Commun. Math. Phys. 256, 239–254 (2005)
Digital Object Identifier (DOI) 10.1007/s00220-005-1319-4

Communications in
Mathematical Physics

Semiclassical Behaviour of Expectation Values in Time Evolved Lagrangian States for Large Times

Roman Schubert

Department of Mathematics, University of Bristol, Bristol, United Kingdom.
E-mail: roman.schubert@bristol.ac.uk

Received: 2 February 2004 / Accepted: 19 October 2004
Published online: 15 March 2005 –

Abstract: We study the behaviour of time evolved quantum mechanical expectation values in Lagrangian states in the limit $\hbar \to 0$ and $t \to \infty$. We show that it depends strongly on the dynamical properties of the corresponding classical system. If the classical system is strongly chaotic, i.e. Anosov, then the expectation values tend to a universal limit. This can be viewed as an analogue of mixing in the classical system. If the classical system is integrable, then the expectation values need not converge, and if they converge their limit depends on the initial state. An additional difference occurs in the timescales for which we can prove this behaviour; in the chaotic case we get up to Ehrenfest time, $t \sim \ln(1/\hbar)$, whereas for integrable system we have a much larger time range.

1. Introduction and Results

A striking property of chaotic dynamical systems is the universality which these systems show in the time evolution for large times. Let $(\Sigma, \Phi^t, \mathrm{d}\mu)$ be a dynamical system, i.e., Σ is the compact phase space, $\Phi^t : \Sigma \to \Sigma$ the flow and $\mathrm{d}\mu$ a normalised invariant measure on Σ. If the system is mixing then for any $\rho, a \in L^2(\Sigma, \mu)$ with $\int \rho \, \mathrm{d}\mu = 1$ one has

$$\int a \circ \Phi^t \rho \, \mathrm{d}\mu \to \int a \, \mathrm{d}\mu \ , \quad \text{for } t \to \infty \ . \tag{1}$$

If we think of ρ as describing a probability distribution of initial states and of a as an observable, then mixing means that the system forgets its initial conditions for large times and so one needs only to know the "equilibrium state" $\mathrm{d}\mu$ in order to predict the behaviour of time evolved observables for large times. If the rate of mixing is fast enough this then often implies other universal statistical features, e.g., a central limit theorem for time means of observables.

We want to explore to what extent this universality shows up in quantum mechanics, too. The analogue of the expectation value in (1) is a quantum mechanical expectation value for a time evolved state. So let $\mathcal{U}(t)$ denote the time evolution operator of our quantum system, $\mathcal{A}$ an observable, i.e. a bounded operator, and ψ a state; we want to know if

$$\langle \mathcal{U}(t)\psi, \mathcal{A}\mathcal{U}(t)\psi\rangle \tag{2}$$

converges to some limit if $\hbar \to 0$ and $t \to \infty$, at least for certain classes of observables and states. We will consider here Lagrangian states as initial states and bounded pseudo-differential operators as observables.

The main difficulty in this problem comes from the fact that we have to perform two limits, $\hbar \to 0$ and $t \to \infty$, and these two limits do not commute. So we have to specify precisely how we take the joint limit and we have to use semiclassical constructions which are to some extent uniform in t. For systems which have some positive Liapunov exponents, it was found in the late 70's in the physics literature [BZ78, Zas81, BBTV79, BB79], that the usual semiclassical constructions apparently can only work up to a timescale which grows logarithmically in $\hbar$, $T_E \sim \ln(1/\hbar)$, the so called Ehrenfest or log-breaking time. That semiclassical constructions actually do work up to that time was rigorously proved in [CR97] for the time evolution of coherent states and in [BGP99] for the time evolution of observables. We will use for our work the results in [BR02] who extended the results by Bambusi, Graffi and Paul.

The time range beyond the Ehrenfest time is not well understood yet. But results by Tomsovic, Heller and coworkers, [TH91, TH93, OTH92], suggest that semiclassical methods might be extended beyond Ehrenfest time. They studied for autocorrelation functions of coherent states the question if one can extend the semiclassical propagator to timescales which are algebraic in $1/\hbar$, and demonstrated numerically that this is possible for the stadium billiard and some quantised maps.

One motivation for this work are the results of Bonechi and De Bièvre for the time evolution of coherent states in cat-maps, [BDB00]. They showed that a coherent state evolved with the quantised cat-map becomes equidistributed just after the Ehrenfest time, but they could control the time evolution only up to a slightly larger time range which is still logarithmic in $1/\hbar$. More precisely, equidistribution holds for times between $\frac{1+\varepsilon}{2\lambda}\ln(1/\hbar)$ and $\frac{1-\varepsilon}{\lambda}\ln(1/\hbar)$, for any $\varepsilon > 0$, where λ is the positive Liapunov exponent of the classical map. Since one expects a coherent state to become stretched along the unstable manifold of the orbit on which it is centred, it might be effectively modelled by a Lagrangian state associated with this unstable manifold. This is one motivation for studying Lagrangian states. Furthermore some particular examples of Lagrangian states have been already considered in [BDB00], namely position eigenstates and their time evolution under the quantised Bakers map, and they are shown to become equidistributed for large times up to the Ehrenfest time. More recently estimates on the time evolution around Ehrenfest time have been used in [FNDB03] to construct scarred eigenstates for the quantised cat map, and in [DBR03] the time evolution of coherent states along the separatrix in one-dimensional systems was investigated.

A typical Lagrangian state on a manifold M is of the form

$$\psi(x) = \rho(\hbar, x)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi(x)} , \tag{3}$$

where φ is a smooth real valued function and $\rho(\hbar, x)$ is a smooth function with compact support with an asymptotic expansion $\rho(\hbar, x) \sim \rho_0(x) + \hbar\rho_1(x) + \cdots$ for $\hbar \to 0$. The important geometrical object associated with ψ is the Lagrangian manifold generated by the phase function φ,

$$\Lambda_\varphi := \{(\varphi'(x), x) \; ; x \in U\} \subset T^*M , \tag{4}$$

where $U \subset M$ is an open set containing the support of the amplitude ρ. We will denote the set of these states with compact support by $I_0(\Lambda)$. The definition can be extended to arbitrary Lagrangian manifolds, i.e., they need not be representable in the form (4). Any Lagrangian submanifold $\Lambda \subset T^*M$ can be represented locally as $\Lambda = \{(\varphi'_x(x,\theta), x)\,;\ \varphi'_\theta(x,\theta) = 0,\ (x,\theta) \in U \times \mathbb{R}^\kappa\}$, where $\varphi(x,\theta)$ is non-degenerate, i.e., the rank of the $d \times (d+\kappa)$ matrix $(\varphi''_{x,x}(x,\theta), \varphi''_{x,\theta}(x,\theta))$ is equal to d at the points (x,θ) with $\varphi'_\theta(x,\theta) = 0$. The corresponding Lagrangian states are given by

$$\psi(x) = \frac{1}{(2\pi\hbar)^{\kappa/2}} \int_{\mathbb{R}^\kappa} \rho(\hbar, x, \theta) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\varphi(x,\theta)}\, \mathrm{d}\theta\ , \tag{5}$$

see [Dui74, BW97] and [Ivr98, Sect. 1.2.1] for more details. Lagrangian states appear quite often in applications, e.g., if $\varphi(x) = \langle p, x\rangle$ we have a localised plane wave with momentum p or if φ depends only on $|x|$ we get circular waves. Since the simultaneous eigenstates of d commuting pseudo-differential operators are typically Lagrangian, this class of states appears quite frequently as the result of the preparation of an experiment, e.g., the above mentioned examples occur if one selects initial states with certain momentum, or certain angular momentum, respectively.

