Estimation of wall effects on floating cylinders

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Abstract

Under the assumptions of classical linearised water wave theory, the time-harmonic two-dimensional motion of a rigid body floating in the surface of a fluid may be characterised by various coefficients which express components of the hydrodynamic forces acting on that body. For a single body in isolation, these are relatively simple to calculate. For bodies placed next to a vertical wall, the corresponding calculations are often much more complicated. In this paper we use the well-known wide-spacing approximation to develop approximations to the hydrodynamic coefficients for a body having a vertical plane of symmetry, but otherwise of arbitrary cross-section, next to a wall solely in terms of the results for an isolated body. Exact results are compared with the wide-spacing approximations for semi-immersed circular cylinders and cylinders of rectangular cross-section. They show that the approximation works remarkably well over all frequency ranges and even when the cylinders are very close to the wall.

1 Introduction

The hydrodynamic coefficients of added mass, radiation damping and exciting force are key ingredients in determining the behaviour of a floating body in waves and powerful computer codes have been developed for estimating them for three-dimensional body shapes of interest to naval architects and marine engineers. A well-established simplified method of computing the hydrodynamic forces on such shapes is to use strip theory which involves dividing the body into sections along its length, solving a two-dimensional problem for each section and then integrating over the length to obtain the overall effect. See Newmann (1977, chapter 7) for details. For this, only the coefficients of added mass, radiation damping and exciting force for a section of a cylinder in two dimensions are required, quantities which are much less computationally expensive to determine.

It is well-known that the behaviour of a floating body in waves is radically affected by the proximity of a rigid boundary such as a harbour wall, and this effect is manifested in major changes to the hydrodynamic coefficients due to the presence of the wall. This is because the isolated portion of free surface between the body and the wall experiences near-resonant behaviour close to certain frequencies. The lowest of the frequencies is associated with the so-called 'Helmholtz' or 'pumping-mode' resonance in which the fluid between the

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body and the wall tends to oscillate as a slug of fluid. At higher frequencies, resonant modal surface oscillations are excited between the wall and the cylinder. Examples of the results obtained for bodies oscillating next to walls are given in Wang & Wahab (1971) and Yeung & Seah (2007) where these resonances are identified with large rapid variations in added mass and radiation damping coefficients. In each of these examples, the problem for a cylinder oscillating next to a wall was replaced by the equivalent problem of the original cylinder plus an image cylinder oscillating in heave (vertically) in tandem.

In this paper we shall study this problem of a cylinder oscillating next to a wall. Specifically we shall consider an arbitrary two-dimensional symmetrical cylindrical section making small time harmonic oscillations in either sway, heave or roll, or fixed in an incoming wave-train, in the presence of a rigid vertical wall a given distance away from it.

In the next section new identities will be derived between the radiation damping coefficients b^w_{jk} $(j, k = 1, 2, 3)$ and the exciting force F^w_j $(j = 1, 2, 3)$, and the far field radiated wave amplitudes A^w_j where $j = 1$ for sway, $j = 2$ for heave, and $j = 3$ for roll. The presence of the wall greatly increases the complexity of the computation of the hydrodynamic coefficients and so it is shown in section 3 how simpler approximate expressions can be derived for the added mass and radiation damping coefficients a^w_{jk} and b^w_{jk} $(j, k = 1, 2, 3)$ in terms of similar quantities in the absence of the wall, which are far easier to compute. The basis of these approximations is the so-called 'wide-spacing approximation' which has been used extensively and to good effect in similar problems. Most closely related to the current problem in this context is the work of Srokosz & Evans (1979). They were interested in using a pair of independently oscillating cylinders to harness wave energy. In particular, they produced numerical and analytical results for pairs of thin barriers making small-amplitude time-harmonic rolling motions about the vertical. In their paper, they use the wide-spacing approximation to develop an approximation to various quantites of interest. However, the hydrodynamic problem Srokosz & Evans (1979) were solving, in which one cylinder is forced to move in the presence of a second fixed body, is not the same as ours. It should also be pointed out that Srokosz & Evans (1979) do not develop their approximation into a form which is easy to implement. In contrast, the expressions developed in this paper are explicit, straightforward to implement and rely only on the solution to certain key properties of wave radiation problems. In order to simplify the approximation to the extent that we do here, a simplifying assumption that the cylinder has a vertical plane of symmetry is made.

As an example of the power of the approximation, a comparison is made with exact results for the cases of circular and rectangular cylinders in Section 4. We consider all three independent motions: heave, sway and roll. The case of a semi-immersed circular cylinder is the simpler of the two examples used as the roll component is zero. The wide-spacing approximation is expected to work well in the regime $Ks \gg 1$, where $K = \omega^2/g$ and ω is angular frequency, s is the spacing between the wall and the cylinder and g is gravity, Our results show that the approximation continues to work remarkably well for values of Ks much less than unity and for certain quantities happens to give the correct results in the limit as $Ks \to 0$.

2 Formulation

A long cylinder of arbitrary, but uniform, cross-section floats on the surface of deep water, density ρ. The boundary of the cylinder in contact with the water is denoted S_B. The origin is centred inside the cylinder with y directed downwards from the level of the undisturbed free surface. The fluid is bounded laterally by a rigid vertical wall at $x = -b$. The cylinder can make small time-harmonic oscillations of angular frequency ω in either sway, heave or roll or be held fixed in an incoming wavetrain of the same frequency from $x = \infty$. The roll axis (termed the metacentre in ship design) is assumed to pass through the point $(0, c)$, where c may be positive or negative. Then on the basis of classical linear water wave theory there exists harmonic velocity potentials describing the two-dimensional fluid motion for the three separate radiation and scattering problems from which we can remove the time dependence by assuming that

$$\Phi_j^w(x, y, t) = \Re\{\phi_j^w(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\} \tag{2.1}$$

$$\Phi_S^w(x, y, t) = \Re\{\phi_S^w(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\}. \tag{2.2}$$

Here $j = 1, 2, 3$ refers to sway, heave, and roll respectively. The superscript w is used to denote the presence of the wall. Each of the time-independent potentials ϕ_j^w, $(j = 1, 2, 3)$ and ϕ_S^w, when represented by ψ, satisfies $\nabla^2\psi = 0$ in the fluid, $|\nabla\psi| \to 0$ as $y \to \infty$, the linearised free surface condition,

$$K\psi + \frac{\partial\psi}{\partial y} = 0, \qquad \text{on } y = 0 \tag{2.3}$$

where $K = \omega^2/g$ is the frequency parameter (g is gravitational acceleration), and the wall condition,

$$\frac{\partial\psi}{\partial x} = 0, \qquad \text{on } x = -b. \tag{2.4}$$

Each potential has its own asymptotic form in the far-field and satisfies different kinematic boundary conditions on the cylinder. For the scattering potential we write

$$\phi_S^w(x, y) \sim (gA/\omega)(\mathrm{e}^{-\mathrm{i}Kx} + R^w\mathrm{e}^{\mathrm{i}Kx})\mathrm{e}^{-Ky}, \qquad x \to \infty \tag{2.5}$$

where A is the complex amplitude of the incident wave and R^w is the complex reflection coefficient, with

$$\frac{\partial\phi_S^w}{\partial n} = 0, \qquad (x, y) \in S_B \tag{2.6}$$

where $\partial/\partial n$ represents the derivative in the direction of the normal.

