

Grid Classes and Partial Well Order

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Abstract

We prove necessary and sufficient conditions on a family of (generalised) gridding matrices to determine when the corresponding permutation classes are partially well-ordered. One direction requires an application of Higman's Theorem and relies on there being only finitely many simple permutations in the only non-monotone cell of each component of the matrix. The other direction is proved by a more general result that allows the construction of infinite antichains in any grid class of a matrix whose graph has a component containing two or more non-monotone-griddable cells. The construction uses a generalisation of pin sequences to grid classes, together with a number of symmetry operations on the rows and columns of a gridding.

1 Introduction

A partial order is *partially well-ordered* if it contains neither an infinite *antichain* (a set of pairwise incomparable elements) nor an infinite descending chain. In the study of classes of combinatorial structures this latter condition is trivially satisfied, thus such a class is partially well-ordered if and only if it contains no infinite antichain. For many combinatorial structures we have only a quasi-ordering rather than a partial ordering, and in this case we call such a class *well quasi-ordered* when it contains no infinite antichain. Celebrated results affirming well quasi-ordering in different contexts range from Kruskal's Tree Theorem [9] to the Robertson–Seymour Theorem [13] for minor-closed classes of graphs, but there are many known examples of quasi-orders that are not well quasi-ordered, such as hereditary properties of graphs. Higman's Theorem (reproduced here in Section 3) is almost the only general tool used to prove that a given quasi-order is well quasi-ordered, but attention has been given more recently to develop a general theory of infinite antichains — see, for example, Gustedt [6] and Cherlin and Latka [5].

In this paper we are concerned with permutations, though there is no particular reason why parts of these results cannot be extended to other structures. A sequence a_1, \dots, a_n of length n of distinct real numbers is said to be *order isomorphic* to another sequence b_1, \dots, b_n if, for all $i, j \in [n] = \{1, 2, \dots, n\}$, $a_i < a_j$ if and only if $b_i < b_j$. In this way every sequence of real

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numbers of length n is order isomorphic to some permutation π of length n : $a_i < a_j$ if and only if $\pi(i) < \pi(j)$. This order isomorphism induces the *containment* ordering on permutations: we say that a permutation α is *contained* in π , $\alpha \leq \pi$, if there is some subsequence of π order isomorphic to α . Such a subsequence of π is called a *copy* of α in π . Conversely, if π does not contain the permutation β , then π is said to *avoid* β . For example, $\pi = 918572346$ contains 51342 because of the subsequence 91572 ($= \pi(1)\pi(2)\pi(4)\pi(5)\pi(6)$), but avoids 3142.

The containment ordering on permutations defines a partial order on the set of all permutations. A *permutation class* is a set of permutations closed downward in this partial order, i.e. if π is a permutation in the class \mathcal{C} and $\alpha \leq \pi$, then $\alpha \in \mathcal{C}$. These classes have received a lot of attention in recent years, and the question of partial well-order has played a central role: there is a vast library of infinite antichains (see, in particular Murphy’s thesis [11]), while Higman’s Theorem has been applied in the other direction by Atkinson, Murphy and Ruškuc [2] and Albert and Atkinson [1].

The traditional description of a class \mathcal{C} is by the unique antichain B that forms its *basis*: we write $\mathcal{C} = \text{Av}(B)$ to mean $\mathcal{C} = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}$. However, in recent years a new description of permutation classes has arisen, namely “grid classes” of matrices whose entries are themselves permutation classes — for formal definitions see Section 2. These have played a role in the development of the “Fibonacci” and “Vatter” dichotomies [8, 14], providing a complete answer to the possible growth rates¹ of permutation classes below $\kappa \approx 2.20557$, and in particular proving that there are only countably many classes below this growth rate. Of particular relevance to this paper is Murphy and Vatter [12] where grid classes and partial well-order first met, and subsequent work in Waton’s thesis [15]. In this paper, we will prove the following:

Theorem 1.1. *Let \mathcal{M} be a gridding matrix whose entries are monotone classes, non-monotone grid-dable classes containing only finitely many simple permutations or empty. Then the permutation class $\text{Grid}(\mathcal{M})$ is partially well-ordered if and only if the graph of \mathcal{M} is a forest, and at most one cell in each component is not monotone.*

The bulk of the work in proving Theorem 1.1 is in showing:

Theorem 1.2. *A grid class $\text{Grid}(\mathcal{M})$ is not partially well-ordered if \mathcal{M} has a component that contains a cycle, or two or more cells that are not monotone grid-dable.*

After introducing the necessary definitions in Section 2, Section 3 presents Higman’s theorem and completes the proof of the right-to-left direction of Theorem 1.1; the remainder of the paper is devoted to proving Theorem 1.2. In Section 4 we introduce a number of symmetries of griddings which reduces the number of classes that have to be considered. In Section 5 we introduce a family of grid matrices and show that they are the only ones we need to consider, and in Section 6 we show that these classes are not partially well-ordered by constructing antichains that lie in them which satisfy the additional properties required by the symmetry arguments.

¹All permutation classes have an *upper growth rate*, $\overline{\text{gr}}(\mathcal{C}) = \limsup_{n \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n|}$ (see [10]) where \mathcal{C}_n is the set of permutations in \mathcal{C} of length n , but it is still not known in general whether the true *growth rate*, $\lim_{n \rightarrow \infty} \sqrt[n]{|\mathcal{C}_n|}$, exists for all permutation classes.

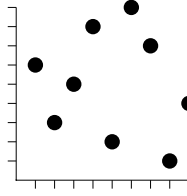


Figure 1: The plot of the permutation $\pi = 635829714$.

2 Definitions

As has become increasingly the case in the study of permutation patterns in recent years, it will prove very useful to view permutations and order isomorphism graphically. Two sets S and T of points in the plane are said to be order isomorphic if we can stretch and shrink the axes for the set S to map the points of S bijectively onto the points of T , i.e. if there are strictly increasing functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that $\{(f(s_1), g(s_2)) : (s_1, s_2) \in S\} = T$. Note that this forms an equivalence relation since the inverse of a strictly increasing function is also strictly increasing. The *plot* of the permutation π is the point set $\{(i, \pi(i))\}$, and every finite point set in the plane in which no two points share a coordinate (often called a *generic* or *noncorectilinear* set) is order isomorphic to the plot of a unique permutation (see Figure 1 for an example). Note that, with a slight abuse of notation, we will say that a point set is order isomorphic to a permutation.

Inflations and Simple Permutations. An *interval* of a permutation π corresponds to a set of contiguous indices $I = [a, b] = \{a, a + 1, \dots, b\}$ such that the set of values $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous. For example, $645 = \pi(345)$ is an interval in $\pi = 72645813$.

We form an *inflation* of σ by the permutations τ_1, \dots, τ_k by replacing the entry $\sigma(i)$ with an interval order isomorphic to τ_i , and denote it by $\sigma[\tau_1, \dots, \tau_k]$. For example, $2413[21, 312, 1, 12] = 32867145$. Two special cases of inflations are the *direct sum* $\tau_1 \oplus \tau_2 = 12[\tau_1, \tau_2]$ and the *skew sum* $\tau_1 \ominus \tau_2 = 21[\tau_1, \tau_2]$. A *lenient inflation* is an inflation $\sigma[\tau_1, \dots, \tau_k]$ where we allow one or more of the τ_ℓ to be empty. A class \mathcal{C} is *substitution-closed* (or, in some texts, *wreath-closed*) if $\sigma[\tau_1, \dots, \tau_k] \in \mathcal{C}$ for all $\sigma, \tau_1, \dots, \tau_k \in \mathcal{C}$. The *substitution closure* of a set X is the smallest substitution-closed class containing X , and is denoted $\langle X \rangle$.

A *simple permutation* is a permutation which has no non-trivial intervals, or equivalently a permutation which cannot be expressed as an inflation of some smaller non-singleton permutation. Conversely:

Proposition 2.1 (Albert and Atkinson [1]). *Every permutation except 1 can be expressed as the inflation of a unique simple permutation of length at least 2.*

This proposition shows how simple permutations can be thought of as the “building blocks” of all other permutations, and consequently they play an important role in the study of permutation classes and have received much attention in recent years — see [3] for a survey. We will denote by $\text{Si}(\mathcal{C})$ the set of simple permutations in the class \mathcal{C} . Note that $\text{Si}(\mathcal{C}) = \text{Si}(\langle \mathcal{C} \rangle)$, and also that $\langle \mathcal{C} \rangle = \langle \text{Si}(\mathcal{C}) \rangle$.

Grid Classes. We will present here only a brief survey of the necessary results, and refer the reader to Vatter [14] for a more complete treatment of this topic. To draw a parallel with

the way we view permutations graphically, we will index matrices and grids starting from the bottom-left corner, and with the order of indices swapped. In other words, the ij th entry of a matrix (respectively, ij th cell of a grid) corresponds to the entry (cell) in column i and row j , and an $m \times n$ matrix has m columns and n rows.

An $m \times n$ -gridding of a permutation π is a collection of $m - 1$ distinct vertical and $n - 1$ distinct horizontal lines that divide the plot of π into mn cells. A permutation equipped with a particular $m \times n$ -gridding is called an $m \times n$ -gridded permutation, and for such a gridded permutation π , π^{st} denotes the set of points contained in the st th cell.