The leading order behaviour of a Lagrangian state ψ for $\hbar \to 0$ is determined by its principal symbol $\sigma(\psi)$ which, modulo phase factors, is a half-density on Λ. In the case that ψ is of the form (3) $\sigma(\psi)$ is the pullback of the half-density $\rho_0(x)|\mathrm{d}x|^{1/2}$ on $\mathbb{R}^d$ by the projection $\pi : \Lambda_\varphi \to \mathbb{R}^d$. We will only encounter its modulus squared, the density $|\sigma(\psi)|^2$, which can be defined more directly by the relation

$$\int_\Lambda a\, |\sigma(\psi)|^2 := \int_{\mathbb{R}^d} a(\varphi'(x), x)|\rho_0(x)|^2\, \mathrm{d}x \tag{6}$$

for any $a \in C^\infty(T^*M)$.

The observables we consider are given by pseudo-differential operators. We will say that $\mathcal{A} \in \Psi^m(M)$ if locally $\mathcal{A} = \mathrm{Op}[a]$ where

$$\mathrm{Op}[a]\psi(x) = \frac{1}{(2\pi\hbar)^d} \iint_{\mathbb{R}^d\times\mathbb{R}^d} a\Big(\frac{x+y}{2}, \xi\Big) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\langle x-y,\xi\rangle} \psi(y)\, \mathrm{d}y\, \mathrm{d}\xi\ , \tag{7}$$

and the symbol $a(\hbar, x, \xi)$ has an asymptotic expansion $a(\hbar, x, \xi) \sim a_0(x,\xi) + \hbar a_1(x,\xi) + \hbar^2 a_2(x,\xi) + \cdots$ and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta a(\hbar, x, \xi)| \le C_{\alpha,\beta}(1 + |x|^2 + |\xi|^2)^{m/2}\ , \tag{8}$$

for $\hbar \in (0,1]$ and all $\alpha, \beta \in \mathbb{N}^d$. One calls $\sigma(a) := a_0$ the principal symbol of a, or of $\mathcal{A}$, and although the full symbol a is only defined locally, the principal symbol defines a function on T^*M, i.e., on phase space. The operators in $\Psi^0(M)$ are bounded, and they will form the set of observables for which we study time evolution. See, e.g., [DS99] for more details.

Our first assumption on the system is that the Hamiltonian fits into the above framework, i.e., is a pseudo-differential operator.

Condition (H). Let M be a C^∞ manifold and let $\mathcal{H} \in \Psi^m(M)$, for some $m \in \mathbb{R}$, be essentially selfadjoint.

A typical example is $\mathcal{H} = -\hbar^2\Delta_g + V$, where Δ_g is the Laplace Beltrami operator associated with a metric g on M, and V is a smooth real valued function (with $|\partial^\alpha V(x)| \leq C_\alpha(1+|x|)^m$ if M is not compact). For conditions on general operators from $\Psi^m(M)$ to be (essentially) selfadjoint see [DS99].

The Hamiltonian flow on T^*M generated by the principal symbol H of $\mathcal{H}$ will be denoted by Φ^t.

Condition (O). There exists an open connected set $\Omega \subset T^*M$ which has compact closure and which is invariant under the flow Φ^t.

Let $\Sigma_E := \{z \in T^*M\,;\ H_0(z) = E\}$ be the energy shell of energy E and denote by $\mathrm{d}\mu_E$ the Liouville measure on Σ_E. Σ_E and $\mathrm{d}\mu_E$ are invariant under the flow. Let us recall the definition of an Anosov flow:

Condition (A). A flow Φ^t on a compact manifold Σ is called Anosov, if for every $x \in \Sigma$ there exists a splitting $T_x\Sigma = E^s(x) \oplus E^u(x) \oplus E^0(x)$ which is invariant under Φ^t and where $E^0(x)$ is one-dimensional and spanned by the generating vectorfield of Φ^t. Furthermore there exist constants $C, \lambda > 0$ such that

$$||\mathrm{d}\Phi^t v|| \leq C\mathrm{e}^{-\lambda t}||v|| \qquad \text{for each } v \in E^s \text{ and } t \geq 0, \tag{9}$$

$$||\mathrm{d}\Phi^t v|| \leq C\mathrm{e}^{\lambda t}||v|| \qquad \text{for each } v \in E^u \text{ and } t \leq 0\ . \tag{10}$$

The two distributions E^s and E^u can be integrated to give the stable and unstable foliations, respectively. We will denote the leaves through x by $W^s(x)$ and $W^u(x)$. If the flow is smooth then the leaves are smooth submanifolds but the dependence of the leaves on x is usually only Hölder continuous, and we will denote the Hölder exponent by α. The corresponding weakly stable and unstable manifolds are defined by $W^{ws/wu}(x) := \bigcup_{t\in\mathbb{R}} \Phi^t(W^{s/u}(x))$. If Σ is an energy-shell of an Hamiltonian system, and Φ^t the Hamiltonian flow, then $W^s(x)$ and $W^u(x)$ have the same dimension, and $W^{ws}(x)$ and $W^{wu}(x)$ are Lagrangian submanifolds.

An example for an Anosov flow is given by the geodesic flow on a compact manifold of negative curvature, see e.g. [Ebe01]. If the Hamilton operator is the Laplace Beltrami operator associated with such a metric, then the flow generated by the principal symbol of this operator is conjugate to the geodesic flow, and its restriction to any equi-energy shell Σ_E is Anosov.

For the time evolution of Lagrangian states the position of Λ relative to the stable foliation will be important. Namely we have to require that $T_x\Lambda$ contains no stable directions for most x, this leads to the following transversality conditions.

Condition (T).

(i) If $\Lambda \subset \Sigma_E$ then assume that $T_x\Lambda \cap E^s(x) = \{0\}$ for all $x \in \Lambda\backslash\Lambda_{sing}$, where $\Lambda_{sing} \subset \Lambda$ has at least codimension 1.
(ii) If $\Lambda \subset \Omega$ and the flow is Anosov on all $\Sigma_E \subset \Omega$ then for all such E assume that $T_x(\Lambda \cap \Sigma_E) \cap (E^s(x) \oplus E^0(x)) = \{0\}$ for all $x \in (\Lambda \cap \Sigma_E)\backslash\Gamma_{E,sing}$, where $\Gamma_{E,sing} \subset (\Lambda \cap \Sigma_E)$ has at least codimension 1.

These conditions are typically fulfilled, in the sense that if a Lagrangian manifold Λ does not satisfy them one can find an arbitrary small perturbation of Λ which does. This would not be true if we would require transversality to the stable foliation everywhere, and this is why we choose this more complicated condition. We can state now the main result of this paper about expectation values of time evolved Lagrangian states.

Theorem 1. *Let M be a C^∞ manifold, and $\mathcal{H} \in \Psi^m(M)$ be a selfadjoint pseudo-differential operator on M, with principal symbol H_0. Let Φ^t be the Hamiltonian flow on T^*M generated by H_0, and assume Condition (O) is fulfilled. Let $\Lambda \subset \Omega$ be a Lagrangian submanifold. Then*

(i) if $\Lambda \subset \Sigma_E \subset \Omega$, the flow on Σ_E is Anosov, and Λ satisfies condition (T)(i), then there exist for every $\psi \in I_0(\Lambda)$ and $\mathrm{Op}[a] \in \Psi^0(M)$ *constants $C, c, \Gamma, \gamma > 0$ such that*

$$\left| \langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle - \int_{\Sigma_E} \sigma(a)\, \mathrm{d}\mu_E \int_\Lambda |\sigma(\psi)|^2 \right| \leq C\hbar \mathrm{e}^{\Gamma|t|} + c\mathrm{e}^{-\gamma t}. \quad (11)$$

(ii) If the flow is Anosov on all $\Sigma_E \subset \Omega$, and $\Lambda \cap \Sigma_E$ satisfies condition (T)(ii), then there exist for every $\psi \in I_0(\Lambda)$ and $\mathrm{Op}[a] \in \Psi^0(M)$ *constants C, c, Γ, γ such that*

$$\left| \langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle - \int \int_{\Sigma_E} \sigma(a)\, \mathrm{d}\mu_E \int_{\Lambda\cap\Sigma_E} |\sigma(\psi)|_E^2 \, \mathrm{d}E \right| \leq C\hbar \mathrm{e}^{\Gamma|t|} + c\mathrm{e}^{-\gamma t}\ , \quad (12)$$

where the density $|\sigma(\psi)|_E^2$ on $\Lambda \cap \Sigma_E$ is defined by $|\sigma(\psi)|^2 = |\sigma(\psi)|_E^2 \otimes |\mathrm{d}E|$.