In the far-field, the radiation potentials are represented by outgoing waves of complex amplitude A_j^w

$$\phi_j^w(x, y) \sim A_j^w\mathrm{e}^{\mathrm{i}Kx-Ky}, \qquad x \to \infty, \qquad (j = 1, 2, 3) \tag{2.7}$$

with

$$\frac{\partial\phi_j^w}{\partial n} = n_j, \qquad (x, y) \in S_B \qquad (j = 1, 2, 3). \tag{2.8}$$

In the case of sway and heave, the n_j are the direction cosines in the x $(j = 1)$ and y $(j = 2)$ directions of the unit normal directed into the cylinder from the fluid. In the case of roll $(j = 3)$ we have $n_3 = xn_2 + (y - c)n_1$.

Consider now a hypothetical harmonic potential ψ_j^w satisfying (2.3), (2.4) and where

$$\psi_j^w(x, y) \sim (C_j\mathrm{e}^{-\mathrm{i}Kx} + D_j\mathrm{e}^{\mathrm{i}Kx})\mathrm{e}^{-Ky}, \qquad x \to \infty. \tag{2.9}$$

If we now apply Green's third identity around the boundary of the fluid region closed by a vertical line at a (large) distance X where the asymptotic behaviour (2.9) applies, we find in the limit as $X \to \infty$, that

$$\int_{S_B} \left(\psi_1^w \frac{\partial \psi_2^w}{\partial n} - \psi_2^w \frac{\partial \psi_1^w}{\partial n} \right) \mathrm{d}s + \mathrm{i}(C_1 D_2 - C_2 D_1) = 0. \tag{2.10}$$

By choosing various combinations of radiation or scattering potentials and using the asymptotic results (2.5) and (2.7) in (2.10), we derive the following results. With $\psi_1^w = \phi_S^w$, $\psi_2^w = \bar{\phi}_S^w$ (where the bar denotes the complex conjugate) we obtain

$$|R^w| = 1 \tag{2.11}$$

which simply confirms that all incident wave energy is reflected. With $\psi_1^w = \phi_S^w$, $\psi_2^w = \phi_j^w - \bar{\phi}_j^w$, since $\partial\psi_2/\partial n = 0$ for $(x, y) \in S_B$, we obtain

$$R^w = -A_j^w / \bar{A}_j^w. \tag{2.12}$$

This remarkable result shows that the phases of each of the radiated amplitudes, $\arg\{A_j^w\} = \theta_j^w$ can only differ by a multiple of π. We turn next to the added inertia and damping coefficients (see, for example, Mei (1983, pp.302/3)). The time-independent restoring force matrix f_{jk}^w representing the hydrodynamic force in the component k due to a forcing in mode j, is

$$f_{jk}^w \equiv -b_{jk}^w + \mathrm{i}\omega a_{jk}^w = \mathrm{i}\rho\omega \int_{S_B} \phi_j^w n_k \mathrm{d}s \tag{2.13}$$

decomposed into added-inertia and radiation damping coefficients a_{jk}^w and b_{jk}^w.

If we let $\psi_1^w = \phi_j^w$, $\psi_2^w = \phi_k^w$ we obtain from (2.10) $f_{jk} = f_{kj}$. Again with the same ψ_1^w and $\psi_2^w = \bar{\phi}_k^w$ we get

$$b_{jk}^w = \tfrac{1}{2}\rho\omega A_j^w \bar{A}_k^w, \qquad (j, k = 1, 2, 3) \tag{2.14}$$

which are all real and it follows that

$$b_{jk}^w b_{kj}^w = b_{jj}^w b_{kk}^w, \qquad (j, k = 1, 2, 3). \tag{2.15}$$

Finally we consider the wave-induced exciting force on the cylinder in direction j (see Mei (1983, p.302)) which is

$$f_{S,j}^w = \mathrm{i}\rho\omega \int_{S_B} \phi_S^w n_j ds. \tag{2.16}$$

With $\psi_j = \phi_S^w$, $\psi_2 = \phi_j^w$ equation (2.10) gives

$$f_{S,j}^w = \rho g A A_j^w. \tag{2.17}$$

3 A wide-spacing approximation to a_{jk}^w, b_{jk}^w and A_j^w

We consider the forced sway, heave or roll motion of the cylinder centred at the origin in the presence of a rigid wall at $x = -b$ on which a Neumann condition is imposed on the potential. Cylinders having a vertical plane of symmetry only are considered here on grounds of simplicity. This allows us to simplify the results obtained.

The wide-spacing approximation is based on the assumption that the wall is far enough from the cylinder for local evanescent terms to be neglected. A full discussion of the approximation, its applications and limitations can be found in Martin (2006). The effect of the wall will be to reflect a radiated wave back towards the cylinder to be partially reflected and transmitted by the cylinder. The reflected part will again be reflected back to the cylinder and the whole process will continue indefinitely. All these partial waves can be combined and the overall effect will be equivalent to the radiated wave field travelling away from the cylinder in the absence of the wall, together with an incident wave of unknown amplitude from the left being scattered by the (fixed) cylinder, which models the effect of the wall. Finally the asymptotic wave field of these combined potentials must satisfy the Neumann condition on $x = -b$. Thus we have

$$\phi_j^w = \phi_j + D_j \phi_S, \qquad j = 1, 2, 3 \tag{3.1}$$

where the first term on the right-hand-side is the radiation potential for a cylinder making sway, heave or roll motions at the origin but in the absence of the wall. The second term is the scattered potential due to a wave incident from $x = -\infty$ on the cylinder held fixed at the origin, again in the absence of the wall.

Note that in (3.1) and throughout this section of the paper, equals signs will be used to indicate that the calculations are exact under the assumptions of the wide-spacing approximation in which interactions of local wave effects are ignored.

We have the far-field expressions for each of the potentials in (3.1) given by

$$\phi_j \sim \begin{cases} (-1)^j A_j \mathrm{e}^{-\mathrm{i}Kx - Ky}, & x \to -\infty \\ A_j \mathrm{e}^{\mathrm{i}Kx - Ky}, & x \to +\infty \end{cases} \tag{3.2}$$

(where left-right symmetry of the cylinder is assumed) and

$$\phi_S \sim \begin{cases} (gA/\omega)(\mathrm{e}^{\mathrm{i}Kx} + R\mathrm{e}^{-\mathrm{i}Kx})\mathrm{e}^{-Ky}, & x \to -\infty \\ (gA/\omega)T\mathrm{e}^{\mathrm{i}Kx - Ky}, & x \to +\infty \end{cases} \tag{3.3}$$

where R and T are the reflection and transmission coefficients for the fixed cylinder, in the absence of the wall, dependent on frequency and A is the complex incident wave amplitude.