Let \mathcal{M} be an $m \times n$ matrix where each entry is either an infinite permutation class or the empty class \emptyset . An \mathcal{M} -gridding of a permutation π is an $m \times n$ gridding of π such that π^{st} lies in the class \mathcal{M}_{st} for all $s \in [m]$ and $t \in [n]$. If π possesses an \mathcal{M} -gridding, then π is said to be \mathcal{M} -griddable, and equipping π with such a gridding gives rise to an \mathcal{M} -gridded permutation. Similarly, a permutation class \mathcal{C} is said to be \mathcal{M} -griddable if every $\pi \in \mathcal{C}$ is \mathcal{M} -griddable. The largest permutation class that is \mathcal{M} -griddable (i.e. the class consisting of all \mathcal{M} -griddable permutations) is called the *grid class* of \mathcal{M} , and is denoted $\text{Grid}(\mathcal{M})$. One special case that has received particular attention has been that of *monotone grid classes*, where \mathcal{M} has only monotone (i.e. the classes $\text{Av}(21)$ and $\text{Av}(12)$) or empty entries.

Now let \mathcal{C} and \mathcal{D} be permutation classes. We say that \mathcal{C} is \mathcal{D} -griddable if there is some matrix \mathcal{M} whose entries are all subclasses of \mathcal{D} for which \mathcal{C} is \mathcal{M} -griddable. The following theorem gives a good characterisation of \mathcal{D} -griddability:

Theorem 2.2 (Vatter [14]). *A permutation class \mathcal{C} is \mathcal{D} -griddable if and only if it does not contain arbitrarily long sums or skew sums of basis elements of \mathcal{D} .*

A particular instance of this theorem is that a permutation class is monotone griddable if and only if it does not contain arbitrarily long sums of 21 or skew sums of 12. Define the *sum completion* of a permutation π to be the permutation class $\oplus\pi = \{\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_k : \alpha_i \leq \pi \text{ for all } i \leq k \in \mathbb{N}\}$, and the *skew completion* $\ominus\pi$ analogously. Thus:

Corollary 2.3. *A permutation class \mathcal{C} is monotone griddable if and only if it contains neither the class $\oplus 21$ nor the class $\ominus 12$.*

Grid classes and partial well-order. The *graph* of the gridding matrix \mathcal{M} is the graph $G_{\mathcal{M}}$ whose vertices are the non-empty cells of \mathcal{M} , with two vertices being adjacent if they share a row or a column of \mathcal{M} and all cells between them are empty. A *component* of \mathcal{M} is a submatrix \mathcal{M}' of \mathcal{M} for which $G_{\mathcal{M}'}$ is a connected component of $G_{\mathcal{M}}$. In determining whether grid classes are partially well-ordered, it is sufficient to look at these components individually:

Proposition 2.4 (Vatter [14]). *$\text{Grid}(\mathcal{M})$ is partially well-ordered if and only if $\text{Grid}(\mathcal{M}')$ is partially well-ordered for every connected component \mathcal{M}' of \mathcal{M} .*

In the case of monotone grid classes, the connection between $G_{\mathcal{M}}$ and partial well-order is well known:

Theorem 2.5 (Murphy and Vatter [12]). *The monotone grid class $\text{Grid}(\mathcal{M})$ is partially well-ordered if and only if $G_{\mathcal{M}}$ is a forest.*

One direction of this theorem is proved by constructing an antichain that “winds around” the cells corresponding to a cycle of $G_{\mathcal{M}}$, while the other requires Higman’s Theorem and has been reproved more efficiently by Waton [15]. Our proof of Theorems 1.1 and 1.2 will borrow a lot from the techniques in these two publications.

3 Partially Well Ordered Grid Classes

We complete one half of the proof of Theorem 1.1 by proving the following theorem.

Theorem 3.1. *Let \mathcal{M} be a gridding matrix whose entries are all permutation classes containing only finitely many simple permutations, and for which $G_{\mathcal{M}}$ is a forest and every component of \mathcal{M} contains at most one cell labelled by a class that is not monotone. Then $\text{Grid}(\mathcal{M})$ is partially well-ordered.*

We begin by giving a complete presentation of Higman’s Theorem, which will form the backbone of the proof of Theorem 3.1. We say that (A, M) is an *abstract algebra* if A is a set of elements and M a set of operations for which each $\mu \in M$ is a k -ary operation, $\mu : A^k \rightarrow A$, for some positive integer k . Denote the set of k -ary operations by M_k , and suppose that M_k is empty for every $k > n$ for some n . (Note that we will allow 0-ary operations.) The abstract algebra (A, M) is said to be *minimal* if no subset B of A allows (B, M) to be an abstract algebra.

A partial order \leq_A on the set of elements A is a *divisibility order* on (A, M) if every operation $\mu \in M_k$, $k = 0, 1, \dots, n$, satisfies,

- $a \leq_A b$ implies $\mu(\mathbf{x}, a, \mathbf{y}) \leq_A \mu(\mathbf{x}, b, \mathbf{y})$,
- $a \leq_A \mu(\mathbf{x}, a, \mathbf{y})$,

where \mathbf{x} and \mathbf{y} are arbitrary sequences comprising elements of A whose lengths sum to $k - 1$. Furthermore, given partial orders \leq_{M_k} on M_k , $k = 0, 1, \dots, n$, we say that \leq_A is *compatible* with these partial orders if, for $\lambda, \mu \in M_k$,

- $\lambda \leq_{M_k} \mu$ implies $\lambda(\mathbf{x}) \leq_A \mu(\mathbf{x})$ for all $\mathbf{x} \in A^k$.

Theorem 3.2 (Higman [7]). *Suppose that (A, M) is a minimal abstract algebra for which, for some n , the set M_k of k -ary operations in M is partially well-ordered for each $k = 0, 1, \dots, n$ and empty for $k > n$. Then (A, M) is partially well-ordered under any divisibility ordering compatible with the orders of M_k .*

When applied to permutation classes, Higman’s Theorem shows that any permutation class which can be described by means of a suitable set of constructions is partially well-ordered. One construction that has been particularly amenable to this approach is the inflation of one permutation by others; inflating a permutation σ of length k by τ_1, \dots, τ_k may be thought of as a k -ary operation that acts on the permutations τ_1, \dots, τ_k . It is clear both that inflation is compatible with the permutation containment ordering and that permutation containment is a divisibility ordering with respect to inflations of this type. To satisfy the conditions of Higman’s Theorem, however, we cannot inflate arbitrarily large permutations. Roughly speaking, if a permutation class \mathcal{C} is a subclass of some substitution-closed class \mathcal{D} that can be expressed as the substitution closure of some finite set X , then Higman’s Theorem can be applied to prove that \mathcal{D} (and consequently \mathcal{C}) is partially well-ordered. Consequently, by Proposition 2.1:

Theorem 3.3 (Albert and Atkinson [1]). *Let \mathcal{C} be a class containing only finitely many simple permutations. Then \mathcal{C} is partially well-ordered.*

On the other hand, since any set X satisfying $\langle X \rangle = \langle \mathcal{C} \rangle$ must contain every permutation in $\text{Si}(\mathcal{C})$, we cannot arrange that X is finite when \mathcal{C} contains infinitely many simple permutations, and Higman's Theorem cannot be used in this way. This, however, does not mean that any class containing infinitely many simple permutations is not partially well-ordered: for example, $\text{Grid}(\text{Av}(21) \text{ Av}(21))$ is partially well-ordered by Theorem 2.5, but contains infinitely long simple permutations of the form $2\ 4\ 6 \cdots 2k\ 1\ 3\ 5 \cdots 2k - 1$.

Let us now extend this use of Higman's Theorem to gridding matrices. We first define an order on the set of $m \times n$ -gridded permutations. For $m \times n$ -gridded permutations α and π of lengths k and ℓ , respectively, we say that α is contained in π , $\alpha \leq_{mn} \pi$, if and only if there is a sequence of indices $1 \leq i_1 < \cdots < i_k \leq \ell$ such that $\pi(i_1) \cdots \pi(i_k)$ is order isomorphic to α as ungridded permutations, and for $j = 1, \dots, k$, $\pi(i_j)$ and $\alpha(j)$ lie in the same cell in the $m \times n$ -griddings. Similarly, for a specific $m \times n$ gridding matrix \mathcal{M} and \mathcal{M} -gridded permutations α and π , we write $\alpha \leq_{\mathcal{M}} \pi$ to mean $\alpha \leq_{mn} \pi$, but also recognising that both α and π are \mathcal{M} -gridded.

Suppose that \mathcal{M} is a gridding matrix consisting of exactly one component, and every non-empty cell of \mathcal{M} is labelled by a monotone class, except for the uv th cell which is labelled by some arbitrary class \mathcal{D} . Viewing $G_{\mathcal{M}}$ as a tree rooted on the uv th cell, each cell other than the uv th is the *child* of some *parent* cell, i.e. the cell lying directly above it in the rooted tree.

Now let τ_1, \dots, τ_k be \mathcal{M} -gridded permutations each with at least one point in cell uv , and let $\sigma \in \mathcal{D}$ be of length k . The \mathcal{M} -inflation of σ by τ_1, \dots, τ_k is the \mathcal{M} -gridded permutation $\pi = \sigma[\tau_1, \dots, \tau_k]_{\mathcal{M}}$, and is formed by first taking the inflation $\pi^{uv} = \sigma[\tau_1^{uv}, \dots, \tau_k^{uv}]$. For every other non-empty cell st of \mathcal{M} , if $\mathcal{M}_{st} = \text{Av}(21)$ then π^{st} is a (possibly lenient) inflation of $1 \cdots k$ by $\tau_1^{st}, \dots, \tau_k^{st}$ in some order, while if $\mathcal{M}_{st} = \text{Av}(12)$ then π^{st} is a (possibly lenient) inflation of $k \cdots 1$ by $\tau_1^{st}, \dots, \tau_k^{st}$ in some order. In either case, the order in which $\tau_1^{st}, \dots, \tau_k^{st}$ appears in the inflation is defined recursively in terms of its parent cell $\pi^{s't'}$ (where either $s = s'$ or $t = t'$) in the tree $G_{\mathcal{M}}$ rooted on cell uv : if cells st and $s't'$ share a column (i.e. if $s = s'$) then the left-to-right order of $\tau_1^{st}, \dots, \tau_k^{st}$ in the specified inflation is the same as the left-to-right order of $\tau_1^{s't'}, \dots, \tau_k^{s't'}$ in $\pi^{s't'}$. Similarly, if $t = t'$ then the bottom-to-top order of $\tau_1^{st}, \dots, \tau_k^{st}$ is the same as the bottom-to-top order of $\tau_1^{s't'}, \dots, \tau_k^{s't'}$. Additionally, for each $i \in [k]$, τ_i^{st} interacts with $\tau_i^{s't'}$ in exactly the same way as in the \mathcal{M} -gridded permutation τ_i , and it interacts with no other $\tau_j^{s't'}$, $j \neq i$. Note also that we must remember the order of $\tau_1^{st}, \dots, \tau_k^{st}$ in every non-empty cell st of \mathcal{M} even if one or more of the τ_i^{st} contains no points, so that we know the order of the cells for any subsequent descendants. See Figure 2 for an illustration.