In order that the right hand sides of the inequalities (11) and (12) tend to zero for $\hbar \to 0$ and $t \to \infty$, we have to have

$$t \leq \frac{1-\varepsilon}{\Gamma} \ln(1/\hbar)\ , \quad (13)$$

for some $\varepsilon > 0$, so up to Ehrenfest time we get convergence. The constant Γ does in fact only depend on the principal symbol of $\mathcal{H}$, it is larger than the largest Liapunov exponent of the classical flow. It seems likely that with some additional effort Γ can be chosen to be the supremum of all Liapunov exponents.

Let us compare this result with mixing for the classical system. To this end assume that $\|\psi\| = 1$, this implies that $\int_\Lambda |\sigma(\psi)|^2 = 1$ and then (11) gives

$$\langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle \to \int_{\Sigma_E} \sigma(a) \mathrm{d}\mu_E \quad (14)$$

for $t \to \infty$ and $\hbar \to 0$ such that $\hbar \mathrm{e}^{\Gamma|t|} \to 0$. So we have the same behaviour as in the classical system, see (1), in particular we obtain the same kind of universality. The limit does not depend any longer on the initial state as long as it satisfies the conditions of part (1) of Theorem 1.

The transversality condition on the Lagrangian manifold is necessary. If Λ is for instance the stable manifold of a periodic orbit γ, then one has for $\psi \in I_0(\Lambda)$,

$$\langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle = \sum_{k\in\mathbb{Z}} b_k \mathrm{e}^{\frac{2\pi \mathrm{i}}{T_\gamma} kt} + O(\hbar \mathrm{e}^{\Gamma|t|}) + O(\mathrm{e}^{-\gamma t}), \quad (15)$$

where T_γ is the period of the orbit, and the coefficients b_k are related to $\sigma(\psi)$ and $\sigma(a)$. We will discuss this in more detail in Sect. 3.

The result in Theorem 1 can be viewed as an analogue for time evolution of the quantum ergodicity results for eigenfunctions [Šni74, Zel87, CdV85, HMR87]. If the classical system is ergodic then almost all eigenfunctions become equi-distributed. Here we obtain equidistribution under time evolution, but we need stronger conditions on the classical system, namely mixing for densities concentrated on certain Lagrangian submanifolds. There seems to be no direct relation to the notion of quantum (weak) mixing introduced by Zelditch [Zel96], since our conditions are much stronger. One of the most interesting open problems now is to try to extend the time range in Theorem 1. This could then in turn be used to improve the quantum ergodicity results for eigenfunctions.

We want to compare now the behaviour found in classically chaotic systems with integrable systems. Following [BR02] we introduce the following integrability condition.

Condition (I). M is analytic, and there exists a symplectic map χ from Ω into $U \times \mathbf{T}^d$, where U is an open set in $\mathbb{R}^d$ and $\mathbf{T}^d$ is an d-dimensional torus such that

$$\chi(\Phi^t(z)) = (I(z), \varphi(z) + t\omega(I(z))) , \qquad \forall z \in \Omega, \forall t \in \mathbb{R} , \tag{16}$$

where $\chi(z) = (I(z), \varphi(z))$. Moreover there exists complex open neighbourhoods $\tilde{\Omega}$, $\tilde{U}$, $\tilde{\mathbf{T}}^d$ of Ω, U, $\mathbf{T}^d$ such that χ is an analytic diffeomorphism from $\tilde{\Omega}$ onto $\tilde{U} \times \tilde{\mathbf{T}}^d$.

According to the Liouville Arnold Theorem this situation occurs if one has d analytic integrals of motion which are in involution and which are independent on Ω.

In the case of integrable systems one can explore larger time scales, and we obtain the following results.

Theorem 2. *Assume Conditions (H), (O) and (I) are fulfilled. Assume furthermore that $\Lambda \subset \Omega$ is an invariant torus with frequency $\omega \in \mathbb{R}^d$, i.e., in action angle coordinates $(I, x) \in U \times \mathbf{T}^d$ from Condition (I) we have $\Lambda = \{I\} \times \mathbf{T}^d$ for a fixed $I \in U$. Let $\psi \in I_0(\Lambda)$ and* $\mathrm{Op}[a] \in \Psi^0$, *and consider the Fourier expansion of the principal symbols, $\sigma(a)|_\Lambda(x) = \sum_{m\in\mathbb{Z}^d} \alpha_m \mathrm{e}^{\mathrm{i}\langle m,x\rangle}$, $|\sigma(\psi)|^2(x) = \sum_{m\in\mathbb{Z}^d} \beta_m \mathrm{e}^{\mathrm{i}\langle m,x\rangle}|\mathrm{d}x|$. Then there are constants $C > 0$, $\beta > 0$ such that*

$$\left| \langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle - \sum_{m\in\mathbb{Z}^d} \alpha_m \beta_{-m} \mathrm{e}^{\mathrm{i}\langle m,\omega(I)\rangle\, t} \right| \leq C\hbar(1 + |t|)^\beta . \tag{17}$$

So in this case expectation values do not converge at all, but keep on oscillating.

If Λ is transversal to the foliation into invariant tori $\mathcal{T}$, then the situation changes. The tori carry a natural invariant density $|\mathrm{d}x|$, which can be combined with a density $|\sigma(\psi)|^2$ on Λ to give a density on Ω. By the transversality assumption there exist local symplectic coordinates $(I, x) \subset U \times V$ such that $\Lambda = \{(I, 0) , I \in U\}$ and the sets $\{(I_0, x) , x \in V\}$ belong to invariant tori. In these coordinates the modulus square of the principal symbol can be written as $|\sigma(\psi)|^2 = |\hat{\rho}(I)|^2|\mathrm{d}I|$ and we define

$$\mu_{\psi,\mathcal{T}} := |\hat{\rho}(I)|^2 |\mathrm{d}I \wedge \mathrm{d}x| . \tag{18}$$

Theorem 3. *Assume Conditions (H), (O) and (I) are fulfilled, and that the system is nondegenerate, i.e.,* $\det \omega'(I) \neq 0$ *on U. If Λ is transversal to the foliation into invariant tori, then for $\psi \in I_0(\Lambda)$ and* $\mathrm{Op}[a] \in \Psi^0$ *there exist constants $C, c > 0$, $\beta > 0$ such that*

$$\left| \langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle - \int \sigma(a)\ \mu_{\psi,T} \right| \leq C\hbar(1 + |t|)^\beta + c\frac{1}{1 + |t|} , \tag{19}$$

where $\mu_{\psi,\mathcal{T}}$ is the density defined in (18).

So in this case we get convergence of the expectation value, but the limit depends strongly on the initial state. Integrating against the density $\mu_{\psi,\mathcal{T}}$ means that we take the mean over each invariant torus, and then integrate these contributions weighted with the principal symbol of the state. This means that the knowledge of the limit density $\mu_{\psi,\mathcal{T}}$ allows to determine the foliation into invariant tori, and the distribution of the mass of the initial state across the tori.

In case of a chaotic system the situation is different. The only information on the initial state which survives is the information on how its mass is distributed among the energy shells. All other information is lost, and so we have the same degree of universality as in the classical system.

Another difference with the Anosov case is that we can control the time evolution for larger time scales, $t \leq 1/\hbar^{\beta-\varepsilon}$, for $\varepsilon > 0$. Since we mainly wanted to give a contrast to the main result in Theorem 1 we have not tried to obtain the optimal bounds on the time scales, for which one probably would need other methods.