It follows from (3.1) that for large positive x

$$\phi_j^w \sim (A_j + (gA/\omega)D_j T)\mathrm{e}^{\mathrm{i}Kx - Ky} \tag{3.4}$$

and for large negative x

$$\phi_j^w \sim \Big((-1)^j A_j + (gA/\omega)D_j R\Big)\mathrm{e}^{-\mathrm{i}Kx - Ky} + (gA/\omega)D_j \mathrm{e}^{\mathrm{i}Kx - Ky}. \tag{3.5}$$

These asymptotic forms are now assumed to hold near the wall along $x = -b$, $y > 0$ where a Neumann condition is now imposed on the potential ϕ_j^w. It follows that

$$(-1)^j A_j + (gA/\omega)D_j R = (gA/\omega)D_j \mathrm{e}^{-\mathrm{i}\lambda} \tag{3.6}$$

with $\lambda = 2Kb$ whence

$$D_j = (\omega/gA)(-1)^j A_j/(\mathrm{e}^{-\mathrm{i}\lambda} - R). \tag{3.7}$$

Substituting (3.7) into (3.4) and comparing with (2.7) gives

$$A_j^w = \delta_j A_j, \qquad \text{where} \qquad \delta_j = \left(\frac{R - (-1)^j T - \mathrm{e}^{-\mathrm{i}\lambda}}{R - \mathrm{e}^{-\mathrm{i}\lambda}} \right). \tag{3.8}$$

Note that this implies $\delta_1 = \delta_3$. According to the decomposition made in (3.1), the restoring force matrix, from (2.13), is approximated under the wide-spacing approximation by

$$f_{jk}^w \equiv -b_{jk}^w + \mathrm{i}\omega a_{jk}^w = f_{jk} + D_j f_{S,k} \tag{3.9}$$

where $f_{S,k}$ is the exciting force on the *fixed* cylinder in the direction k due to an incident wave of unit amplitude from $x = -\infty$. Also,

$$f_{jk} \equiv -b_{jk} + \mathrm{i}\omega a_{jk} \tag{3.10}$$

is the force matrix for a cylinder in the absence of a wall in the direction k due to forced motion in mode j decomposed into its usual added-inertia and radiation damping coefficients, which we assume are known. Notice that (3.9) only holds provided $j + k$ is *even* since if $j + k$ is odd, then the symmetry of the cylinder implies that the term f_{jk} is identically zero (for example, a heave motion induces neither sway force nor roll moment on a symmetric cylinder). On the other hand f_{11}, f_{22} and f_{33} are clearly non-zero in general and so is $f_{13} = f_{31}$ since a sway motion will produce a roll moment and a roll motion a sway force.

We make use of reciprocal relations satisfied for a symmetric cylinder oscillating or fixed in an incident wave train, in the absence of the wall. See, for example, Mei (1983, pp.301/2). Thus we have

$$f_{S,k} = \rho g (-1)^k A A_k \tag{3.11}$$

and

$$b_{jk} = \tfrac{1}{2}\rho\omega(1 + (-1)^{j+k}) A_j \bar{A}_k \tag{3.12}$$

(clearly zero if $j + k$ is odd) and, finally, the Newman/Bessho relations (see Mei (1983), p.328)

$$R + (-1)^j T = -A_j/\bar{A}_j = -\mathrm{e}^{2\mathrm{i}\theta_j}, \tag{3.13}$$

where θ_j is the phase of the far-field radiated amplitude in the jth mode; all three relations hold for $j, k = 1, 2, 3$. Notice from (3.13) that $\theta_1 = \theta_3 (\mathrm{mod}\pi)$. Also it follows that

$$R = -\mathrm{e}^{\mathrm{i}(\theta_j + \theta_2)} \cos(\theta_j - \theta_2), \tag{3.14}$$

$$T = \mathrm{i}\mathrm{e}^{\mathrm{i}(\theta_j + \theta_2)} \sin(\theta_j - \theta_2) \tag{3.15}$$

where $j = 1$ or $j = 3$ (incidently demonstrating that the energy density flux condition $|R|^2 + |T|^2 = 1$ is satisfied). Furthermore,

$$\chi \equiv -\mathrm{i}R/T = \cot(\theta_j - \theta_2) \tag{3.16}$$

is real.

We first assume $j + k$ is even so that (3.12) simplifies. Then we obtain from (3.9), using (3.7) and (3.10)–(3.13),

$$b_{jk}^w - \mathrm{i}\omega a_{jk}^w = b_{jk}\gamma_k - \mathrm{i}\omega a_{jk} \tag{3.17}$$

where

$$\gamma_k \equiv \alpha_k + \mathrm{i}\beta_k = \frac{\mathrm{e}^{-\mathrm{i}\lambda} + (-1)^k T}{\mathrm{e}^{-\mathrm{i}\lambda} - R} \tag{3.18}$$

and we have

$$b^w_{jk} = b_{jk}\alpha_k \qquad \text{and} \qquad a^w_{jk} = a_{jk} - \beta_k b_{jk}/\omega. \tag{3.19}$$

When $j+k$ is odd, there is no contribution from f_{jk} in (3.9) and we have

$$f^w_{jk} = D_j f_{S,k}. \tag{3.20}$$

So from (3.7) and (3.11) we find

$$b^w_{jk} - \mathrm{i}\omega a^w_{jk} = \frac{\rho\omega A_j A_k}{\mathrm{e}^{-\mathrm{i}\lambda} - R}. \tag{3.21}$$

To summarize then, on the basis of the wide-spacing assumption, approximate expressions for the added inertia and radiation damping coefficients a^w_{jk} and b^w_{jk} for a symmetric cylindrical section oscillating next to a wall, are given by (3.17) and (3.18) when $j+k$ is even and by (3.21) when $j+k$ is odd; the far field complex radiation amplitude A^w_j is given by (3.8) which also provides an expression for the wave-induced exciting force $f^w_{S,j}$ given by (2.17).

Now it is clear from (3.12) to (3.15) that R, T, and b_{jk} can all be expressed in terms of the complex far-field radiated amplitudes A_j. It follows from (3.8), (3.17), (3.18) and (3.21) that the same is true of A^w_j, b^w_{jk} and also a^w_{jk} (except when $j+k$ is even, when it involves a_{jk}), since these equations involve the complex constants γ_k and δ_k, defined in (3.18), (3.8) which only depend on R and T. These constants may in turn be expressed solely in terms of the A_j as follows.

