Surface wave interaction with submerged ridges and steps

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August 15, 2006

Abstract

The scattering and trapping of oblique surface gravity waves by two classes of submerged topography is considered in this paper. Attention initially focuses on ridges of arbitrary, but uniform, cross-section and finite extent which protude from an otherwise flat bed. A more general shoaling topography is also considered in which an arbitrary bed profiles connects two semi-infinite regions of constant depth. In both cases, the aim has been to develop a new approach to solving these problems based on integral equation techniques. These are designed to be exact, in the context of the linear theory being used, and to generate integral equations which are only weakly singular by use of a novel transformation which converts normal derivatives to tangential derivatives. Two methods of numerical solution are implemented. A boundary element approach provides a remarkably effective method which can typically achieve three significant figure accuracy by inverting a 100×100 system of equations whose elements are easy and quick to calculate. In contrast, a Rayleigh-Ritz approach typically achieves higher accuracy by inverting a much smaller 10×10 real symmetric system of equations whose elements require more work to calculate. The methods are applied to a range of examples.

Keywords: Surface waves, topography, oblique incidence, integral equations.

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1 Introduction

The problem of determining the effect that submerged bodies or topographic features have upon the propagation of surface gravity waves is one of considerable interest to engineers designing coastal or offshore structures.

The three-dimensional problem of wave interaction with variable topography is extremely complicated, even under the simplifying assumptions of linearised water wave theory, primarily due to the impermeable bed condition which must be imposed over the undulating section of topography. To date, only one explicit solution has been found for a specific family of varying (two-dimensional) bed profiles by Roseau [1]. Inevitably the complicated nature of the boundary-value problem which holds in a region of varying depth and has the no-flow condition placed upon its lower variable boundary has demanded approximations to be made in order to progress. One obvious approach is to perform a direct numerical assault on the equations of motion, an approach reviewed extensively by Mei [2] to whom the reader is referred. Typical examples of the issues involved are highlighted in Davis [3] who investigated two-dimensional oscillations in a canal of arbitrary cross section, and Fenton [4] who considered the forces on axisymmetric bodies of revolution. Both of these papers illustrate the type of significant numerical issues arising in a direct approach and which are avoided in our method.

The best known commercially-available code for solving water wave scattering problems is WAMIT [5], which is a panel-based boundary element code. Although such codes are powerful in their flexibility of use, their range of application can be limited. For example, WAMIT is unable to treat problems in which there are different depths at infinity, as in the problems being considered in this paper.

In order to extract some of the key features of the topographical scattering problem numerous papers have been written on the subject of the scattering of waves by a step, or sill of constant height. Lamb [6] first dealt with scattering by a vertical step using the shallow water equations, an approach repeated in Mei [7]. Miles [8] used a variational approach in conjunction with an eigenfunction matching technique to deal with this same problem and in the course of doing so, introduced the scattering matrix formulation that we also find ourselves using in the present paper.

Mei & Black [9] extended these ideas to deal with scattering by rectangular obstacles whereas Kirby & Dalrymple [10] solved the similar problem of oblique diffraction by a rectangular trench. Porter [11] revisited Miles' problem for oblique scattering by a step, developing Miles' eigenfunction expansion matching approach and solving the resulting integral equations by a Galerkin method. More recently Rhee [12], [13] has looked at the scattering of oblique waves over a step to second-order.

Given the complicated nature of three-dimensional problems, one approach has been to consider the simpler two-dimensional problem of normal plane-wave incidence upon topography of uniform cross-section. Devillard *et al.* [14] approximated the arbitrary step problem by using a step discretisation, approximating the bed profile by a series of piecewise constant steps. Fitz-Gerald [15] looking at the two-dimensional step problem, used complex-variable techniques to convert the problem into one defined on an infinite strip, albeit with a more complicated free-surface boundary condition. As a by product of this approach he was also able to prove uniqueness for the problem. Evans & Linton [16] combined the approaches of Devillard *et al.* [14] and Fitz-Gerald [15] to derive an alternative step approximation and this has recently been revisited by Porter & Porter [17]; however, this approach remains limited to two dimensions due to the use of complex-variable techniques, and furthermore the mapping function must be known.

A common approach is to use the so-called Mild Slope Equations (MSE) derived independently by Smith & Sprinks [18] and Berkhoff [19] which essentially perform a depth averaging to reduce the dimension of the problem. Many papers followed using the MSE and variants thereof and it still remains popular due to its simplicity and ease of implementation. Booij [20] produced a significant paper investigating accuracy of the MSE over a linear change in depth, suggesting that it can be used for bed slopes of up to $45°$, and this profile continues to be used to benchmark the accuracy of the various MSE-based methods. Various improvements to the MSE have subsequently been proposed. For example, Chamberlain & Porter [21] who showed that terms proportional to the bed gradient and curvature could be important and termed their improvement the modified mild slope equations (MMSE). Porter & Staziker [22] investigated the inclusion of evanescent modes in the MMSE approach and further modifications were proposed by [23] who include an extra bed-mode term in the approximation to satisfy the bottom condition exactly. Kim & Bai [24] also managed this feat using a one-term approximation by developing an elegant complementary mild-slope equation based on the use of streamfunction methods.