A *lenient \mathcal{M} -inflation* of σ by τ_1, \dots, τ_k is defined in exactly the same way, except that we do not stipulate that each τ_i^{uv} be non-empty.

Proof of Theorem 3.1. By Proposition 2.4, we may assume that $G_{\mathcal{M}}$ consists of exactly one component. Thus \mathcal{M} is an $m \times n$ gridding matrix such that $G_{\mathcal{M}}$ is a tree and every non-empty cell of \mathcal{M} is labelled by a monotone class, except for the uv th cell which is labelled by some infinite class \mathcal{D} containing only finitely many simple permutations. We will also assume that \mathcal{D} is substitution closed, as otherwise we may replace it with $\langle \mathcal{D} \rangle$ and prove the result for this larger class.

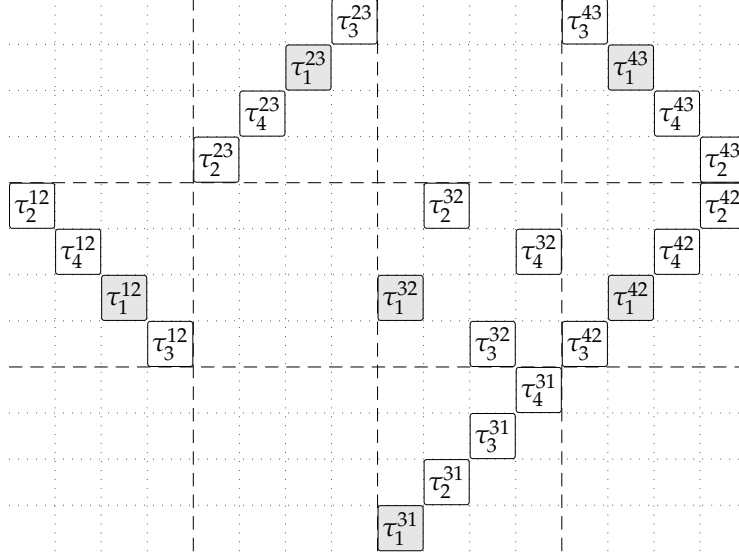


Figure 2: Forming the \mathcal{M} -inflation $2413[\tau_1, \tau_2, \tau_3, \tau_4]_{\mathcal{M}}$ for a 4×3 gridding matrix \mathcal{M} . The highlighted cells correspond to the \mathcal{M} -gridded permutation τ_1 .

For each $\sigma \in \text{Si}(\mathcal{D})$ of length k , we view an \mathcal{M} -inflation of σ as a k -ary operation. We claim that $\text{Grid}(\mathcal{M})$ is generated by this finite list of \mathcal{M} -inflations and all the \mathcal{M} -griddings of the singleton permutation 1. It will then follow by Higman's Theorem 3.2 that $\text{Grid}(\mathcal{M})$ is partially well-ordered.

We proceed by induction on the length of \mathcal{M} -gridded permutations. As we already have all the \mathcal{M} -gridded permutations of length 1, it is enough to show that any $\pi \in \text{Grid}(\mathcal{M})$ with $|\pi| \geq 2$ can be expressed as an \mathcal{M} -inflation of some $\sigma \in \text{Si}(\mathcal{D})$. Given one such π , suppose first that π^{uv} contains at least two points. By Proposition 2.1 there exists some $\sigma \in \text{Si}(\mathcal{D})$ such that π^{uv} is an inflation of σ , i.e. $\pi^{uv} = \sigma[\tau_1^{uv}, \dots, \tau_k^{uv}]$, for some permutations $\tau_1^{uv}, \dots, \tau_k^{uv}$. Label each point of π^{uv} with the symbol from $1, \dots, k$ corresponding to which of $\tau_1^{uv}, \dots, \tau_k^{uv}$ it belongs. We now label each cell recursively, working down the tree $G_{\mathcal{M}}$ rooted at the cell uv . Consider a cell st whose parent rw has been labelled. We will label each point p in π^{st} as follows:

- If the child shares a column with its parent (i.e. $r = s$), then p is assigned the same label as the rightmost point in π^{rw} that lies to its left. If there is no point in π^{rw} to the left of p , give p the label of the leftmost point of π^{rw} . If there are no points in π^{rw} , label every point of π^{st} with the label 1.
- If the child shares a row with its parent (i.e. $t = w$), then p is assigned the same label as the highest point of π^{rw} that lies below it. If there is no such point in π^{rw} , give p the label of the lowest point of π^{rw} . If there are no points in π^{rw} , label every point of π^{st} with the label 1.

For each $i \in [k]$, now create the \mathcal{M} -gridded permutation τ_i by taking all points of π with label i . It is now clear to see that $\pi = \sigma[\tau_1, \dots, \tau_k]_{\mathcal{M}}$, as required.

This leaves the case where π^{uv} contains a singleton or is empty. Since $|\pi| \geq 2$, either there is a cell of π containing at least two points, or there are at least two non-empty cells. If there is

a cell π^{st} containing at least two points, label the leftmost point with the label 1 and all other points in this cell with label 2. Then view $G_{\mathcal{M}}$ as a tree rooted at the cell st and label the points in the cells of π recursively as described above. Using these labels, now form τ_1 and τ_2 as before, and observe that π is a lenient \mathcal{M} -inflation of 12 or 21 with τ_1 and τ_2 , in some order.

Finally, if all of the non-empty cells of π contain only one point, then label the point in any one non-empty cell of π with the symbol 1 and the point in any other non-empty cell with the symbol 2. Now assign every other point in every other cell either the label 1 or 2 in such a way that, forming the permutations τ_1 and τ_2 from the labels, π can be expressed as a lenient inflation of 12 or 21 by the gridded permutations τ_1 and τ_2 in some order. \square

4 Grid Classes by Symmetry

For the remainder of this paper we will be showing that certain types of grid class are not partially well-ordered by exhibiting antichains that lie in them. Among these non-partially well-ordered grid classes will be those needed to prove the remaining direction of Theorem 1.1. We begin by showing how we may divide grid classes into families using “grid mappings”, defined by appealing to three of the eight symmetries of permutations that preserve the usual containment ordering.

Let \mathcal{M} be an $m \times n$ gridding matrix, and let π be an \mathcal{M} -gridded permutation. Recall that the *inverse* of a permutation π is π^{-1} , defined by $\pi^{-1}(i) = j$ if and only if $\pi(j) = i$, and we extend this in two ways: first to an \mathcal{M} -gridded permutation π by mapping any vertical line between positions i and $i + 1$ ($i = 0, \dots, n$) to a horizontal line between values i and $i + 1$ and vice versa, and second to a permutation class \mathcal{C} by setting $\mathcal{C}^{-1} = \{\pi^{-1} : \pi \in \mathcal{C}\}$. We consider the effect of taking the inverse of π on the gridding of π , and consequently the effect on \mathcal{M} of taking the inverse of $\text{Grid}(\mathcal{M})$.

Lemma 4.1. *Let \mathcal{M} be an $m \times n$ gridding matrix. Then $\text{Grid}(\mathcal{M})^{-1} = \text{Grid}(\phi(\mathcal{M}))$ where $\phi(\mathcal{M})$ is defined by $(\phi(\mathcal{M}))_{ij} = \mathcal{M}_{ji}^{-1}$.*

We will call the map ϕ the *grid inverse map*.

Proof. First note that $\phi(\phi(\mathcal{M})) = \mathcal{M}$, so it suffices to show that $\text{Grid}(\mathcal{M})^{-1} \subseteq \text{Grid}(\phi(\mathcal{M}))$. Let π be any permutation in $\text{Grid}(\mathcal{M})^{-1}$, so $\pi^{-1} \in \text{Grid}(\mathcal{M})$ is \mathcal{M} -griddable. Pick any \mathcal{M} -gridding of π^{-1} , and apply the inverse operation to this gridded matrix to recover a gridding of π . By definition, all points of the ij th cell of the gridded version of π^{-1} are mapped under inverse to the ji th cell of the gridded π . Moreover, if σ represents the permutation order isomorphic to the points in the ij th cell of π^{-1} , then it is clear that σ^{-1} represents the permutation order isomorphic to the points in the ji th cell of π , from which it follows that $\pi \in \text{Grid}(\phi(\mathcal{M}))$. \square

Given a permutation π of length k , the *reverse* of π — $r(\pi)$ — is the permutation obtained by reading the entries of π from right to left, i.e. for $i \in [k]$, we have $r(\pi(i)) = \pi(k + 1 - i)$. Similarly, the *complement* of π — $c(\pi)$ — is formed by reading the permutation from top to bottom, i.e. $c(\pi(i)) = k + 1 - \pi(i)$. Accordingly, the *reverse* of a set of permutations X is $r(X) = \{r(\pi) : \pi \in X\}$, and the *complement* is $c(X) = \{c(\pi) : \pi \in X\}$. Note in particular that if $\mathcal{C} = \text{Av}(B)$ is a permutation class with basis B then $r(\mathcal{C}) = \text{Av}(r(B))$ and $c(\mathcal{C}) = \text{Av}(c(B))$.