First (3.8) can be reduced to

$$\delta_k = \frac{2\mathrm{e}^{\mathrm{i}(\theta_l - Kb)}\cos\mu_l}{\mathrm{e}^{-\mathrm{i}\lambda} - R} \tag{3.22}$$

where $\mu_k = \theta_k + Kb$ and l is either $k+1$ or $k-1$ provided that number falls in the set $\{1,2,3\}$. Also, we can write

$$\gamma_k = \frac{1 - (-1)^k \bar{R}T - (R - (-1)^k T)\mathrm{e}^{\mathrm{i}\lambda} - \mathrm{e}^{-\mathrm{i}\lambda}\bar{R} + \mathrm{e}^{\mathrm{i}\lambda}R}{|\mathrm{e}^{-\mathrm{i}\lambda} - R|^2} \tag{3.23}$$

and it follows, using (3.16) to give $\chi|T|^2 = \mathrm{i}\bar{R}T$, that

$$\Re\{\gamma_k\} = \alpha_k = \frac{2\cos^2\mu_l}{|\mathrm{e}^{-\mathrm{i}\lambda} - R|^2} \tag{3.24}$$

and, again, l takes the values $k-1$ or $k+1$ provided that number falls in the set $\{1,2,3\}$. A similar manipulation shows that

$$\Im\{\gamma_k\} = \beta_k = \frac{\frac{1}{2}(-1)^k \sin 2(\theta_l - \theta_2) - \sin 2\mu_k}{|\mathrm{e}^{-\mathrm{i}\lambda} - R|^2} \tag{3.25}$$

where l is either 1 or 3. A further simplification, using (3.12), is

$$\mathrm{e}^{-\mathrm{i}\lambda} - R = \mathrm{e}^{-\mathrm{i}Kb}(\mathrm{e}^{\mathrm{i}\theta_k}\cos\mu_k + \mathrm{e}^{\mathrm{i}\theta_2}\cos\mu_2) \tag{3.26}$$

so that

$$|e^{-i\lambda} - R|^2 = \cos^2 \mu_l + \cos^2 \mu_2 + 2\cos\mu_l \cos\mu_2 \cos(\theta_l - \theta_2) \tag{3.27}$$

where $l = 1$ or 3.

With these simplifications we obtain, for $j + k$ even,

$$b^w_{jk} = \frac{2b_{jk}\cos^2 \mu_l}{|e^{-i\lambda} - R|^2} \tag{3.28}$$

where $\mu_l = \theta_l + Kb$, and l is either $k + 1$ or $k - 1$ provided that number falls in the set $\{1, 2, 3\}$, and

$$a^w_{jk} = a_{jk} - (\beta_k/\omega)b_{jk} \tag{3.29}$$

where β_k is given by (3.25), whilst for $j + k$ odd, we obtain, from (3.12), and (3.21)

$$b^w_{jk} = \frac{2\cos\mu_j \cos\mu_k (b_{jj}b_{kk})^{\frac{1}{2}}}{|e^{-i\lambda} - R|^2} \tag{3.30}$$

and

$$\omega a^w_{jk} = \frac{-(b_{jj}b_{kk})^{\frac{1}{2}} \sin(\mu_j + \mu_k)}{|e^{-i\lambda} - R|^2}. \tag{3.31}$$

In the next section we examine the accuracy of these approximations in two distinct cases. Before that we shall verify that our approximations satisfy the exact reciprocal relations derived in section 2.

We begin with the relation

$$\arg\{A^w_j\} = \arg\{A^w_k\}(\mathrm{mod}\pi), \qquad (j \neq k). \tag{3.32}$$

For $j + k$ even, the result is trivial if $j = k$ and follows immediately from (3.8), (3.13), for $j \neq k$. For $j + k$ odd, we have from (3.8), (3.22)

$$\frac{A^w_j}{A^w_k} = \frac{\delta_j A_j}{\delta_k A_k} = \frac{|A_j|\cos\mu_k}{|A_k|\cos\mu_j}, \tag{3.33}$$

the realness of the right-hand side confirming (3.32).

Next we consider the identity (2.14). For $j + k$ even

$$b^w_{jk} = \alpha_k b_{jk} = \rho\omega\alpha_k A_j \bar{A}_k = \rho\omega\alpha_k \frac{A^w_j \bar{A}^w_k}{\delta_j \bar{\delta}_k} = \tfrac{1}{2}\rho\omega A^w_j \bar{A}^w_k \tag{3.34}$$

where (3.12), (3.8), (3.22) and (3.24) have been used. For $j + k$ odd, we have from (3.21)

$$b^w_{jk} = \frac{2\rho\omega|A_j||A_k|\cos\mu_j \cos\mu_k}{|e^{-i\lambda} - R|^2}. \tag{3.35}$$

But $A^w_j A^w_k = \delta_j \bar{\delta}_k A_j \bar{A}_k$ follows from (3.33) and so

$$b^w_{jk} = \frac{4A_j \bar{A}_k \cos\mu_j \cos\mu_k (e^{i(\theta_k - \theta_j)})}{|e^{-i\lambda} - R|^2} \tag{3.36}$$

from (3.24). Thus $b^w_{jk} = \frac{1}{2}\rho\omega A^w_j \bar{A}^w_k$ for $j + k$ also.

4 Comparison with exact results

In this section we consider two simple cylindrical sections oscillating next to a wall, for which powerful semi-analytical results can be obtained for the various hydrodynamic coefficients using full linear theory. These are compared with the wide-spacing approximations developed in the previous section which blend the hydrodynamic coefficients for the corresponding cylindrical section in the absence of a wall.

The first set of results are for a half-immersed circular cylinder. In the absence of a wall, the theory was first given by Ursell (1949). The problem was been revisited by Martin and Dixon (1983) and recently by Porter & Evans (2008*b*) using the same method which expands the potential in terms of sources, dipoles, and wave-free potentials. When a half-immersed circular cylinder is placed next to a wall, the solution method becomes much more complicated. The effect of the wall can be modelled by including a second identical cylinder moving in equal and opposite manner in the image of the wall. This problem was considered by Porter & Evans (2008*b*).

The second set of results are concerned with the motion of a vertical partially-immersed rectangular cylinder. The method of solution with and without the wall is similar. In each rectangular subdomain of the fluid, the potential can be expressed by separation solutions and matching across common interfaces allow integral equations to be formulated for unknown functions relating to horizontal fluid velocities. The integral equations are solved by using a Galerkin scheme in which the unknown functions are approximated by series which incorporate the anticipated singularity in the velocity field at the corner of the rectangular cylinder. This scheme provides provides accurate and efficient numerical approximations to the hydrodynamic coefficients. For cylinders in heave, both in isolation and next to a vertical wall, the method of solution has been outlined in Porter & Evans (2008*a*). An alternative 'mode-matching' technique which does not take explicit account of singularities in the velocity field which had previously been adopted for the problem of a heaving cylinder next to a wall by Yeung & Seah (2007).

It is not conceptually difficult to adapt the problems considered by Porter & Evans (2008*a*) to consider the additional sway and roll motions, though the details are both tedious and complicated. For this reason, and since there is no other literature reporting these calculations, a technical report outlining the method of solution for the full set of independent motions is provided as an online supplement to the current paper in Porter (2008).