The organisation of the paper is as follows. In Sect. 2 we reduce the quantum mechanical problem to one in classical mechanics, here the limitations on the time range occur. In Sect. 3 we extend previous results on mixing in Anosov systems and use them to prove Theorem 1. In Sect. 4 we discuss the integrable case and give proofs of Theorems 2 and 3.

2. Reduction to Classical Dynamics

Our aim in this section is to reduce the quantum mechanical problem to a problem in classical mechanics. This is obtained in

Proposition 1. *Assume Conditions (H) and (O), and let $\Lambda \subset \Omega$ be a Lagrangian manifold, $\psi \in I_0(\Lambda)$ and $\mathrm{Op}[a] \in \Psi^0(M)$. Then there exists a constant $\Gamma > 0$, independent of Λ and a, and $C > 0$ such that*

$$\left| \langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle - \int_\Lambda \sigma(a) \circ \Phi^t \, |\sigma(\psi)|^2 \right| \leq C\hbar \mathrm{e}^{\Gamma|t|} \, . \tag{20}$$

When Condition (I) is fulfilled in addition then there exists a constant $\beta > 0$ and $C > 0$ such that

$$\left| \langle \mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle - \int_\Lambda \sigma(a) \circ \Phi^t \, |\sigma(\psi)|^2 \right| \leq C\hbar (1+|t|)^\beta \, . \tag{21}$$

The first step in the proof of this proposition is the following simple lemma. Here and in the following $|\cdot|_\infty$ will denote the sup-norm.

Lemma 1. *Let $\psi \in I_0(\Lambda)$ be a Lagrangian state with compact support on M, then there exists $C > 0$ and an integer $k > 0$ such that for all $\mathrm{Op}[a] \in \Psi^0(M)$,*

$$\left| \langle \psi, \mathrm{Op}[a]\psi\rangle - \int_\Lambda \sigma(a) \, |\sigma(\psi)|^2 \right| \leq C \sum_{|\beta|\leq k} |\partial^\beta a|_\infty \, \hbar. \tag{22}$$

This is a standard result which follows from the results about application of pseudodifferential operators on Lagrangian states, see e.g. [Hör94, BW97], we have only made the dependence on a of the right-hand side more explicit. Since this lemma is an application of the method of stationary phase, the remainder follows from the remainder estimates in this method, see [Hör90].

The second ingredient in the proof of Proposition 1 is an Egorov theorem which is valid up to Ehrenfest time. The problem of time evolution of observables with remainder estimates uniform in time has been studied by Ivrii and Kachalkina in [Ivr98, Chap. 2.3]. Independently [BGP99] obtained a proof of the validity of Egorov up to Ehrenfest time for analytic observables and Hamiltonians. These results were then extended in the work of Bouzouina and Robert, [BR02]. In the formulation of the result we need the notion of essential support of an operator $\mathrm{Op}[a] \in \Psi^0(M)$. Recall that $z \in T^*M$ is not in the essential support of $\mathrm{Op}[a]$ if there is a neighbourhood U of z such that $|a(z)| \leq C_N \hbar^N$ for all $N \in \mathbb{N}$ and $z \in U$. So $\mathrm{Op}[a]$ is semiclassically negligible outside of its essential support.

Theorem 4 ([BR02]). *Assume Conditions (H) and (O). Then there exists a constant* $\Gamma_1 > 0$ *such that for any* $\mathrm{Op}[a] \in \Psi^0(M)$ *with essential support in* Ω *there is a* $C > 0$ *such that*

$$\|\mathcal{U}(t)^* \operatorname{Op}[a]\mathcal{U}(t) - \operatorname{Op}[a \circ \Phi^t]\| \leq C\hbar \mathrm{e}^{\Gamma_1 t} \ . \tag{23}$$

A much stronger version of this theorem was proved for $M = \mathbb{R}^n$ in [BR02], but the generalisation of their result to manifolds is complicated since the higher order terms of the symbol are not invariantly defined on T^*M. But we only need the leading order term, i.e. the principal symbol, and since this is a function on T^*M the result generalises to the case of manifolds.

In case of integrable systems we will use instead the stronger Theorem 1.13 from [BR02].

Theorem 5 ([BR02]). *Assume Conditions (H), (O) and (I), then for every* $\mathrm{Op}[a] \in \Psi^0(M)$ *with essential support in* Ω *there exist constants* $C > 0$ *and* $\beta_d \leq 5d + 4$ *such that*

$$\|\mathcal{U}(t)^* \operatorname{Op}[a]\mathcal{U}(t) - \operatorname{Op}[a \circ \Phi^t]\| \leq C\hbar (1 + |t|)^{\beta_d} \ . \tag{24}$$

We can now conclude the proof of Proposition 1.

Proof (Proposition 1). We will first assume that the essential support of $\mathrm{Op}[a]$ is contained on Ω. Then by Theorem 4 we have that

$$|\langle \mathcal{U}(t)\psi, \operatorname{Op}[a]\mathcal{U}(t)\psi\rangle - \langle \psi, \operatorname{Op}[a \circ \Phi^t]\psi\rangle| \leq C\hbar \mathrm{e}^{\Gamma_1 |t|} \ , \tag{25}$$

and Lemma 1 gives

$$\left| \langle \psi, \operatorname{Op}[a \circ \Phi^t]\psi, \rangle - \int_\Lambda \sigma(a) \circ \Phi^t |\sigma(\psi)|^2 \right| \leq C \sum_{|\beta| \leq k} |\partial^\beta (a \circ \Phi^t)|_\infty \ \hbar \ . \tag{26}$$

But as is well known, $\sum_{|\alpha|\leq k} |\partial^\alpha (a \circ \Phi^t)|_\infty \leq C \mathrm{e}^{\Gamma_2 |t|} \sum_{|\alpha| \leq k} |\partial^\alpha a|_\infty$ for some $\Gamma_2 > 0$, see e.g. [BR02, Lemma 2.4], and combining these estimates gives (20) with $\Gamma = \max\{\Gamma_1, \Gamma_2\}$. For the proof of Eq. (21) we use Theorem 5 together with Lemma 1 to get

$$\begin{aligned} &\left| \langle \mathcal{U}(t)\psi, \operatorname{Op}[a]\mathcal{U}(t)\psi\rangle - \int_\Lambda \sigma(a) \circ \Phi^t |\sigma(\psi)|^2 \right| \\ &\quad \leq C\hbar (1 + |t|)^{\beta_d} + C' \sum_{|\alpha| \leq k} |\partial^\alpha (a \circ \Phi^t)|_\infty \ \hbar \end{aligned} \tag{27}$$

and with the estimate $\sum_{|\alpha|\leq k}|\partial^\alpha(a\circ\Phi^t)|_\infty \leq C''\sum_{|\alpha|\leq k}|\partial^\alpha a|_\infty(1+|t|)^{\beta'_d}$, see [BR02, Lemma 4.2], the proof is complete if we take $\beta = \max\{\beta_d, \beta'_d\}$.

We finally show that we can reduce the case of an arbitrary observable $\mathrm{Op}[a] \in \Psi^0(M)$ to the case of observables with essential support in Ω. Let $I_0 := H_0^{-1}(\Omega)$ and $I_1 := H_0^{-1}(\mathrm{supp}(\sigma(\psi)))$, where H_0 is the principal symbol of $\mathcal{H}$, be the energy-ranges of Ω and the support of $\sigma(\psi)$ on Λ, respectively. Then I_0 is an open interval, I_1 is a closed interval with $I_1 \subset I_0$, and so there exists a function $f \in C_0^\infty(I_0)$ with $f|_{I_1} \equiv 1$. Then by the functional calculus, see [DS99], the operator $f(\mathcal{H})$ is in $\Psi^0(M)$, has essential support in Ω, commutes with $\mathcal{U}(t)$, and satisfies $\|f(\mathcal{H})\psi - \psi\| \leq C\hbar$. Therefore

$$|\langle\mathcal{U}(t)\psi, \mathrm{Op}[a]\mathcal{U}(t)\psi\rangle - \langle\mathcal{U}(t)\psi, f(\mathcal{H})\,\mathrm{Op}[a]\mathcal{U}(t)\psi\rangle| \leq C\hbar\ , \tag{28}$$

and since the essential support of $f(\mathcal{H})\,\mathrm{Op}[a]$ is contained in Ω we are done. □

3. Chaotic Systems

By Proposition 1 the proof of Theorem 1 is now reduced to the study of

$$\int_\Lambda \sigma(a)\circ\Phi^t\,|\sigma(\psi)|^2, \tag{29}$$

and this expression is very similar to a correlation function like in (1). The only difference is that the density ρ is replaced by a density concentrated on the submanifold Λ. Our aim in this section is to extend existing results on mixing of Anosov flows to this modified correlation functions. It is clear that we need a condition on the manifold Λ, as the example of a weakly stable manifold shows. Because if Λ is the weakly stable manifold of a periodic trajectory, then the mass of a will become more and more concentrated on that trajectory and will not become equidistributed. This example will be discussed in more detail at the end of this section.