Now let \mathcal{M} be any $m \times n$ gridding matrix. For fixed $i \in [m]$, let $r_i(\mathcal{M})$ be the *i th column reverse* of \mathcal{M} , formed by applying the reverse map r to every cell in column i . Thus for all

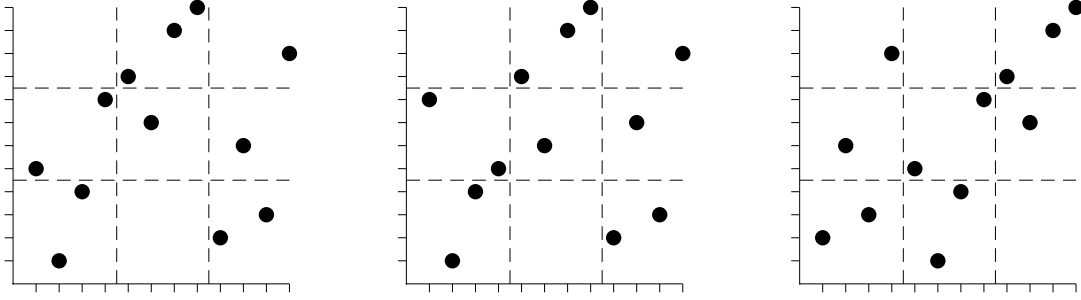


Figure 3: From left to right, the 3×3 gridded permutation $\pi = 5\ 1\ 4\ 8\ 9\ 7\ 11\ 12\ 2\ 6\ 3\ 10$, the 2nd row complement $c_2(\pi) = 8\ 1\ 4\ 5\ 9\ 6\ 11\ 12\ 2\ 7\ 3\ 10$, and the permutation $\mu(\pi) = 2\ 6\ 3\ 10\ 5\ 1\ 4\ 8\ 9\ 7\ 11\ 12$ where $\mu = 312$ is a permutation of the columns.

$j \in [n]$, for any $i' \neq i$ we have $(r_i(\mathcal{M}))_{i'j} = \mathcal{M}_{i'j}$, while $(r_i(\mathcal{M}))_{ij} = r(\mathcal{M}_{ij})$. We define the j th row complement analogously: $(c_j(\mathcal{M}))_{i'j} = \mathcal{M}_{i'j}$ whenever $j' \neq j$, and $(c_j(\mathcal{M}))_{ij} = f(\mathcal{M}_{ij})$ for all $i \in [m]$. Next, if μ is a permutation of length m , then let $\mu(\mathcal{M})$ be the gridding matrix formed by permuting the columns of \mathcal{M} as prescribed by μ , so that $(\mu(\mathcal{M}))_{ij} = \mathcal{M}_{\mu(i)j}$. We say that μ is a permutation of the columns of \mathcal{M} . Similarly, a permutation of the rows of \mathcal{M} is a permutation ν of length n satisfying $(\nu(\mathcal{M}))_{ij} = \mathcal{M}_{i\nu(j)}$.

We also extend the definitions of complements, reverses and permutations to gridded permutations in the obvious way. For example, if π is a gridded permutation for which the set of points in row j have values $a, a+1, \dots, b$, then the j th row complement of π is $c_j(\pi)$ defined by $c_j(\pi)(i) = b+a-\pi(i)$ if $(i, \pi(i))$ lies in row j , and $c_j(\pi)(i) = \pi(i)$ otherwise. See Figure 3.

A grid mapping is any composition of grid inverse, row complements, column reverses and row and column permutations, and we say that two matrices \mathcal{M} and \mathcal{N} are equivalent under the grid mapping f if $f(\mathcal{M}) = \mathcal{N}$. Grid mappings do not in general preserve the normal permutation containment ordering, but they do respect gridded containment (defined in Section 3).

Lemma 4.2. *Let \mathcal{M} be a gridding matrix, α and π two \mathcal{M} -gridded permutations and f any grid mapping of \mathcal{M} . Then $\alpha \leq_{\mathcal{M}} \pi$ if and only if $f(\alpha) \leq_{f(\mathcal{M})} f(\pi)$.*

Proof. It suffices to show that $\alpha \leq_{\mathcal{M}} \pi$ implies $f(\alpha) \leq_{f(\mathcal{M})} f(\pi)$ where f is a grid inverse, row complement, column reverse, or a row or column permutation. Suppose that α is of length k and π of length ℓ , and that the indices $1 \leq i_1 < \dots < i_k \leq \ell$ give rise to a subsequence $\pi(i_1) \dots \pi(i_k)$ that witnesses the gridded containment $\alpha \leq_{\mathcal{M}} \pi$. First, if $f = \phi$ is the grid inverse mapping then $\alpha \leq_{\mathcal{M}} \pi$ immediately implies $f(\alpha) = \alpha^{-1} \leq \pi^{-1} = f(\pi)$ as this is the normal inverse for permutations. Moreover, if $\pi(i_j)$ and $\alpha(j)$ ($j = 1, \dots, k$) lie in cell st of \mathcal{M} , then their images under f both lie in cell ts of $f(\mathcal{M})$, from which we conclude that $f(\alpha) \leq_{f(\mathcal{M})} f(\pi)$.

With the result for grid inverse established, we now only need to check the cases where f is a column reverse or column permutation, as row complements and permutations can then be derived by composition of these functions. Let us consider first a column permutation, and note (again by composition) that we need only show this is true when the column permutation is a transposition. Thus suppose f swaps columns u and v . It is clear that the images of $\pi(i_j)$ and $\alpha(j)$ under f both lie in the same cell, so it remains to show that $f(\alpha) \leq f(\pi)$ as ungridded permutations. This, however, is also straightforward: f simply swaps the segments of α that lie in columns u and v , and it does likewise in π . In particular, f swaps the two subsequences

of $\pi(i_1) \cdots \pi(i_k)$ lying in columns u and v , and this image is a copy of $f(\alpha)$ in $f(\pi)$. A similar argument can be applied when f is a column reversal, completing the proof. \square

We next make a simple observation, which allows us to pass between grid containment and normal permutation containment.

Lemma 4.3. *Let α and π be \mathcal{M} -griddable permutations with $\alpha \leq \pi$ as ungridded permutations. Then for any \mathcal{M} -gridding of π , there exists an \mathcal{M} -gridding of α such that $\alpha \leq_{\mathcal{M}} \pi$.*

Proof. Since $\alpha \leq \pi$ as ungridded permutations, there is a subsequence $i_1 < i_2 < \dots < i_k$ where $k = |\alpha|$ such that $\pi(i_1) \cdots \pi(i_k)$ is order isomorphic to α . Now, for any \mathcal{M} -gridding of π , it is clear that α can be \mathcal{M} -gridded to satisfy $\alpha \leq_{\mathcal{M}} \pi$ by restricting the \mathcal{M} -gridded permutation π to the \mathcal{M} -gridded subsequence $\pi(i_1) \cdots \pi(i_k)$. \square

We will use Lemma 4.3 on permutations that have a unique gridding: if α and π are two permutations which have unique \mathcal{M} -griddings for some matrix \mathcal{M} , then $\alpha \not\leq_{\mathcal{M}} \pi$ implies $\alpha \not\leq \pi$. However, unique griddability is not in general preserved by grid mappings. For example, 135246 has a unique gridding in $\text{Grid}(\text{Av}(21) \text{ Av}(21))$, but applying a column reverse to the first column yields the permutation 531246, which can be gridding in two different ways in $\text{Grid}(\text{Av}(12) \text{ Av}(21))$. Thus, for a gridding matrix \mathcal{M} , we say that an \mathcal{M} -gridded permutation π is *strongly uniquely \mathcal{M} -griddable* if the given \mathcal{M} -gridding of π is unique and, for every grid mapping f of \mathcal{M} , $f(\pi)$ is also the unique $f(\mathcal{M})$ -gridding of $f(\pi)$. This extra condition gives us what we need:

Theorem 4.4. *Let \mathcal{M} be a gridding matrix, and let A be an antichain for which infinitely many elements are strongly uniquely \mathcal{M} -griddable. Then the grid class of any gridding matrix \mathcal{N} that is equivalent to \mathcal{M} under some grid mapping is not partially well-ordered.*

Proof. First, we may assume that A consists only of strongly uniquely \mathcal{M} -griddable permutations, as we may discard any elements that are not. Note that $\text{Grid}(\mathcal{M})$ contains A and so is not partially well-ordered. Let f be any grid mapping of \mathcal{M} , and let $\mathcal{N} = f(\mathcal{M})$. Take any pair of distinct permutations $\alpha, \beta \in A$ (noting that $\alpha \not\leq \beta$), and equip each permutation with its unique \mathcal{M} -gridding. With these griddings $f(\alpha)$ and $f(\beta)$ are \mathcal{N} -gridded permutations, and since α and β are strongly uniquely \mathcal{M} -griddable these \mathcal{N} -griddings are the unique griddings of the underlying permutations of $f(\alpha)$ and $f(\beta)$. Now, since $\alpha \not\leq \beta$ we have $\alpha \not\leq_{\mathcal{M}} \beta$, and consequently $f(\alpha) \not\leq_{\mathcal{N}} f(\beta)$ by Lemma 4.2. Additionally, we have $f(\alpha) \not\leq f(\beta)$ as ungridded permutations by Lemma 4.3. Similarly, $\beta \not\leq \alpha$ implies $f(\beta) \not\leq f(\alpha)$, and so $f(\alpha)$ and $f(\beta)$ are incomparable permutations lying in $\text{Grid}(\mathcal{N})$, completing the proof. \square

5 A Family of Grid Matrices

Let $\mathcal{C} = \mathcal{D}^+ = \oplus 21$ and $\mathcal{D}^- = \ominus 12$. For $k \in \mathbb{N}$ define \mathcal{M}^k recursively as follows:

- $\mathcal{M}^1 = (\mathcal{C} \ \mathcal{D}^-)$.
- When $k = 4\ell + 1$, \mathcal{M}^k is a $(2\ell + 2) \times (2\ell + 1)$ matrix with $\mathcal{M}_{ij}^k = \mathcal{M}_{ij}^{k-1}$ for all $i \in [1, 2\ell + 1], j \in [2, 2\ell + 1]$; $\mathcal{M}_{(2\ell+2)1}^k = \mathcal{D}^-$; $\mathcal{M}_{11}^k = \text{Av}(21)$; and all other entries are \emptyset .