Throughout the results section, we report results on the non-dimensionalised added masses and radiation damping coefficients, defined by

$$\mu_{ij}^w = a_{ij}^w/M, \qquad \nu_{ij}^w = b_{ij}^w/(M\omega), \qquad i,j = 1,2$$

where M is the mass of the floating cylinder (as we are assuming a freely-floating cylinder, this is determined by Archimedes' principle). For $i = j = 3$, M in the above is replaced by I, the moment of inertia of the cylinder about the point of roll, whilst when either $i = 3$ or $j = 3$ (but not both) M is replaced by $(MI)^{1/2}$.

Semi-immersed circular cylinders

Returning to circular cylinders oscillating in the free surface, we show in figures 1 and 2, the comparison between the exact values (lines) and the wide-spacing approximations (points) of the non-dimensional added mass and radiation damping for a cylinders next to

	wide-spacing approximation	exact ($\epsilon = a/b \ll 1$)
μ_{11}^w	1	$1 + \frac{1}{2}\epsilon^2 + \frac{1}{8}\epsilon^4 + \frac{3}{16}\epsilon^6 + \ldots$
μ_{12}^w	0	$-(4/\pi)(\frac{1}{2}\epsilon + \frac{1}{24}\epsilon^3 + \frac{3}{40}\epsilon^5 + \ldots)$
μ_{22}^w	$(-8/\pi^2)(\log(4Ka) + \gamma - \frac{3}{2})$	$(-8/\pi^2)(\log(8K^2ab) + 2\gamma - \frac{3}{2} - \frac{1}{3}\epsilon^2 + \ldots)$
ν_{11}^w	0	0
ν_{12}^w	0	0
ν_{22}^w	$16/\pi$	$16/\pi$

Table 1: The analytically-derived asymptotic behaviour of the hydrodynamic coefficients as $Ka \to 0$ for a circular cylinder next to a wall, using (i) the wide-spacing approximation and (ii) the exact calculations (the added-mass asymptotics are derived on the additional assumption that $a/b \ll 1$).

a wall. Since there is no roll component to the motion of a circular cylinder, there are just three sets of curves to present. Moreover, there are just two independent parameters in this problem, namely the non-dimensional frequency parameter Ka and the wall to cylinder ratio a/b.

Thus for $a/b = \frac{1}{2}$ the inside edge of the cylinder is just half a diameter from the wall, and the wide-spacing approximation shows close agreement to the exact results over the range of frequencies shown (see figure 1*a,b*). The wide-spacing approximation is designed to work when the wavelength, λ, is much smaller than the spacing, $s = b - a$, between the cylinder and the wall. We have $\lambda/s = 2\pi(a/b)/(Ka(1-a/b))$ which equates to $2\pi/Ka$ in figure 1 (and $8\pi/Ka$ in figure 2 where $a/b = \frac{4}{5}$). In both figures 1 and 2 the wavelength is significantly greater than the spacing over all values of Ka shown and therefore well outside the range of values over which the wide-spacing approximation is designed to be effective. Remarkably, the approximation works well in the prediction of radiation damping as $Ka \to 0$. This limit is now investigated further.

For a single circular cylinder, the following results are known (Ursell 1949, 1976) in the limit as $Ka \to 0$

$$\mu_{11} \sim 1, \qquad \nu_{11} \sim 0, \qquad \text{and} \qquad \mu_{22} \sim -\frac{8}{\pi^2}(\log(Ka) + \gamma - \tfrac{3}{2} + \log 4), \qquad \nu_{22} \sim \frac{8}{\pi} \quad (4.1)$$

where $\gamma = 0.5772\ldots$ is Euler's constant. These asymptotic results are straightforward to derive by taking the limit as $Ka \to 0$ in the infinite system of equations that define the solution to the heaving and swaying problems constructed by the use of sources, dipoles and wave-free potentials. See the technical report of Porter (2008*a*) for details of this process.

The low frequency asymptotics (4.1) for cylinders in isolation can be used in the wide-spacing expressions of section 3 to derive explicit low-frequency results based on the wide-spacing approximations to the added-mass and radiation damping coefficients. These are given in the middle column of table 1. In deriving these expressions, we have also had to use the results $R \sim -2\mathrm{i}Ka$, $T \sim 1 - 2\mathrm{i}Ka$ and $\theta_1 \to 0$, $\theta_2 \to -\frac{1}{2}\pi$ as $Ka \to 0$. Again, these are readily established from the infinite systems of equations for cylinders in isolation.

We have attempted to extend Ursell's low frequency results (4.1) to a cylinder in motion next to a wall. The addition of the wall complicates the systems of equations that arise in the limit of $Ka \to 0$ and prohibits the same sort of progress being made. However, we are able to determine the exact leading order behaviour of the radiation damping coefficients as

$\epsilon \equiv a/b$	μ_{11}^w	μ_{12}^w	μ_{22}^w
$\frac{1}{4}$	9.8×10^{-7}	8.3×10^{-6}	5.6×10^{-6}
$\frac{1}{2}$	2.6×10^{-4}	6.0×10^{-4}	1.1×10^{-4}
$\frac{3}{4}$	8.0×10^{-3}	8.7×10^{-3}	6.0×10^{-4}

Table 2: The relative error between the 'exact' computed numerical results and analytically-derived asymptotics given in the right-hand column of table 1 extrapolated back to $Ka = 0$.

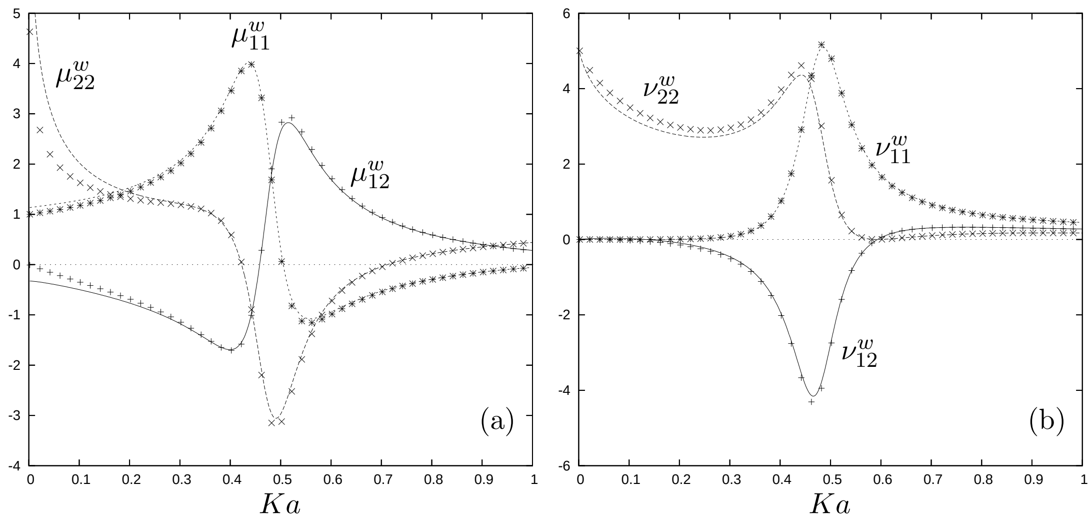

Figure 1: Variation of non-dimensional (a) added mass and (b) radiation damping coefficients for a circular cylinder, radius a, whose centre is a distance $b = 2a$ from a wall as a function of Ka. Solid lines are exact results, points represent corresponding results from the wide-spacing approximation.