Recall that a function a on a set X with metric $d(x, y)$ is Hölder continuous with Hölder exponent $\alpha \in (0, 1)$ if $|a(x) - a(y)| \leq Cd(x, y)^\alpha$ and the smallest constant C is called a Hölder constant $|a|_\alpha$. The set of Hölder continuous functions on a set X will be denoted by $C^\alpha(X)$. Following the usual conventions we will fix a metric on the energy shell Σ_E, which then in turn induces metrics on submanifolds of Σ_E.

We will rely mainly on Liverani's recent result on mixing for contact Anosov flows, [Liv04]. He shows that for any $\alpha \in (0, 1)$ there exist constants $C, \gamma > 0$ such that for $a, b \in C^\alpha(\Sigma)$ one has

$$\left|\int a\circ\Phi^t\,b\,\mathrm{d}\mu - \int a\,\mathrm{d}\mu\int b\,\mathrm{d}\mu\right| \leq C|a|_\alpha|b|_\alpha\mathrm{e}^{-\gamma t}\ . \tag{30}$$

Quantitative results on the decay of correlations for Anosov flows are rather recent, the main results prior to [Liv04] were obtained by Chernov [Che98] and Dolgopyat [Dol98], see the introduction of [Liv04] for more details on the history of this problem. Since the restriction of a Hamiltonian flow to an energy shell is a contact flow, the result of Liverani applies to the systems we are interested in.

We want to extend the result of Liverani to the case that one of the functions in the correlation integral is a density concentrated on a smooth submanifold. Such results have been obtained previously for geodesic flows on manifolds of negative curvature with certain measures concentrated on the unstable manifolds by Sinai and Chernov.

Sinai showed in [Sin95] that mixing holds and Chernov, [Che97], showed that the correlations decay at least like $\mathrm{e}^{-\gamma\sqrt{t}}$. On manifolds of constant negative curvature Eskin and McMullen, [EM93], derived mixing if one of the functions is concentrated on certain submanifolds. They reduced this to the classical mixing results for functions by using the hyperbolicity of the flow. We will follow their approach, where the only additional difficulty coming in is that the stable foliation is no longer smooth but only Hölder continuous if the curvature is no longer constant. To overcome this we use the absolute continuity property of the stable foliation.

In the following we will assume that non-vanishing smooth densities σ_Λ and σ_Γ have been fixed on the submanifolds Λ and Γ, so that every density can be written as $\sigma = \hat{\sigma}\sigma_\lambda$ or $\sigma = \hat{\sigma}\sigma_\Gamma$. We say then that $\sigma \in C^\alpha(\Lambda)$ if $\hat{\sigma} \in C^\alpha(\Lambda)$ and analogously $\sigma \in C^\alpha(\Gamma)$ if $\hat{\sigma} \in C^\alpha(\Gamma)$.

Theorem 6. *Let S be a symplectic manifold of dimension $2d$, and $\Phi^t : S \to S$ be a Hamiltonian flow on S with Hamilton-function $H \in C^\infty(S)$. Denote by $\Sigma_E := \{z \in S\,;\, H(z) = E\}$ the energy shell with energy E and by $\mathrm{d}\mu_E$ the Liouville measure on Σ_E. Assume Σ_E is compact and connected, and Φ^t is Anosov on Σ_E and the stable foliation has Hölder exponent α.*

(i) Let $\Lambda \subset \Sigma_E$ be a d-dimensional submanifold which is transversal to the stable foliation of Σ_E except on a subset of codimension at least 1. Then there exist $\gamma_1 > 0$ and for every density $\sigma \in C_0^\alpha(\Lambda)$ a constant C_1 such that for every function $a \in C^\alpha(\Sigma_E)$ we have

$$\left| \int_\Lambda a \circ \Phi^t \, \sigma - \int_{\Sigma_E} a \, \mathrm{d}\mu_E \int_\Lambda \sigma \right| \leq C_1 |a|_\alpha \mathrm{e}^{-\gamma_1 t}. \tag{31}$$

(ii) Let $\Gamma \subset \Sigma_E$ be a $(d-1)$-dimensional submanifold which is transversal to the weakly-stable foliation of Σ_E, except on a subset of codimension at least 1. Then there exist $\gamma_2 > 0$ and for every density $\sigma \in C_0^\alpha(\Gamma)$ a C_2 such that for every function $a \in C^\alpha(\Sigma_E)$ we have

$$\left| \int_\Gamma a \circ \Phi^t \, \sigma - \int_{\Sigma_E} a \, \mathrm{d}\mu_E \int_\Gamma \sigma \right| \leq C_2 |a|_\alpha \mathrm{e}^{-\gamma_2 t}. \tag{32}$$

(iii) Let $\Lambda \subset S$ be a d-dimensional submanifold and assume that the flow is Anosov on all Σ_E with $\Sigma_E \cap \Lambda \neq \emptyset$. Assume furthermore that for all these E $\Lambda \cap \Sigma_E$ is transversal to the weakly stable foliation of Σ_E, except on a subset of codimension at least one in $\Lambda \cap \Sigma_E$. Then there exist $\gamma_3 > 0$ and for every density $\sigma \in C_0^\alpha(\Lambda)$ a constant C_3 such that for every function $a \in C_0^\alpha(S)$ we have

$$\left| \int_\Lambda a \circ \Phi^t \, \sigma - \int \int_{\Sigma_E} a \, \mathrm{d}\mu_E \int_{\Lambda \cap \Sigma_E} \sigma_E \, \mathrm{d}E \right| \leq C_3 |a|_\alpha \mathrm{e}^{-\gamma_3 t}\ , \tag{33}$$

where σ_E is a density on $\Lambda \cap \Sigma_E$ defined by $\sigma = \sigma_E \otimes |\mathrm{d}E|$.

Proof. In order to prove (i), we will relate the behaviour of

$$\int_\Lambda a \circ \Phi^t \, \sigma \tag{34}$$

to the behaviour of the standard correlation function

$$\int_{\Sigma_E} a \circ \Phi^t \rho \, \mathrm{d}\mu_E, \tag{35}$$

where $\rho \in C^\alpha(\Sigma_E)$ is supported in a neighbourhood of Λ. The heuristic idea is that since a neighbourhood of Λ converges exponentially fast along the stable manifolds to Λ, the integral (35) will become close to the integral (34) for appropriately chosen ρ. But to (35) we can then apply the result (30) by Liverani.