- When $k = 4\ell + 2$, \mathcal{M}^k is a $(2\ell + 2) \times (2\ell + 2)$ matrix with $\mathcal{M}_{ij}^k = \mathcal{M}_{ij}^{k-1}$ for all $i, j \in [1, 2\ell + 1]$; $\mathcal{M}_{(2\ell+2)(2\ell+2)}^k = \mathcal{D}^+$; $\mathcal{M}_{(2\ell+2)1}^k = \text{Av}(12)$; and all other entries are \emptyset .
- When $k = 4\ell + 3$, \mathcal{M}^k is a $(2\ell + 3) \times (2\ell + 2)$ matrix with $\mathcal{M}_{ij}^k = \mathcal{M}_{(i-1)j}^{k-1}$ for all $i \in [2, 2\ell + 3], j \in [1, 2\ell + 1]$; $\mathcal{M}_{1(2\ell+2)}^k = \mathcal{D}^-$; $\mathcal{M}_{(2\ell+3)(2\ell+2)}^k = \text{Av}(21)$; and all other entries are \emptyset .
- When $k = 4\ell + 4$, \mathcal{M}^k is a $(2\ell + 3) \times (2\ell + 3)$ matrix with $\mathcal{M}_{ij}^k = \mathcal{M}_{i(j-1)}^{k-1}$ for all $i \in [1, 2\ell + 2], j \in [2, 2\ell + 3]$; $\mathcal{M}_{11}^k = \mathcal{D}^+$; $\mathcal{M}_{1(2\ell+3)}^k = \text{Av}(12)$; and all other entries are \emptyset .

Suppressing the labels of empty cells, the first few such matrices are:

$$\begin{aligned}
\mathcal{M}^1 &= (\mathcal{C} \ \mathcal{D}^-) \\
\mathcal{M}^2 &= \begin{pmatrix} & \mathcal{D}^+ \\ \mathcal{C} & \text{Av}(12) \end{pmatrix} \\
\mathcal{M}^3 &= \begin{pmatrix} \mathcal{D}^- & \text{Av}(21) \\ & \mathcal{C} \ \text{Av}(12) \end{pmatrix} \\
\mathcal{M}^4 &= \begin{pmatrix} \text{Av}(12) & \text{Av}(21) \\ & \mathcal{C} \ \text{Av}(12) \\ \mathcal{D}^+ & & \end{pmatrix} \\
\mathcal{M}^5 &= \begin{pmatrix} \text{Av}(12) & \text{Av}(21) & \\ & \mathcal{C} \ \text{Av}(12) & \\ \text{Av}(21) & & \mathcal{D}^- \end{pmatrix}.
\end{aligned}$$

Note that $G_{\mathcal{M}^k}$ is a path of length k , one end of which is labelled by \mathcal{C} and the other by either \mathcal{D}^- or \mathcal{D}^+ , and whose internal vertices are labelled by $\text{Av}(21)$ or $\text{Av}(12)$.

We now show that all gridding matrices of the desired form for the proof of Theorem 1.1 are equivalent to one of the matrices \mathcal{M}^k .

Theorem 5.1. *Let \mathcal{M} be a gridding matrix such that*

- (1) \mathcal{M} has no completely empty rows or columns,
- (2) every row and column of \mathcal{M} contains at most two non-empty cells, and
- (3) $G_{\mathcal{M}}$ is a path of length k whose internal vertices are all labelled by a class of monotone permutations and whose leaves are each labelled by either $\oplus 21$ or $\ominus 12$.

Then \mathcal{M} and \mathcal{M}^k are equivalent under some grid mapping.

Proof. We will form the grid mapping $f : \mathcal{M}^k \rightarrow \mathcal{M}$ in three stages. First, we check whether we need to apply the grid inverse map to \mathcal{M}^k so that it has the same dimensions as \mathcal{M} . Next, we permute the rows and columns of \mathcal{M}^k (or $\phi(\mathcal{M}^k)$) to form an intermediate matrix \mathcal{N} that has empty cells in exactly the same positions as \mathcal{M} . Finally, we use row complements and column reversals on \mathcal{N} to match the non-empty cells to those of \mathcal{M} .

Write $G_{\mathcal{M}} = v_1 v_2 \cdots v_{k+1}$ and $G_{\mathcal{M}^k} = u_1 u_2 \cdots u_{k+1}$, and assign a labelling to the edges of $G_{\mathcal{M}}$ and $G_{\mathcal{M}^k}$ to distinguish between edges that correspond to a pair of vertices which share a

row (*row edges*), and edges whose vertices share a column (*column edges*). Note that the edges of both $G_{\mathcal{M}}$ and $G_{\mathcal{M}^k}$ must alternate between row and column edges as there are at most two non-empty cells in each row and column, and by the construction we can assume that u_1u_2 is a row edge.

If $k = 2\ell$ then the conditions of the theorem imply that \mathcal{M} is an $(\ell + 1) \times (\ell + 1)$ matrix, while if $k = 2\ell + 1$ then \mathcal{M} is either an $(\ell + 2) \times (\ell + 1)$ matrix or an $(\ell + 1) \times (\ell + 2)$ matrix. The first two of these cases give matrices whose dimensions coincide with the dimensions of \mathcal{M}^k , and moreover we can assume that v_1v_2 is a row edge. For the final case we must apply the inverse map ϕ , since $\phi(\mathcal{M}^k)$ has the same dimensions as \mathcal{M} and both $G_{\phi(\mathcal{M}^k)}$ and $G_{\mathcal{M}}$ will begin and end with column edges. In any case, we now have a matrix derived from \mathcal{M}^k with the same dimensions as \mathcal{M} , and whose graphs have the same edge labellings.

Suppose without loss that v_1v_2 (and consequently u_1u_2) is a row edge, so we did not need to apply the grid inverse mapping. We need to apply a sequence of row and column permutations to \mathcal{M}^k to map the cells corresponding to vertices u_1, \dots, u_{k+1} to the cells for vertices v_1, \dots, v_{k+1} . We do this by first finding the row transposition and column transposition that sends u_1 to the correct cell. Since u_2 shares a row with u_1 , we now only need to apply the column transposition to place it — and u_3 — in the correct column. We continue in this way until all of u_1, \dots, u_k have been moved to the correct cells, noting that each row or column transposition cannot displace any correctly-placed cell arising earlier in the sequence. Furthermore, since u_{k+1} lies in the same row or column as u_k , it is also now in the correct place, and so we have shown how to obtain the matrix \mathcal{N} from \mathcal{M}^k .

All that remains is to fix the non-empty cells of \mathcal{N} to have the same labels as \mathcal{M} . Write $G_{\mathcal{N}} = w_1 \cdots w_{k+1}$. The cells corresponding to vertices v_1 and w_1 have entries that are either $\oplus 21$ or $\ominus 12$. If they do not match, apply the row complement to the row containing w_1 , since $c(\oplus 21) = \ominus 12$ and $c(\ominus 12) = \oplus 21$, noting that this will also change the entry of the cell corresponding to w_2 . Now, if v_2 and w_2 (or the image of w_2 if we applied the row complement) do not have the same entry, we apply the column reverse to the column of \mathcal{N} containing w_2 so that they do match. (Recall that $r(\text{Av}(21)) = c(\text{Av}(21)) = \text{Av}(12)$ and $r(\text{Av}(12)) = c(\text{Av}(12)) = \text{Av}(21)$.) We proceed in this way until the entries of all the cells corresponding to w_1, \dots, w_k have been mapped so that they match the cells v_1, \dots, v_k . Finally, if the entry of the cell corresponding to w_{k+1} does not match v_{k+1} , since it is the sole entry in either its row or column we can use (respectively) a row complement or column reverse so that it does match, without affecting any other non-empty cell. \square

6 Grid Pin Sequences and Antichains

To complete the proof of Theorem 1.2, all we require by Theorems 4.4 and 5.1 is to find a strongly uniquely \mathcal{M}^k -griddable antichain for each $k \in \mathbb{N}$. The reason we chose the matrices \mathcal{M}^k is that they admit antichains that are easily described in terms of “grid pin sequences”. We now define these pin sequences, and prove some elementary results about them that should assist in our description of the antichains we wish to construct — it is not our aim here to produce a complete theory of these sequences.