$Ka \to 0$ which are independent of a/b. Additionally, by making the assumption $a/b \ll 1$, we have derived explicit expressions to the exact low frequency behaviour of the added-mass coefficients. These results are derived in Porter (2008*a*) and reported in the final column of table 1.

The accuracy of the three approximate added-mass formulae in table 1 is demonstrated in table 2 where they where exact numerical computations for small decreasing values of Ka are compared to results of using the expressions in table 2 and extrapolated back to $Ka = 0$. The results show how the small-a/b approximation improves as a/b decreases and is particularly effective for μ_{11}^w and μ_{22}^w even when a/b is relatively large.

It is not the purpose of the paper to dwell on these asymptotics. Table 1 is provided to show that, whilst the wide-spacing approximation does not predict the exact low-frequency behaviour of the added mass coefficients (nor should it be expected to do so), remarkably it *does* predict the exact value of radiation damping as $Ka \to 0$. This behaviour can be seen clearly in figures 1*b* and 2*b*. The fact that the approximation to the radiation damping is correct for both small and large values of Ka means that the approximation works better than might otherwise be expected over intermediate values of Ka, as seen in both figures 1*b* and 2*b*.

In figure 2*a,b* we repeat the calculations described above but with $a/b = \frac{4}{5}$, so that there

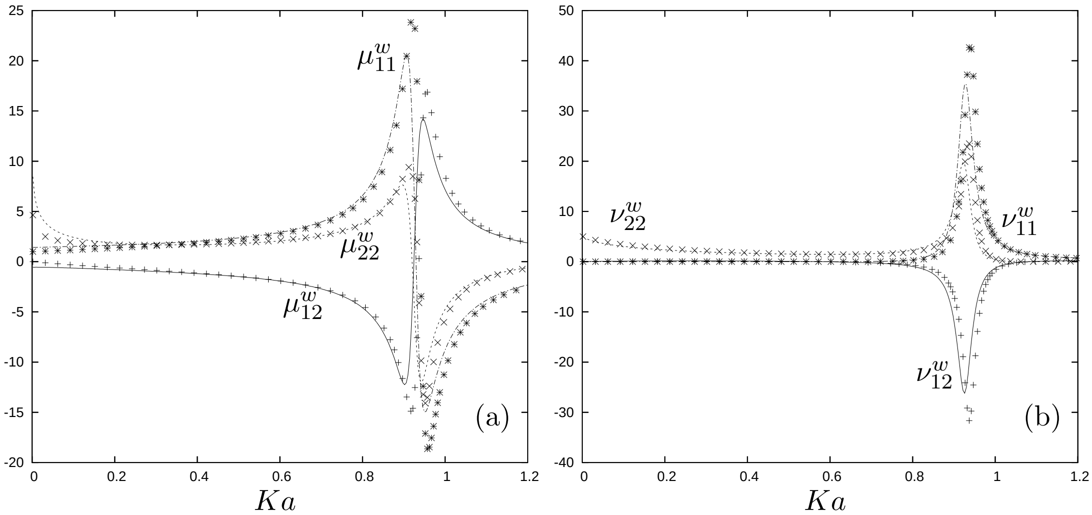

Figure 2: Variation of non-dimensional (a) added mass and (b) radiation damping coefficients for a circular cylinder, radius a, whose centre is a distance $b = \frac{5}{4}a$ from a wall as a function of Ka. Solid lines are exact results, points represent corresponding results from the wide-spacing approximation. The line- and point-styles correspond to those in figure 1.

is now just one eighth of a cylinder diameter of free surface between the wall and the cylinder. Over most of the range of values of Ka, the agreement between the wide-spacing results and the exact results is excellent, with the only significant discrepancy occurring over the range of values of Ka at which the resonant "Helmholtz" or pumping-mode operates.

Surface-piercing cylinders of rectangular cross-section

In the next set of results presented, we compare the performance of the wide-spacing approximation against exact results for rectangular cylinders in motion next to walls. There are differences between this geometry and the previous one. First, the fluid is of constant finite depth h (as opposed to infinite depth in the circular cylinder examples) which is essential to the method of solution. This changes none of the theory, outlined in the body of the paper, except that the infinite depth wavenumber $K = \omega^2/g$ should be replaced everywhere by k, the wavenumber in fluid of depth h, where $K = k \tanh kh$. Note also, that the depth variation of propagating modes is no longer e^{-Ky}, but this is not important in the derivation of the results of the paper.

Second, the rectangular cylinder can assume all three modes of motion including roll. We assume the cylinder rolls about the depth $y = c$ in the midplane of the cylinder. The value of the roll point c is arbitrary and can be both negative or positive. The cylinder is assumed to have a width $2a$ and a draft d and its midplane is a distance b from the wall so that $s = b - a$ is the separation between the wall and the cylinder as before.

We have chosen the point of roll to satisfy $c/d = \frac{1}{2} - \frac{1}{3}(a/d)^2$, which defines the natural point of roll (termed the metacentre in ship hydrodynamics) for the rectangular cylinder. For the purposes of non-dimensionalisation we have also defined the moment of inertia as

$$I = \frac{M}{3}\left(\frac{(d-c)^3 - (l-c)^3}{(d-l)} + a^2\right)$$

which allows for the cylinder of infinitely thin massless sidewalls to be filled between the depths l and d with a medium of uniform density different to the surrounding fluid. Thus, when $l = 0$, that density will in fact be the same as the fluid as dictated by Archmedes' principle. When $l > 0$, the medium is more dense than the fluid and when $l < 0$, it is less dense.

In both sets of numerical results presented here, we have chosen $a/d = \frac{1}{2}$ (so that $c/d = \frac{5}{12}$ and the submerged section is square), $d/h = \frac{1}{5}$, $l/d = -\frac{1}{10}$ (so that the mass of the cylinder above mean waterline is one tenth the submerged mass).

In figures 3 and 4 we have plotted the results of the wide-spacing approximation as points against the exact results computed for cylinders next to walls. Here, there are six independent results for both added mass and radiation damping, allowing for the roll component in addition to heave and sway. The particular spacings chosen in figures 3 and 4 are $b/a = 4$ and $b/a = 2$. In the latter case, the distance between the cylinder and the wall is half the width of the cylinder. In the first set of figures (figures 3*a,b,c,d*), the Helmholtz mode is excited around $Ka = 0.4$, whilst a higher order near-resonant sloshing mode is excited around $Ka = 2.1$. The agreement between wide-spacing and exact results is excellent and the peaks and spikes in the added mass and damping are accurately picked out by the approximation.

In the second set of figures (figures 4*a,b,c,d*) where the spacing between the cylinder and the wall is closer, the wide-spacing results lose accuracy as the non-dimensional wavenumber kd tends to zero. However, this loss of accuracy less prominent in the results for the radiation damping. As for the circular cylinder, numerically it appears that the limiting values of radiation damping as $kd \to 0$ (all zero apart from ν_{22}^w) are accurately captured by the wide-spacing approximation.