We will formalise this idea now and treat first the case that Λ is transversal to the stable foliation. By using a partition of unity we can assume that the support of σ is in a small compact set $\Lambda_0 \subset \Lambda$, such that there is a neighbourhood $\hat{\Lambda}_0 \subset \Sigma_E$ of Λ_0 in Σ_E in which we can choose coordinates $(x, y) \in U \times W \subset \mathbb{R}^d \times \mathbb{R}^{d-1}$ with the property that $\Lambda = \{(x, 0), x \in U\}$ and $W^s(x) = \{(x, y); y \in W\}$. This is where we use the transversality assumption. Since Λ_0 is compact, W can be chosen to be bounded. Notice that since the stable foliation is usually only Hölder continuous, the transformation to this coordinate system is only Hölder continuous, too. But Anosov showed that the stable foliation is absolutely continuous, which means that there is a measurable function $\delta_x(y)$ (basically the Jacobian of the holonomy associated with the stable foliation) which depends measurably on x and satisfies $1/C < \delta_x(y) < C$ for some $C > 0$ and all $(x, y) \in U \times W$, such that

$$\int_{\Sigma_E} a \circ \Phi^t \rho \, \mathrm{d}\mu_E = \int_U \int_W \rho(x, y) a \circ \Phi^t(x, y) \delta_x(y) \, \mathrm{d}y \mathrm{d}x \ , \tag{36}$$

where we have assumed that ρ is supported in $U \times W$, see [BS02, Chap. 6.2]. Furthermore, the dependence of $\delta_x(y)$ on x is Hölder continuous, see Eq. (A.3) in [Liv04]. We will now show that ρ can be chosen to be in $C^\alpha(\Sigma_E)$ and such that

$$\int_W \rho(x, y) \delta_x(y) \, \mathrm{d}y = \hat{\sigma}(x), \tag{37}$$

where $\sigma(x) = \hat{\sigma}(x)\mathrm{d}x$. To this end, set $\rho(x, y) = \rho_1(x)\rho_2(x, y)\hat{\sigma}(x)$ with $\rho_2(x, y) > 0$ on Λ_0, Hölder and supported in $\hat{\Lambda}_0$, and with

$$\rho_1(x) = \left(\int_W \rho_2(x, y) \delta_x(y) \, \mathrm{d}y \right)^{-1} \tag{38}$$

on Ω. Then ρ_1 is Hölder, since $\delta_x(y)$ is bounded and depends Hölder continuously on x, and so $\rho(x, y)$ is Hölder and satisfies (37) by construction.

By Hölder continuity, and since W is bounded, we get now

$$|a \circ \Phi^t(x, y) - a \circ \Phi^t(x, 0)| \leq C|a|_\alpha d(\Phi^t(x, y), \Phi^t(x, 0))^\alpha \leq C'|a|_\alpha \mathrm{e}^{-\alpha\gamma t} \ , \tag{39}$$

since the flow is contracting along the stable leaves, i.e., $d(\Phi^t(x, y), \Phi^t(x, 0)) \leq C\mathrm{e}^{-\gamma t}$ for some constants $C, \gamma > 0$. Therefore we obtain with (37),

$$\begin{aligned} &\left| \int_U \int_W \rho(x, y) a \circ \Phi^t(x, y) \delta_x(y) \, \mathrm{d}y \mathrm{d}x - \int_U \int_W \rho(x, y) a \circ \Phi^t(x, 0) \delta_x(y) \, \mathrm{d}y \mathrm{d}x \right| \\ &\quad \leq C'|a|_\alpha \int_U |\hat{\sigma}(x)| \, \mathrm{d}x \, \mathrm{e}^{-\alpha\gamma t} \end{aligned} \tag{40}$$

and

$$\int_U \int_W \rho(x,y) a \circ \Phi^t(x,0) \delta_x(y) \, \mathrm{d}y \mathrm{d}x = \int_U a \circ \Phi^t(x,0) \hat{\sigma}(x) \, \mathrm{d}x = \int_\Lambda a \circ \Phi^t \, \sigma \ . \tag{41}$$

On the other hand we have by (30),

$$\left| \int_{\Sigma_E} a \circ \Phi^t \rho \, \mathrm{d}\mu_E - \int_{\Sigma_E} \rho \, \mathrm{d}\mu_E \int_{\Sigma_E} a \, \mathrm{d}\mu_E \right| \leq C |a|_\alpha |\rho|_\alpha \mathrm{e}^{-\gamma' t}, \tag{42}$$

and by (37) ,

$$\int_{\Sigma_E} \rho \, \mathrm{d}\mu_E = \int_\Lambda \sigma \ , \quad |\rho|_\alpha \leq C_\Lambda \, |\hat{\sigma}|_\alpha, \tag{43}$$

so finally we obtain

$$\left| \int_\Lambda a \circ \Phi^t \sigma - \int_\Lambda \sigma \int_{\Sigma_E} a \ \mathrm{d}\mu_E \right| \leq C(|\sigma|_\alpha + ||\sigma||_{L^1(\Lambda)}) |a|_\alpha \mathrm{e}^{-\gamma t}. \tag{44}$$

This completes the proof of (i) in case the manifolds are transversal.

We will now extend this result to the non-transversal case. Let $\Lambda_{sing} = \{x \in \Lambda; \dim T_x \Lambda \cap T_x W^s(x) \geq 1\}$ be the set of point on Λ where the intersection is not transversal, and define $\Lambda_{sing,\varepsilon} := \{x \in \Lambda; d(x, \Lambda_{sing}) \leq \varepsilon\}$. Choose $\varphi_\varepsilon \in C^\alpha(\Lambda)$ with $\operatorname{supp} \varphi_\varepsilon \subset \Lambda_{sing,\varepsilon}$ and $\varphi_\varepsilon \equiv 1$ on $\Lambda_{sing,\varepsilon/2}$. Then

$$\left| \int_\Lambda \varphi_\varepsilon a \circ \Phi^t |\sigma(\psi)|^2 \right| \leq C|a| \varepsilon^{d - d_{sing}} \ , \tag{45}$$

where d_{sing} is the dimension of Λ_{sing}.

To the integral $\int_\Lambda (1-\varphi_\varepsilon) a \circ \Phi^t |\sigma(\psi)|^2$ we can apply the previous results, we only have to pay attention to the ε-dependence of the constants. The second estimate in (43) has to be refined. By the definition of ρ we have $|\rho(1-\varphi_\varepsilon)|_\alpha \leq |\rho_1(1-\varphi_\varepsilon)|_\alpha |\rho_2|_\alpha |\hat{\sigma}|_\alpha$ and since the Jacobian $\delta_y(x)$ becomes degenerate when x approaches Λ_{sing} we get

$$|\rho_1(1-\varphi_\varepsilon)|_\alpha \leq C \varepsilon^{-\gamma'}, \tag{46}$$

where $\gamma' > 0$ depends on α and d_{sing}. Collecting the estimates yields

$$\left| \int_\Lambda a \circ \Phi^t \sigma - \int_\Lambda \sigma \int_{\Sigma_E} a \mathrm{d}\mu_E \right| \leq C \varepsilon^{-\gamma'} (|\sigma|_\alpha + ||\sigma||_{L^1(\Lambda)}) |a|_\alpha \mathrm{e}^{-\gamma t} + C' |a| \varepsilon^{d - d_{sing}}, \tag{47}$$

and choosing $\varepsilon = \mathrm{e}^{-\gamma'' t}$ with $\gamma'' = \gamma/(\gamma' + (d - d_{sing}))$ gives

$$\left| \int_\Lambda a \circ \Phi^t \sigma - \int_\Lambda \sigma \int_{\Sigma_E} a \ \mathrm{d}\mu_E \right| \leq C(|a|_\alpha + |a|) \ \mathrm{e}^{-\gamma_1 t} \tag{48}$$

with $\gamma_1 = \gamma(d - d_{sing})/(\gamma' + (d - d_{sing}))$.