Given points p_1, p_2, \dots in the plane, denote by $\text{rect}(p_1, p_2, \dots)$ the smallest axes-parallel rectangle containing them. A *grid pin sequence* is a sequence of points (called *pins*) p_1, p_2, \dots in

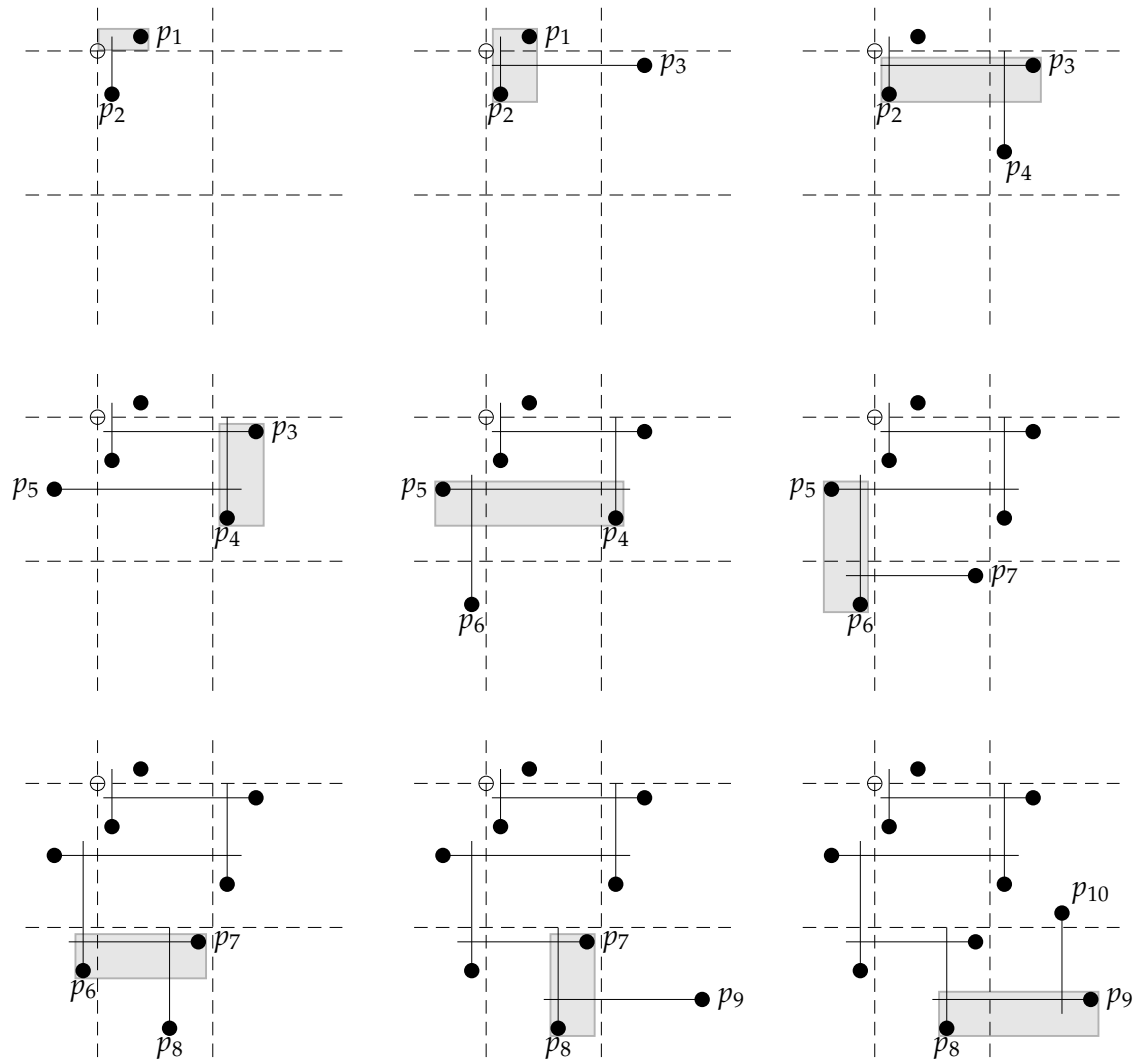


Figure 4: A grid pin sequence on the 3×3 grid.

an $m \times n$ gridded plane which for $i \geq 2$ must satisfy four conditions:

- *Local separation*: Each pin p_{i+1} separates p_i from p_{i-1} by position or by value.
- *Local externality*: Each pin p_{i+1} lies outside all of $\text{rect}(p_0, p_1), \text{rect}(p_1, p_2), \dots, \text{rect}(p_{i-1}, p_i)$. The *direction* of a pin, being one of left, right, up or down, is the position in which it lies relative to $\text{rect}(p_{i-1}, p_i)$.
- *Row-column agreement*: If p_{i+1} is an up or a down pin, it must lie in the same column as p_i , while if p_{i+1} is a left or a right pin, it must lie in the same row.
- *Non-interaction*: Each pin p_{i+1} , could not have been used as a grid pin earlier in the pin sequence. I.e. for every $2 \leq j < i$ the pin p_{i+1} must violate at least one of local separation, row-column agreement or local externality with respect to p_j and p_{j-1} .

Note that, reducing to the 2×2 grid case, local separation and row-column agreement combine to form the separation condition of [4], and local externality and non-interaction combine to give the externality condition. Thus grid pin sequences should be thought of as a generalisation of proper pin sequences.

It still remains to explain how to initiate a grid pin sequence. As with normal pin sequences, we begin by placing a fictional pin p_0 corresponding to an *origin* at the intersection of two chosen perpendicular grid lines. Our next pin, p_1 , is then placed in one of the four cells adjacent to this origin and has two directions given by its position relative to p_0 . For example, if p_1 lies below and to the left of p_0 , then p_1 is both a left pin and a down pin. The second pin is then placed to satisfy the four above conditions relative to p_0 and p_1 .

The directions of pins p_2, p_3, \dots must alternate:

Lemma 6.1. *In a grid pin sequence, p_{i+1} cannot lie in the same or opposite direction as p_i for $i > 1$.*

Proof. Suppose without loss that p_i is a left pin. By local externality and local separation, p_{i+1} must extend from $\text{rect}(p_{i-1}, p_i)$. However, if p_{i+1} lies in the same or opposite direction as p_i then p_{i+1} also either extends from $\text{rect}(p_{i-2}, p_{i-1})$ contradicting non-interaction, or it lies in $\text{rect}(p_{i-2}, p_{i-1})$ contradicting local externality. \square

Lemma 6.2. *If p_{i+1} is a left pin for the grid pin sequence p_1, \dots, p_i , then p_{i+1} lies further left than all previous left pins in its column, and to the right of all previous right pins in its column. Analogous statements hold if p_{i+1} is a right, up or down pin.*

Proof. We prove both statements of the first sentence simultaneously by induction on the total number of left and right pins in a given column. The base case, where there is just a single left or right pin, is trivial. So now suppose for a contradiction to the first statement that p_{i+1} is a left pin which lies to the right of some earlier left pin p_j ($j < i$) in the same column, and assume without loss that there are no other left pins between p_{i+1} and p_j . If p_{j-1} (the predecessor of p_j) lies to the right of p_{i+1} , then p_{i+1} separates p_{j-1} from p_j , is not contained in any of $\text{rect}(p_{j-1}, p_j), \dots, \text{rect}(p_0, p_1)$ and shares a column with p_j , and hence is a pin for p_1, \dots, p_j , contradicting non-interaction. Thus p_{j-1} lies between p_j and p_{i+1} and so lies in the same column. Now consider the pin p_{j-2} , which was a left or a right pin, or p_0 . (Note that p_1 is always either a left or a right pin.) It cannot be a left pin as it would necessarily have to lie by

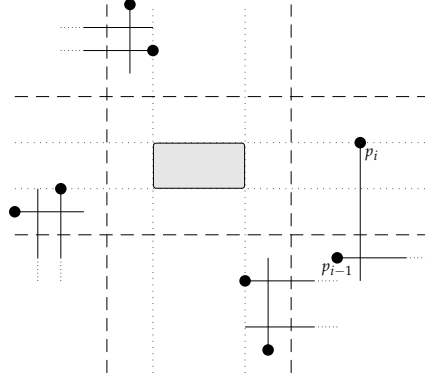


Figure 5: The shaded region denotes the area where pin p_{i+1} can be placed.

position between p_j and p_{i+1} but by our assumption there are no left pins between p_j and p_{i+1} ; it cannot be a right pin since by row-column agreement it must lie in the same column as p_{j-1} but to the right of p_j , contradicting the inductive hypothesis; finally, it cannot be p_0 as then, in order to lie on an adjacent grid line and ensure that $p_j = p_2$ extends from $\text{rect}(p_0, p_1)$, it must lie to the right of p_{i+1} , but then p_{i+1} either lies in $\text{rect}(p_0, p_1)$ contradicting local externality or it satisfies the conditions to be a pin for p_0, p_1 contradicting non-interaction.

A similar argument may be applied to show that p_{i+1} lies to the right of all previous right pins in its column, and so by induction the first sentence of the lemma is true. Finally, symmetry proves the analogous statements in the other three directions. \square

Unlike the 2×2 case, for an arbitrary $m \times n$ grid the direction of the pin is not sufficient to describe the placement of the pin, and instead we need to be more specific. A *horizontal pin* is either a left or a right pin, while a *vertical pin* is either an up or a down pin.

Lemma 6.3. *Let p_1, p_2, \dots, p_i be a grid pin sequence of length $i \geq 2$ in an $m \times n$ grid. Then if p_{i+1} is a horizontal pin, its placement relative to p_1, \dots, p_i is uniquely determined (up to order isomorphism) by the column in which it lies. Similarly, if p_{i+1} is a vertical pin, its placement relative to p_1, \dots, p_i is uniquely determined by the row in which it lies.*

Proof. We prove only the case where p_{i+1} is a horizontal pin and p_i is an up pin. By row-column agreement, p_{i+1} must be made to lie in the same row as p_i , so coupling this information with knowing the column that is to contain p_{i+1} is enough to determine the cell into which p_i is placed. In particular, if the column that is to contain p_{i+1} is to the left (respectively, right) of the column containing p_i , then p_{i+1} is a left (resp. right) pin. If p_{i+1} is to lie in the same column as p_i , then the direction of p_{i+1} must match the direction of p_{i-1} to satisfy Lemma 6.3. (Note that if $p_{i-1} = p_1$, then the direction of p_{i+1} matches the horizontal direction of p_1 .)