A low-frequency analysis of the integral equations used to calculate the solutions for rectangular cylinders which is contained within the technical report of Porter (2008*b*) shows that, for an isolated cylinder, $\nu_{22} \sim a/(khd)$ as $kd \to 0$ with other geometric parameters fixed. According to the approximation (3.28) in which we used the asymptotic formulae $R \to 0$ and $\theta_1 \to 0$ as $kd \to 0$ (exactly as for circular cylinders), the wide-spacing formula predicts that $\nu_{22}^w \sim 2a/(khd)$. (In fact, (3.28) will always give the result $\nu_{22}^w \sim 2\nu_{22}$ as the wavenumber tends to zero, irrespective of the cylinder cross-section). Again, in Porter (2008*b*), it was possible to show that the *exact* low frequency asymptotics for a rectangular cylinder next to a wall is given by the exactly same formula, $\nu_{22}^w \sim 2a/(khd)$.

The fact that the wide-spacing approximation has been shown to capture the low-frequency asymptotic behaviour of the radiation damping for two completely different cylinder sections is intriguing and it is tempting to speculate that it might apply universally to cylinders of arbitrary symmetric cross-section.

5 Conclusions

In this paper the following new results have been obtained. First, various well-known reciprocal results for an arbitrary two-dimensional cylinder oscillating at a given frequency next to a rigid wall, or fixed in an incident plane wave, described for example in Mei (1983, pp.302/3), have been extended to include the presence of a fixed rigid vertical wall. These include an extension of the Haskind relation (equation (2.17)), and expressions for the radiation damping coefficients (equation (2.14)) and the reflection coefficient (equation (2.12)), both expressed in terms of the far field wave-making coefficients due to forced motion of the

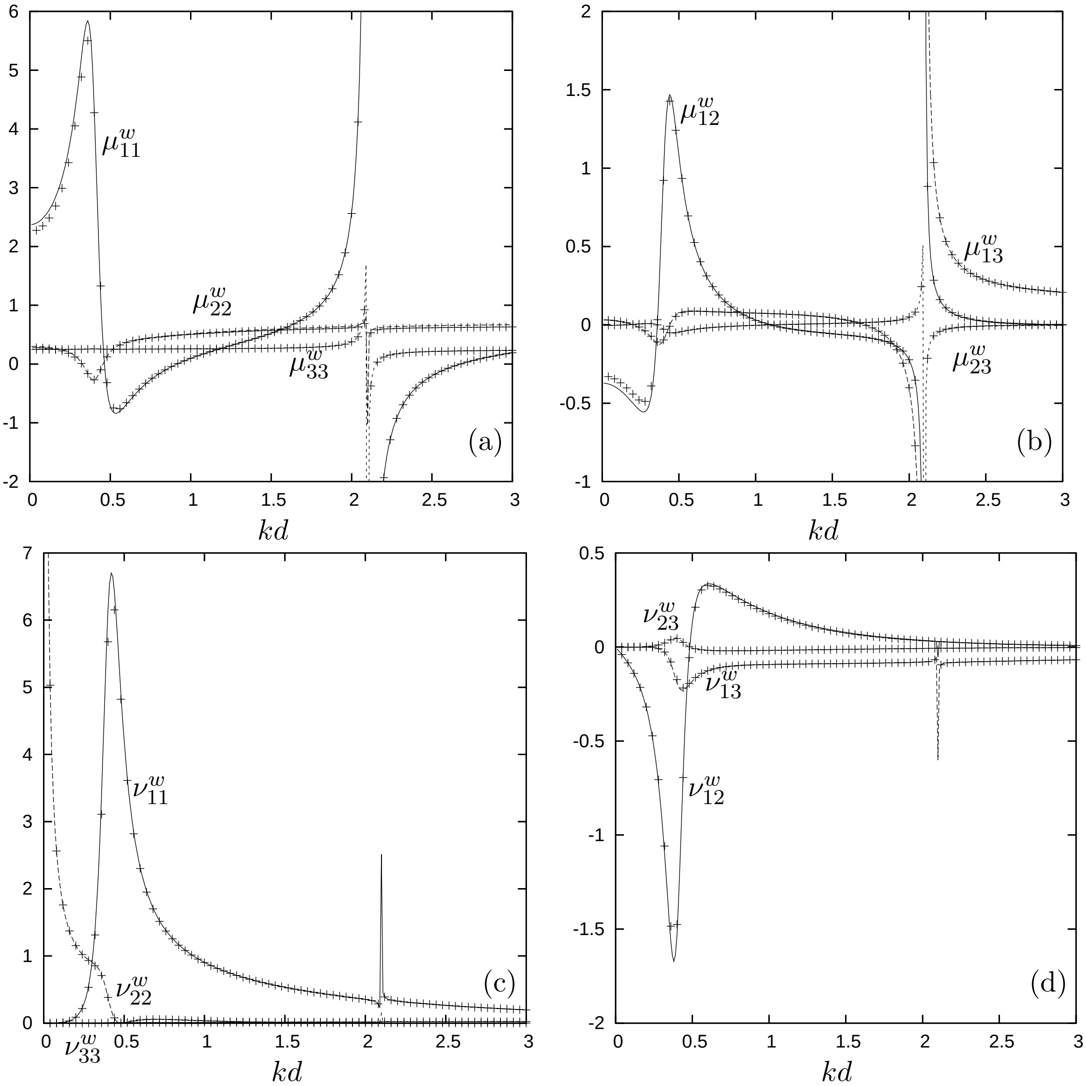

Figure 3: Variation of non-dimensional (a,b) added-mass and (c,d) radiation damping coefficients for a rectangular cylinder of width $2a$, submerged to a depth d and centreline a distance b from a wall in water of finite depth h as a function of dimensionless wavenumber kd for $d/h = 0.2$, $a/d = \frac{1}{2}$, $a/b = \frac{1}{4}$. Solid lines are exact results, points represent corresponding results from the wide-spacing approximation.