The proof of (ii) is based on (i). Define for some $\delta > 0$ $\Lambda := \bigcup_{|t|<\delta} \Phi^t(\Gamma) \subset \Sigma_E$, then Λ is transversal to the stable foliation except on a subset of codimension at least one. If $s \in U \subset \mathbb{R}^{d-1}$ are local coordinates on Γ, then (r, s) $|r| < \delta$ are local coordinates on Λ. Let ρ be a smooth function with compact support in $|r| < \delta$, $\int \rho(r)\, \mathrm{d}r = 1$, and define $\rho_\varepsilon(r) := \frac{1}{\varepsilon}\rho(\varepsilon r)$. If we write $\sigma = \hat{\sigma}(s)\,\mathrm{d}s$ and $\sigma_\varepsilon := \hat{\sigma}(s)\rho_\varepsilon(r)\mathrm{d}s\mathrm{d}r$, we have

$$\left| \int_\Gamma a \circ \Phi^t\, \sigma - \int_\Lambda a \circ \Phi^t\, \sigma_\varepsilon \right| = \left| \int_U a(t, s)\, \hat{\sigma}\, \mathrm{d}s - \int_U \int_{\mathbb{R}} a(r + t, s)\, \rho_\varepsilon(r)\hat{\sigma}(s)\, \mathrm{d}r\mathrm{d}s \right|$$
$$\leq \int_U \int_{\mathbb{R}} \rho_\varepsilon(r)|a(t, s) - a(r + t, s)|\, \mathrm{d}r\, \hat{\sigma}(s)\, \mathrm{d}s \tag{49}$$

but

$$\int_{\mathbb{R}} \rho_\varepsilon(r)|a(t, s) - a(r + t, s)|\, \mathrm{d}r = \int_{\mathbb{R}} \rho(r)|a(t, s) - a(\varepsilon r + t, s)|\, \mathrm{d}r \leq C|a|_\alpha \varepsilon^\alpha, \tag{50}$$

and therefore

$$\left| \int_\Gamma a \circ \Phi^t\, \sigma - \int_\Lambda a \circ \Phi^t\, \sigma_\varepsilon \right| \leq C||\sigma||_{L^1(\Sigma)}|a|_\alpha \varepsilon^\alpha\ . \tag{51}$$

On the other hand with $|\sigma_\varepsilon|_\alpha \leq C|\sigma|_\alpha \varepsilon^{\alpha-1}$ and $||\sigma_\varepsilon||_{L^1(\Lambda)} = ||\sigma||_{L^1(\Gamma)}$ we obtain from (i) that

$$\left| \int_\Lambda a \circ \Phi^t\, \sigma_\varepsilon - \int_{\Sigma_E} a\, \mathrm{d}\mu_E \int_\Gamma \sigma \right| \leq C|a|_\alpha(|\sigma|_\alpha \varepsilon^{\alpha-1} + ||\sigma||_{L^1(\Gamma)})\mathrm{e}^{-\gamma_1 t}\ . \tag{52}$$

If we now choose $\varepsilon = \mathrm{e}^{-\gamma' t}$ with $\gamma' > 0$ and $(1 - \alpha)\gamma' > \gamma_1$, the proof of (ii) is complete.

Part (iii) then follows immediately by writing

$$\int_\Lambda a \circ \Phi^t \sigma = \int \int_{\Lambda \cap \Sigma_E} a \circ \Phi^t \sigma_E\, \mathrm{d}E, \tag{53}$$

and applying (ii) to the integral over $\Lambda \cap \Sigma_E$ on the right-hand side. □

Theorem 1 is now a straightforward consequence of Proposition 1 and Theorem 6.

Let us end this section by discussing the meaning of the transversality condition. Let us first look at the example that Λ is the stable manifold of an periodic orbit γ with period T_γ. Let $(r, x) \in S^1 \times \mathbb{R}^{d-1}$ be coordinates on Λ such that γ is given by $x = 0$ and $\Phi^t(r, x) = (r + t \mod T_\gamma, x(t))$, then

$$\int_\Lambda a \circ \Phi^t\, \sigma = \int_0^{T_\gamma} \int_{\mathbb{R}^{d-1}} a(r + t, x(t))\hat{\sigma}(r, x)\, \mathrm{d}r\mathrm{d}x\ . \tag{54}$$

With $|a(r+t, x(t)) - a(r+t, 0)| \leq C\mathrm{e}^{-\gamma t}$ and by inserting the Fourier series $a(r, 0) = \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\frac{2\pi}{T_\gamma}\mathrm{i}kr}$ we obtain

$$\int_\Lambda a \circ \Phi^t\, \sigma = \sum_{k\in\mathbb{Z}} a_k \tilde{\sigma}_k \mathrm{e}^{\frac{2\pi}{T_\gamma}\mathrm{i}t} + O(\mathrm{e}^{-\gamma t}) \tag{55}$$

with $\tilde{\sigma}_k = \int_0^{T_\gamma} \int_{\mathbb{R}^{d-1}} \hat{\sigma}(r, x)\, \mathrm{d}x\, \mathrm{e}^{\frac{2\pi}{T_\gamma}\mathrm{i}kr}\, \mathrm{d}r$. So in this case we do not get convergence for large times, and together with Proposition 1 this gives (15). This example shows that some condition on the position of Λ with respect to the stable foliation is necessary.

4. Integrable Systems

In this section we give the proofs of Theorem 2 and Theorem 3 and discuss the situation for integrable systems.

Proof (Theorem 2). By Proposition 1 we have to study the behaviour of

$$\int_\Lambda \sigma(a)\circ\Phi^t\,|\sigma(\psi)|^2\ , \tag{56}$$

for large t. In action angle coordinates $(I,x)\in U\times\mathbf{T}^d$ we have $\Lambda=\{(I,x), x\in V\subset\mathbf{T}^d\}$, for a fixed $I\in U$, and so with $|\sigma(\psi)|^2=|\rho(x)|^2|\mathrm{d}x|$ we get

$$\int_\Lambda \sigma(a)\circ\Phi^t\,|\sigma(\psi)|^2=\int_{\mathbf{T}^d}\sigma(a)(I,x+t\omega(I))|\rho(x)|^2\,\mathrm{d}x\ . \tag{57}$$

If we insert now the Fourier expansion in x, $\sigma(a)(I,x)=\sum_{m\in\mathbb{Z}^d}\alpha_m(I)\mathrm{e}^{\mathrm{i}\langle m,x\rangle}$, we obtain,

$$\int_\Lambda \sigma(a)\circ\Phi^t\,|\sigma(\psi)|^2=\sum_{m\in\mathbb{Z}^d}\alpha_m(I)\int_{\mathbf{T}^d}\mathrm{e}^{\mathrm{i}\langle x,m\rangle}|\rho(x)|^2\,\mathrm{d}x\,\mathrm{e}^{\mathrm{i}t\langle\omega(I),m\rangle}\ , \tag{58}$$

which is Eq. (17) in Theorem 2. □

Proof (Theorem 3). In order to prove Eq. (19) we notice that the transversality assumption on Λ with respect to the foliation in invariant tori implies that action angle coordinates $(I,x)\subset U\times V$ can be chosen such that Λ can be locally represented by a generating function $\varphi:U\to\mathbb{R}$,

$$\Lambda=\{(I,\varphi'(I))\,,\ I\in U\}. \tag{59}$$

Therefore we have

$$\int_\Lambda \sigma(a)\circ\Phi^t\,|\sigma(\psi)|^2=\int_U\sigma(a)(I,\varphi'(I)+t\omega(I))|\hat\rho(I)|^2\,\mathrm{d}I\ , \tag{60}$$

where we have written $|\sigma(\psi)|^2=|\hat\rho(I)|^2|\mathrm{d}I|$. Inserting for $\sigma(a)$ again the Fourier expansion in x leads to

$$\int_\Lambda \sigma(a)\circ\Phi^t\,|\sigma(\psi)|^2=\sum_{m\in\mathbb{Z}^d}\int_U\alpha_m(I)\mathrm{e}^{\mathrm{i}\langle m,\varphi'(I)\rangle}\mathrm{e}^{\mathrm{i}t\langle m,\omega(I)\rangle}|\hat\rho(I)|^2\,\mathrm{d}I\ . \tag{61}$$