By Lemma 6.2, p_i must be placed in the region of the cell to the left of all earlier left pins in its column, and to the right of all right pins in the column. This defines a vertical strip extending the length of the column that is devoid of points. Similarly, p_{i+1} must lie below p_i and above p_{i-1} to satisfy separation, and additionally it must lie above all up pins other than p_i in its row to satisfy non-interaction. This defines a horizontal strip extending to the ends of the row which is devoid of points.

The intersection of the horizontal strip and the vertical strip defines a rectangular region in the correct cell in which p_{i+1} can be placed — see Figure 5. By its construction, there are no points among p_1, \dots, p_i separating this region, and so all placements of p_{i+1} within this region produce the same permutation up to order isomorphism. \square

Note that the above lemma can be extended to include pin p_2 , but this requires a little further thought. It is not sufficient to state which cell it is to be placed in as there are two different placements of p_2 if it is to lie in the same cell as p_1 : one horizontal, one vertical. However, if the placement of p_2 is specified by a row, then we know p_2 is to be a vertical pin lying in the same column as p_1 , and if specified by a column then p_2 is a horizontal pin lying in the same row as p_1 .

Before we embark on constructing our antichain, we recall the definition of an inflation from Section 2 and extend this to grid pin sequences. Letting p_1, \dots, p_n be a grid pin sequence, the *grid pin sequence inflation of the permutation corresponding to p_1, \dots, p_n by the permutations $\alpha_1, \dots, \alpha_n$* is the permutation formed by replacing each point p_i ($i = 1 \dots n$) with the permutation α_i . This is denoted $p_1[\alpha_1], p_2[\alpha_2], \dots, p_i[\alpha_i]$, but whenever $\alpha_i = 1$ we denote the trivially inflated pin $p_i[1]$ simply by p_i . We call such a permutation an *inflated grid pin permutation*.

For each k , we now use inflated grid pin sequences to construct an infinite set of permutations A^k lying in $\text{Grid}(\mathcal{M}^k)$. This construction is accompanied by Figure 6. We begin by showing how to construct the infinite uninflated grid pin sequence p_1, p_2, \dots that will be used to construct all the permutations of A^k : First place the imaginary pin p_0 in the top-right corner of the cell labelled \mathcal{C} lying in the middle of \mathcal{M}^k , and the pin p_1 as a left and down pin (also in the cell labelled by \mathcal{C}). This cell is the only one in its column, but there is one other non-empty cell in the same row, into which we place a right pin p_2 . We then recursively place each pin p_{i+1} so that it does not lie in the same cell as p_i , but shares a row or column with p_i . (Note that by Lemma 6.3, this is a sufficient description, as we know whether p_i was a horizontal or a vertical pin.) Once we have placed our first pin p_j in the cell labelled by \mathcal{D}^+ or \mathcal{D}^- , we place the next pin p_{j+1} in the same cell, and then p_{j+2} can be placed in the cell that contained p_{j-1} . Again we place one pin per cell back around until we reach the cell labelled by \mathcal{C} . Once in the cell with label \mathcal{C} , we place a second point in this cell to “turn around”, and repeat.

Finally, $A^k = \{\alpha_1, \alpha_2, \dots\}$, where $\alpha_i = p_1[21], p_2, \dots, p_{(2i-1)k-1}, p_{(2i-1)k}[\beta]$ is an inflated grid pin permutation of length $(2i-1)k+2$, with $\beta = 21$ if k is even and $\beta = 12$ otherwise.

Lemma 6.4. *Every permutation in A^k is \mathcal{M}^k -griddable.*

Proof. This is clear by considering the permitted region in which to place successive pins described in Lemma 6.3. In particular, cells labelled by monotone classes contain only monotone sequences of the right type, and the non-monotone cells contain permutations from $\oplus 21$ or $\ominus 12$ as required. \square

We need an infinite subset of A^k that is both an antichain and strongly uniquely \mathcal{M}^k -griddable. We will in fact find the latter first: knowing the uniqueness of the \mathcal{M}^k -gridding will assist us in proving that elements of A^k are incomparable. The methods of our proofs are similar in flavour to those used by Murphy and Vatter [12]. We begin by making the following straightforward observation, which we will use repeatedly.

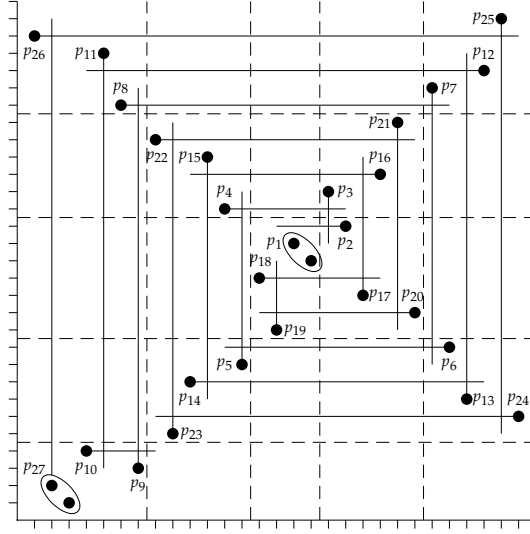


Figure 6: An element of A^8 in the grid class of \mathcal{M}^8 .

Lemma 6.5. *Let $\alpha \in A^k$ be of length $n + 2$, α' be the \mathcal{M}^k -gridded permutation of length n corresponding to the uninflated grid pin sequence p_1, \dots, p_n of α , and f be any grid mapping of \mathcal{M}^k . Then, if in $f(\alpha')$ the pins p_i and p_j ($1 < i < j < n$) lie in the same cell and are adjacent by position, they are separated precisely by the pins p_{i-1} and p_{j+1} by value. The same holds swapping “position” and “value”.*

Proof. First, by composition of functions it is sufficient to prove the statement when f is the grid inverse map, a row or column permutation, a row complement, a column reversal or the identity map. The effect of the grid inverse mapping ϕ of \mathcal{M}^k on α' is merely to swap the terms “position” and “value” in the statement of the lemma, so we can discount this case. Moreover, every other grid mapping that we need to consider preserves the relative orderings of points by position and value in any given row or column, except possibly to reverse the order. Thus the lemma is true if we can show it is true when f is the identity grid mapping, and this is easily seen by considering the placement of successive pins as described in Lemma 6.3. \square

This lemma is all that is required to prove what we need.

Lemma 6.6. *Every permutation of length at least $2(k + 1)^2 + 3$ in A^k is strongly uniquely \mathcal{M}^k -griddable.*

Proof. First, for any gridding matrix \mathcal{M} and any permutation π that possesses a unique \mathcal{M} -gridding, observe that $\phi(\pi)$ must possess a unique $\phi(\mathcal{M})$ -gridding. Thus, to show that a permutation π is strongly uniquely \mathcal{M} -griddable, it suffices to show that $f(\pi)$ has a unique $f(\mathcal{M})$ -gridding for every grid mapping f that does not use the grid inverse map.

Given $\alpha \in A^k$ of length $n + 2 \geq 2(k + 1)^2 + 3$, consider the permutation α' corresponding to the uninflated grid pin sequence of length $n \geq 2(k + 1)^2 + 1$ used to create α . We will prove the result for α' , from which the required result easily follows. Label the points of $f(\alpha')$ with p_1, p_2, \dots, p_n according to the grid pin sequence used to construct α' , i.e. the point with label p_i in $f(\alpha')$ is the image under f of the pin p_i . Suppose that $f(\alpha')$ has two $f(\mathcal{M})$ -griddings, the first being the gridding inherited from the original gridding of α' under the grid mapping f ,

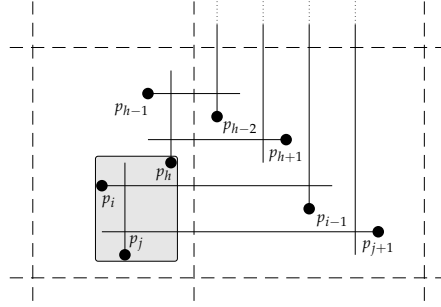


Figure 7: Finding points adjacent by position or by value in the end cell for the proof of Lemma 6.6.

and the second some other gridding. Since $f(\alpha')$ contains at least $2(k+1)^2 + 1$ points, there must be one cell of the second gridding that contains 3 points — p_h , p_i and p_j with $h < i < j$ — which also lie together in a common cell in the first gridding. (Note that these two cells do not yet necessarily correspond to the same cell of the gridding matrix.) Since $\text{rect}(p_h, p_i, p_j)$ is necessarily contained in the nominated cell in each gridding, so also must any other points inside $\text{rect}(p_h, p_i, p_j)$. Thus, by shrinking the rectangle and relabelling if necessary, we can assume that $\text{rect}(p_h, p_i, p_j)$ contains only the points p_h , p_i and p_j .

We claim that two of these three points must be adjacent either by position or by value. This is immediate if they lie in one of the monotone cells in the first gridding by the grid pin sequence construction, so suppose they lie in one of the non-monotone cells. Up to grid mappings, we now have a situation such as the one depicted in Figure 7. In particular either $p_h = p_{i-1}$ or $p_i = p_{j-1}$ from which it follows that one of these two pairs are adjacent by position or by value as they are consecutive pins. Note also that this excludes the possibility that either of the two pins is p_1 or p_n .