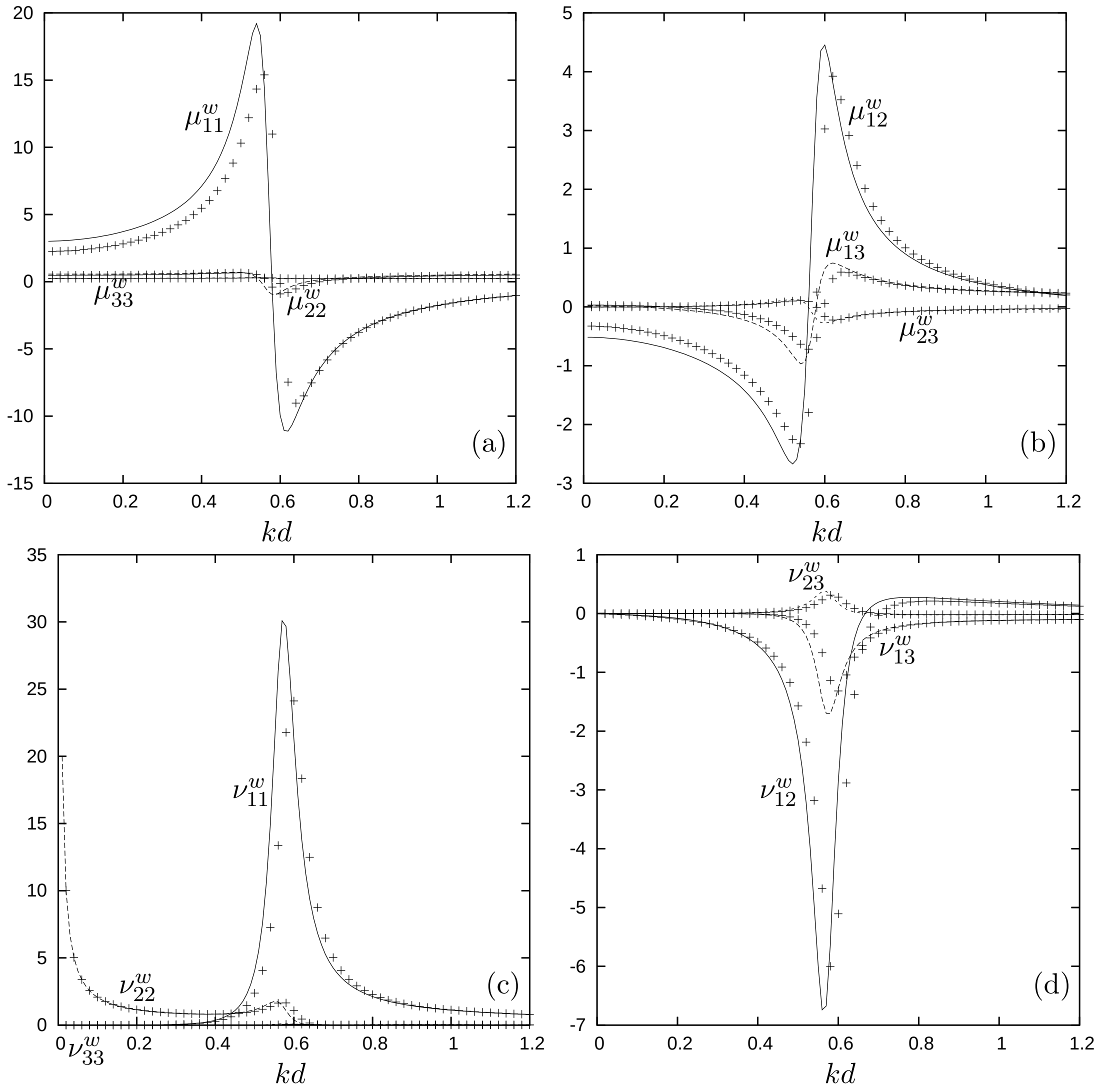

Figure 4: Variation of non-dimensional (a,b) added-mass and (c,d) radiation damping coefficients for a rectangular cylinder of width $2a$, submerged to a depth d and centreline a distance b from a wall in water of finite depth h as a function of dimensionless wavenumber kd for $d/h = 0.2$, $a/d = \frac{1}{2}$, $a/b = \frac{1}{2}$. Solid lines are exact results, points represent corresponding results from the wide-spacing approximation.

cylinder in the presence of the wall.

Next, a wide-spacing approximation has been used to derive explicit expressions for the hydrodynamic coefficients of added mass or inertia, radiation damping, and wave- making coefficients for a symmetric but otherwise arbitrary cylinder next to a wall, in terms of the hydrodynamic coefficients for such a cylinder in the absence of the wall, which in turn are then expressed solely in terms of the far-field radiated amplitudes or wave- making coefficients in the absence of the wall. These are given in equations (3.24) through (3.31).

In order to check the accuracy of these approximations two simple cylindrical sections oscillating next to a wall were chosen for which powerful semi-analytical results can be derived. These were a half-immersed circular cylinder in deep water and a rectangular cylinder in finite water depth. In both cases a comparison of added mass or inertia, and radiation damping coefficients, based on the new expressions derived using the wide-spacing approximation, with computations based on the exact linear theory, showed excellent agreement even when the cylinders were close to the wall and the assumptions of wide spacing were clearly not valid. It was also shown analytically, that for the circular cylinder, the radiation damping coefficients for Ka small are identical on both exact and approximate theories and independent of distance from the wall. The same appears to be true from the computations for the rectangular cylinder.

From these test cases it would appear reasonable to assume that computations of the hydrodynamic coefficients for more realistic cylindrical ship sections close to a wall based on the approximations provided here which only involve the wave-making coefficients in the absence of the wall, will prove to be accurate over a wide range of parameters.

References

[1] Martin, P. A., 2006. *Multiple Scattering: Interaction of Time-Harmonic Waves with N Obstacles.* Cambridge University Press.

[2] Martin, P. A. & Dixon, A. G., 1983. The scattering of regular surface waves by a fixed, half-immersed, circular cylinder. *Appl. Ocean Res.* **5**(1), 13–23.

[3] Mei, C. C., 1983, *The Applied Dynamics of Ocean Surface Waves.* Wiley-Interscience.

[4] Newman, J. N., 1977, *Marine Hydrodynamics.* MIT Press, Cambridge, Massachusetts.

[5] Porter, R., 2008*a*, The solution to water wave scattering and radiation problems involving semi-immersed circular cylinders (`http://www.maths.bris.ac.uk/~marp/abstracts/semicyl_techrep.html`)

[6] Porter, R., 2008*b*, Calculation of the hydrodynamic coefficients for cylinders of rectangular cross-section oscillating in the free surface. (`http://www.maths.bris.ac.uk/~marp/abstracts/rectcyl_hydrocalc.html`)

[7] Porter, R. & Evans, D. V., 2008*a*, Examples of trapped modes in the presence of freely-floating structures. *J. Fluid. Mech.*, **606**, 189–207.

[8] Porter, R. & Evans, D. V., 2008*b*, Water wave trapping by floating circular cylinders. *Submitted for publication*

[9] Srokosz, M. A. & Evans, D. V., 1979, A theory for wave-power absorption by two independently oscillating bodies. *J. Fluid. Mech.* **90**, 367–392.

[10] Ursell, F., 1949, On the heaving motion of a circular cylinder on the surface of a fluid. *Quart. J. Mech. Appl. Math.*, **2** 218–231.

[11] Ursell, F., 1976, On the virtual-mass and damping coefficients for long waves in water of finite depth, *J. Fluid Mech.* **76** 17–28.

[12] Yeung, R. W. & Seah, R. K. M., 2007, On Helmholtz and higher-order resonance of twin floating bodies. *J. Eng. Math.*, **58**, 251–265.

[13] Wang, S. & Wahab, R., 1971, Heaving oscillations of twin cylinders in a free surface. *J. Ship Res.*, **15**(1), 33–48.