The non-degeneracy condition $\det\omega'(I)\neq 0$ implies that there exist a constant $C>0$,

$$|\nabla_I\langle\omega(I),m\rangle|\geq C|m|\ , \tag{62}$$

for all $I\in\operatorname{supp}\hat\rho$. Now by the non-stationary phase estimates, see, e.g., [Hör90, Theorem 7.7.1], one gets

$$\left|\int_U\alpha_m(I)\mathrm{e}^{\mathrm{i}\langle m,\varphi'(I)\rangle}\mathrm{e}^{\mathrm{i}t\langle m,\omega(I)\rangle}|\hat\rho(I)|^2\,\mathrm{d}I\right|\leq C|m||\alpha_m|_1|\rho|_1^2\,\frac{1}{1+|t|} \tag{63}$$

for $m \neq 0$. Since $\sigma(a)$ is C^∞ the Fourier-coefficients satisfy $|\alpha_m|_1 = O_N(|m|^{-N})$, for all $N \in \mathbb{N}$, and therefore we finally obtain

$$\int_\Lambda \sigma(a) \circ \Phi^t \, |\sigma(\psi)|^2 = \int_U \alpha_0(I)|\hat{\rho}(I)|^2 \, \mathrm{d}I + O(1/t) \ . \tag{64}$$

But $\alpha_0(I) = \int \sigma(a)(I, x) \, \mathrm{d}x$ and so the proof of Theorem 3 is complete. □

There are a couple of directions in which one probably can extend and improve Theorems 2 and 3. We have only studied the two extreme cases of the position of Λ relative to the foliation into invariant tori. Certainly the transversal case is (locally) generic, but the case that the intersections are clean can be studied without much additional effort, one would expect an oscillatory behaviour in this case. It appears as well to be very interesting to investigate the behaviour of the time evolution close to singularities of the foliation into invariant tori.

Another direction where one can extend some of the results is to more general classes of systems. Namely by using normal forms around invariant tori in a general system one can extend Theorem 2 to that case. Such invariant tori occur typically in a situation described by KAM theory, e.g., for perturbed integrable systems, and close to elliptic orbits.

Acknowledgements. This work is a result of research during my stays at the SPhT in Saclay/Paris and the MSRI in Berkeley. I would like to thank Stéphane Nonnemacher, Boris Gutkin and Andre Voros for interest and support. Furthermore I thank Stephan De Bièvre and the referee for helpful comments. This work has been fully supported by the European Commission under the Research Training Network (Mathematical Aspects of Quantum Chaos) n$^{\mathrm{o}}$ HPRN-CT-2000-00103 of the IHP Programme.

References

[BB79] Berry, M.V., Balazs, N.L.: Evolution of semiclassical quantum states in phase space. J. Phys. A **12**(5), 625–642 (1979)

[BBTV79] Berry, M.V., Balazs, N.L., Tabor, M., Voros, A.: Quantum maps. Ann. Physics **122**(1), 26–63 (1979)

[BDB00] Bonechi, F., De Bièvre, S.: Exponential mixing and $|\ln \hbar|$ time scales in quantized hyperbolic maps on the torus. Commun. Math. Phys. **211**(3), 659–686 (2000)

[BGP99] Bambusi, D., Graffi, S., Paul, T.: Long time semiclassical approximation of quantum flows: a proof of the Ehrenfest time. Asymptot. Anal. **21**(2), 149–160 (1999)

[BR02] Bouzouina, A., Robert, D.: Uniform semiclassical estimates for the propagation of quantum observables. Duke Math. J. **111**(2), 223–252 (2002)

[BS02] Brin, M., Stuck, G.: *Introduction to dynamical systems*. Cambridge: Cambridge University Press, 2002

[BW97] Bates, S., Weinstein, A.: Lectures on the geometry of quantization. Berkeley Mathematics Lecture Notes, Vol. **8**, Providence, RI: Amer. Math. Soc. 1997

[BZ78] Berman, G. P., Zaslavsky, G. M.: Condition of stochasticity in quantum nonlinear systems. Phys. A **91**(3–4), 450–460 (1978)

[CdV85] Colin de Verdière, Y.: Ergodicité et fonctions propres du laplacien. Commun. Math. Phys. **102**(3), 497–502 (1985)

[Che97] Chernov, N. I.: On Sinai-Bowen-Ruelle measures on horocycles or 3-D Anosov flows. Geom. Dedicata **68**(3), 359–369 (1997)

[Che98] Chernov, N. I.: Markov approximations and decay of correlations for Anosov flows. Ann. of Math. (2) **147**(2), 269–324 (1998)

[CR97] Combescure, M., Robert, D.: Semiclassical spreading of quantum wave packets and applications near unstable fixed points of the classical flow. Asymptot. Anal. **14**(4), 377–404 (1997)

[DBR03] De Bièvre, S., Robert, D.: Semiclassical propagation on $|\log \hbar|$ time scales. Int. Math. Res. Not. Vol. **12**, 667–696 (2003)

[Dol98] Dolgopyat, D.: On decay of correlations in Anosov flows. Ann. of Math. (2) **147**(2), 357–390 (1998)
[DS99] Dimassi, M., Sjöstrand, J.: *Spectral asymptotics in the semi-classical limit*. London Mathematical Society Lecture Note Series, Vol. **268**, Cambridge: Cambridge University Press, 1999
[Dui74] Duistermaat, J. J.: Oscillatory integrals, Lagrange immersions and unfolding of singularities. Comm. Pure Appl. Math. **27** , 207–281 (1974)
[Ebe01] Eberlein, P.: Geodesic flows in manifolds of nonpositive curvature. In: *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, Proc. Sympos. Pure Math., Vol. **69**, Providence, RI: Amer. Math. Soc. 2001, pp. 525–571
[EM93] Eskin, A., McMullen, C.: Mixing, counting, and equidistribution in Lie groups. Duke Math. J. **71**(1), 181–209 (1993)
[FNDB03] Faure, F., Nonnenmacher, S., De Bièvre, S.: Scarred eigenstates for quantum cat maps of minimal periods. Commun. Math. Phys. **239**(3), 449–492 (2003)
[HMR87] Helffer, B., Martinez, A., Robert, D.: Ergodicité et limite semi-classique. Commun. Math. Phys. **109**(2), 313–326 (1987)
[Hör90] Hörmander, L.: The analysis of linear partial differential operators. I. second ed., Grundlehren der Mathematischen Wissenschaften, Vol. **256**, Berlin: Springer-Verlag 1990
[Hör94] Hörmander, L.: The analysis of linear partial differential operators. IV. Grundlehren der Mathematischen Wissenschaften, Vol. **275**, Berlin: Springer-Verlag, 1985
[Ivr98] Ivrii, V.: Microlocal analysis and precise spectral asymptotics. Springer Monographs in Mathematics, Berlin: Springer-Verlag, 1998
[Liv04] Liverani, C.: On contact Anosov flows. Ann. of Math. **159**, 1275–1312 (2004)
[OTH92] O'Connor, P. W., Tomsovic, S., Heller, E. J.: Semiclassical dynamics in the strongly chaotic regime: breaking the log time barrier. Phys. D **55**(3–4), 340–357 (1992)
[Sin95] Sinai, Ya. G.: Geodesic flows on manifolds of negative curvature. In: *Algorithms, fractals, and dynamics (Okayama/Kyoto, 1992)*, New York: Plenum, 1995, pp. 201–215
[Šni74] Šnirelʹman, A. I.: Ergodic properties of eigenfunctions. Uspehi Mat. Nauk **29**, 6(180), 181–182 (1974)
[TH91] Tomsovic, S., Heller, E. J.: Semiclassical dynamics of chaotic motion: unexpected long-time accuracy. Phys. Rev. Lett. **67**(6), 664–667 (1991)
[TH93] Tomsovic, S., Heller, E. J.: Long-time semiclassical dynamics of chaos: the stadium billiard. Phys. Rev. E (3) **47**(1), 282–299 (1993)
[Zas81] Zaslavsky, G. M.: Stochasticity in quantum systems. Phys. Rep. **80**(3), 157–250 (1981)
[Zel87] Zelditch, S.: Uniform distribution of eigenfunctions on compact hyperbolic surfaces. Duke Math. J. **55**(4), 919–941 (1987)
[Zel96] Zelditch, S.: Quantum mixing. J. Funct. Anal. **140**, 68–86 (1996)

Communicated by P. Sarnak