[22] Hejhal D A and Rackner B N 1992 On the topography of Maass waveforms for PSL(2, $\mathbb{Z}$) *Exp. Math.* **1** 275

[23] Aurich R and Steiner F 1993 Statistical properties of highly excited quantum eigenstates of a strongly chaotic system *Physica* D **64** 185

[24] Li B and Robnik M 1994 Statistical properties of high-lying chaotic eigenstates *J. Phys. A: Math. Gen.* **27** 5509

[25] Aurich R, Bäcker A, Schubert R and Taglieber M 1999 Maximum norms of chaotic quantum eigenstates and random waves *Physica* D **129** 1

[26] Bohigas O, Tomsovic S and Ullmo D 1993 Manifestations of classical phase space structures in quantum mechanics *Phys. Rep.* **223** 43

[27] Müller K, Mehlig B, Milde F and Schreiber M 1995 Statistics of wave functions in disordered and in classically chaotic systems *Phys. Rev. Lett.* **78** 215

[28] Mehlig B, Müller K and Eckhardt B 1999 Phase space localization and matrix element distributions in systems with mixed classical phase space *Phys. Rev.* E **59** 5272

[29] Robnik M 1983 Classical dynamics of a family of billiards with analytic boundaries *J. Phys. A: Math. Gen.* **16** 3971

[30] Robnik M 1984 Quantising a generic family of billiards with analytic boundaries *J. Phys. A: Math. Gen.* **17** 1049

[31] Li B and Robnik M 1995 Geometry of high-lying eigenfunctions in a plane billiard system having mixed-type classical dynamics *J. Phys. A: Math. Gen.* **28** 2799

[32] Tualle J M and Voros A 1995 Normal modes of billiards portrayed in the stellar (or nodal) representation *Chaos, Solitons Fractals* **5** 1085

[33] Simonotti F P, Vergini E and Saraceno M 1997 Quantitative study of scars in the boundary section of the stadium billiard *Phys. Rev.* E **56** 3859

[34] Bäcker A and Schubert R 2002 Autocorrelation function of eigenstates in chaotic and mixed systems *J. Phys. A: Math. Gen.* **35** 539 (following paper) (nlin.cd/0106018)