Thus, suppose we have pins p_i and p_j that are adjacent by value and in both griddings lie together in a cell. We now apply Lemma 6.5 repeatedly: the points p_{i-1} and p_{j+1} separate p_i from p_j by position. If the cell containing p_i and p_j in the first gridding is not one of the two cells with non-monotone labels (we will encounter the other case shortly), then p_{i-1} and p_{j+1} cannot lie in the same cell as p_i and p_j (as $p_i p_{i-1} p_{j+1} p_j$ is not a monotone sequence), and so must lie in the only other non-empty cell in the column. In the second gridding, this requires that there must be another non-empty cell in the same column as the first cell, and that it must contain the points p_{i-1} and p_{j+1} . We now apply Lemma 6.5 to p_{i-1} and p_{j+1} , and proceed in this way until we encounter the pins p_{i-l} and p_{j+l} which lie in one of the two non-monotone cells in the first gridding.

Without loss suppose that these pins are adjacent by position (an analogous argument holds when the pins are adjacent by value), so that for each gridding the pins p_{i-l+1} and p_{j+l-1} lie together in a different cell but in the same column of \mathcal{M} as p_{i-l} and p_{j+l} . In particular, this means that the non-monotone cell containing p_{i-l} and p_{j+l} in the first gridding is the unique non-empty cell in its row of \mathcal{M} . Since p_{i-l} and p_{j+l} are adjacent by position, they are separated by value by the pins p_{i-l-1} and p_{j+l+1} . Moreover, reading $f(\alpha')$ from left to right, they appear in the order $p_{i-l-1} p_{i-l} p_{j+l} p_{j+l+1}$ or its reverse, and form one of the patterns 2143 or 3412. We claim that in the second gridding these points all lie in one of the non-monotone cells. If not, then since p_{i-l} and p_{j+l} lie in the same (monotone) cell in the second gridding, neither p_{i-l-1}

nor $p_{j+\ell+1}$ can also lie in this cell as this would give rise to a non-monotone pattern. However, $p_{i-\ell-1}$ and $p_{j+\ell+1}$ must lie in the same row of the gridding, but on opposing sides of $p_{i-\ell}$ and $p_{j+\ell}$, and this is impossible as there are at most 2 non-empty cells in each row.

We now have two pairs of points which are adjacent by value, namely $p_{i-\ell-1}, p_{i-\ell}$ and $p_{j+\ell}, p_{j+\ell+1}$. to which we can apply Lemma 6.5. After following these pairs around by using Lemma 6.5 k times, we reach the other non-monotone cell in the first gridding with the two pairs of pins $p_{i-\ell-k-1}, p_{i-\ell+k}$ and $p_{j+\ell-k}, p_{j+\ell+k+1}$. Note that the cells that the second gridding uses are now forced to be the same as the first gridding, i.e. all pins encountered so far lie in the same cells in both griddings. Moreover by our original assumptions on i and j we have $i - \ell + k = j + \ell - k - 1$, and so we have shown that each of $p_{i-\ell-k-1}, p_{i-\ell-k}, \dots, p_{j+\ell+k+1}$ is placed in the same cell in both griddings.

Now repeat the argument given earlier to handle non-monotone cells on the two pairs $p_{i-\ell-k-1}, p_{i-\ell+k}$ and $p_{j+\ell-k}, p_{j+\ell+k+1}$. Each of these two pairs gives rise to two new pairs which can be followed back around $f(\mathcal{M})$ via Lemma 6.5, although note that we need only follow the pairs $p_{i-\ell-k-2}, p_{i-\ell-k-1}$ and $p_{j+\ell+k+1}, p_{j+\ell+k+2}$, being the only ones that give rise to pins that we haven't yet seen. We repeat this process of collecting pins, until we encounter either p_1 or p_n , whence we follow just one pair of points around until that reaches the other end of the pin sequence.

Thus all points p_1, \dots, p_n must be placed in the same cells in both griddings, as required. The extension to $f(\alpha)$ is trivial: when we encountered pin p_1 or p_n in the above argument, we now encounter two points, both of which have their cell placements forced. \square

Lemma 6.7. *The set of permutations of length at least $2(k+1)^2 + 3$ in A^k is an antichain with respect to permutation containment.*

Proof. Let α, β be two permutations in A^k of lengths m and n respectively, both of length at least $2(k+1)^2 + 3$. Assuming $m < n$, suppose for a contradiction that $\alpha \leq \beta$, and fix one such embedding. Since both α and β have unique \mathcal{M}^k -griddings, this implies not only that $\alpha \leq_{\mathcal{M}^k} \beta$, but that our fixed embedding witnesses this gridded containment. By their construction, we can write α and β as inflated grid pin permutations, thus $\alpha = p_1[21], p_2, \dots, p_m[\gamma]$ and $\beta = q_1[21], q_2, \dots, q_n[\gamma]$, where $\gamma = 12$ or 21 depending on the parity of k . Moreover, since the gridgings must match up, the fictive pin p_0 is placed in exactly the same position as q_0 , and so we will assume that p_0 is mapped to q_0 .

We claim that the inflated pin $p_1[21]$ must be mapped to $q_1[21]$. If not, then $p_1[21]$ must be mapped to two consecutive pins in the same cell, this being the only other way to form a 21 pattern in the cell labelled by \mathcal{C} . Thus suppose $p_1[21]$ is mapped to the pins q_{2ki} and q_{2ki+1} . Then the left pin p_2 must be mapped to some left pin with index at most $2kj + 2$, where $j < i$, since all later pins do not separate q_{2ki} from q_0 . Next, by a similar argument, p_3 can be mapped to a pin in β with index at most $2kj + 3$, and so on, until we find that pin p_{2k} (which is the next pin of α we encounter in the cell labelled by \mathcal{C}) must be mapped to a pin with index at most $2kj + 2k \leq 2ki$. This, however, is impossible because p_{2k} must lie below and to the left of $p_1[21]$, but yet in β its image cannot.

Now, since p_0 and q_0 , and $p_1[21]$ and $q_1[21]$ coincide, by the properties of grid pin sequences we conclude that p_2 must be mapped to q_2 , p_3 to q_3 , and so on. This, however, becomes impossible when we try to map $p_m[\gamma]$ into q_m : by non-interaction, there are no points other than q_m in β that separate $\text{rect}(q_{m-2}, q_{m-1})$, but yet we need two in α to separate $\text{rect}(p_{m-2}, p_{m-1})$. \square

Thus we have:

Proof of Theorem 1.2. First note that if $G_{\mathcal{M}}$ contains a cycle then it contains a non-partially well-ordered class by Theorem 2.5, so we may assume that $G_{\mathcal{M}}$ is acyclic and has a component with at least two non-monotone griddable entries. It suffices to prove that $\text{Grid}(\mathcal{M})$ is not partially well-ordered for a gridding matrix \mathcal{M} satisfying the conditions of Theorem 5.1, since every other grid class that needs to be considered contains such a class. Thus \mathcal{M} is equivalent under grid mappings to some \mathcal{M}^k .

By Lemmas 6.6 and 6.7, A^k contains an infinite antichain of strongly uniquely \mathcal{M}^k -griddable permutations. By Theorem 5.1, \mathcal{M} can be obtained from one of the matrices \mathcal{M}^k for some k via a grid mapping, and so $\text{Grid}(\mathcal{M})$ is not partially well-ordered by Theorem 4.4. \square

The proof of Theorem 1.1 now follows by combining Theorems 1.2 and 3.1.

7 Concluding Remarks

Monotone griddable classes. Theorem 1.1 cannot immediately be extended by replacing “monotone classes” with “monotone gridable classes”. For example, if $\mathcal{M} = (\mathcal{C} \ \mathcal{D})$ where $\mathcal{C} = \begin{pmatrix} \text{Av}(12) \\ \text{Av}(21) \end{pmatrix}$ and $\mathcal{D} = \begin{pmatrix} \text{Av}(21) \\ \text{Av}(12) \end{pmatrix}$, then $\text{Grid}(\mathcal{M})$ contains $\text{Grid} \begin{pmatrix} \text{Av}(12) & \text{Av}(21) \\ \text{Av}(21) & \text{Av}(12) \end{pmatrix}$ which is not partially well-ordered by Theorem 2.5. (Note also that both \mathcal{C} and \mathcal{D} contain only finitely many simple permutations, so adding this restriction would not help.) However, Theorem 1.1 can be used indirectly for such gridding matrices by refining the gridding until all cells are monotone or non-monotone griddable — the details of such a refinement are beyond the scope of this paper, but see Vatter [14] for more details on griddability.

Grid pin sequences and antichains. Every currently known antichain in the permutation containment order can be built, via grid symmetries, from an infinite grid pin sequence. Note, however, that not every known antichain is constructed simply by inflating the first and last points of a grid pin sequence: see Murphy’s thesis [11] for some other “anchoring” constructions. This naturally leads one to wonder whether there are antichains that cannot be formed in this way. The *closure* of a set A of permutations is the class of permutations contained in the permutations of A , $\text{Cl}(A) = \{\pi : \pi \leq \alpha \text{ for some } \alpha \in A\}$.

Question 7.1. *Does there exist an antichain A for which $\text{Cl}(A)$ does not contain arbitrarily long grid pin sequences?*

Partial well-order decidability. A first step in answering more general questions of decidability could be to consider the following question.

Question 7.2. *Is it decidable whether a given gridding matrix whose entries are partially well-ordered permutation classes defines a grid class that is partially well-ordered or not?*

Theorems 1.1 and 1.2 make some progress towards answering this, particularly in the extension to monotone griddable classes discussed earlier. However, a complete answer would also need to consider gridding matrices where each component is a tree with entries given by monotone classes except for one cell, which is labelled by a non-monotone-griddable class with

arbitrarily long simple permutations. This situation is currently amenable neither to Higman's Theorem nor grid pin sequences.

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