In the two-dimensional (normal incidence) problem of waves incident on a submerged ridge Staziker *et al.* [25] presented a different approach; one that we generalise in this paper to three dimensions. They formulated the problem first as a standard integral equation and subsequently converted two normal derivatives of the Greens function into two tangential derivatives of a related function a process which allows integrating by parts to transfer derivatives from the most singular part of the operator onto other functions. Subsequently this technique was extended by Porter & Porter [26], to two-dimensional wave scattering by a step of arbitrary profile connecting different depths at either infinity. In both [25] and [26] the integral equations that result are only weakly-singular and are self-adjoint, both properties which are advantageous when finding numerical solutions. For example, by using a variational principle (the Rayleigh-Ritz approximation) quantities of interest are second-order accurate with respect to first order errors in the unknown functions, whilst conservation of energy is ensured automatically, whatever level of approximation is applied.

However, the switching of normal to tangential derivatives presented in [25] and [26] was essentially an expression of the Cauchy-Riemann equations, and therefore restricted to two-dimensional problems, or normally-incident waves. In this paper we implement a method which generalises these two pieces of work to three-dimensions and, in particular, oblique waves. The main idea is still to make a connection between normal derivatives and tangential derivatives, but we find that it has to be done more carefully than in the two-dimensional counterparts mentioned above. Thus, instead of deriving a standard type of integral equation, we derive self-adjoint integro-differential equations. The differential operators involved are transferred onto other functions via integration by parts in the implementation of the Rayleigh-Ritz approximation, which is therefore intrinsic to the formualtion, ultimately giving rise to weakly-singular kernels. It is confirmed that setting the obliqueness to zero reduces the formulation to that in either [25] and [26].

The majority of the paper is devoted to the ridge problem in which the main ideas are presented and issues addressed. The much more complicated problem which includes the possibility of different depths on either side of the varying part of the bed is only briefly outlined later in section 6 of the paper. We present the governing equations and the framework of the problem in section 2, before presenting the standard integral equation formulation of the ridge problem in section 3, which introduces the Green's functions. Section 4 deals with the main development of the new integral equation approach to and we implement the method of approximation and discuss numerical issues in Section 5. Section 6 concentrates on the application of the new techniques to the problem of oblique waves passing from one depth to another over a variable bed. Section 7 presents a selection of results for all the problems considered. Appendix A includes a brief description of an independent method based upon multipole potentials which are specific to semi-circular protrusions and have been used to test the numerical results of our work.

2 Statement of the problem

We consider the motion of an ideal incompressible fluid within a domain bounded by a fixed impermeable bed and having a free surface with the atmosphere. We choose a Cartesian co-ordinate system with the x and y axes lying in the undisturbed free surface and the z axis oriented vertically downwards.

The fluid is bounded below by $S_b : \{z = h(x), -\infty < x, y < \infty\}$ where $h(x)$ is assumed to be a continuous function with $h(x) = h_0$, a constant, for $x \notin (0, a)$ and $h(x) \leq h_0$ for $x \in (0, a)$. That is, the topography consists of an infinitely-long ridge with constant cross-section in the (x, z)-plane and which protrudes from an otherwise flat bed of depth h_0.

Under the assumptions of linearised theory and assuming a time harmonic variation of frequency $\omega/2\pi$, the motion may be described in terms of a velocity potential, being the real part of $\phi(x, y)\mathrm{e}^{\mathrm{i}(ly-\omega t)}$ which satisfies the three-dimensional Laplace's equation, whence we have

$$(\nabla^2 - l^2)\phi = 0, \qquad (x, z) \in D \tag{2.1}$$

where $\nabla = (\partial/\partial x, \partial/\partial z)$. The uniformity of the geometry in the y-direction implies the quasi-periodicity in the y-direction encapsulated by the wavenumber l, which will be related to the incident wave field shortly. Also, ϕ satisfies the linearised free surface condition

$$\frac{\partial \phi}{\partial z} + K\phi = 0, \quad \text{on } z = 0,\ -\infty < x < \infty \tag{2.2}$$

where $K = \omega^2/g$ along with the no-flow conditions on fixed boundaries,

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on } z = h_0 \text{ for } x < 0 \text{ and } x > a \tag{2.3}$$

being the semi-infinite regions of constant depth, and

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on } \Gamma : \{0 < x < a,\ z = h(x)\} \tag{2.4}$$

on the protrusion. Here, $\partial/\partial n \equiv \mathbf{n}.\nabla$ represents the derivative (out of the fluid) in the direction normal to the curve Γ where

$$\left.\begin{aligned} \mathbf{n} &= (-h'(x), 1)/\sigma(x) \\ \mathbf{s} &= (1, h'(x))/\sigma(x) \end{aligned}\right\} \qquad \sigma(x) = \sqrt{1 + [h'(x)]^2} \tag{2.5}$$

define local orthonormal unit vectors normal and tangential to Γ. We shall also denote the tangential derivative on Γ by $\partial/\partial s \equiv \mathbf{s}.\nabla$.

To complete the formulation of the problem, we need radiation conditions at infinity, which are written as

$$\phi(x,z) \sim \begin{cases} A_-\phi_0^+(x,z) + B_-\phi_0^-(x,z), & x \to -\infty \\ A_+\phi_0^-(x,z) + B_+\phi_0^+(x,z), & x \to \infty. \end{cases} \tag{2.6}$$

Here $\phi_0^\pm(x,z)$ define waves propagating obliquely towards $x = \pm\infty$ in water of constant depth h_0, whilst $A_\pm$ and $B_\pm$ represent wave amplitudes associated with waves that are incoming and outgoing (respectively) on the ridge from $x = \pm\infty$. More specifically,

$$\phi_0^\pm(x,z) = \mathrm{e}^{\pm \mathrm{i}\alpha x}\psi_0(z) \tag{2.7}$$

where

$$\alpha = k\cos\theta, \qquad \text{and} \qquad l = k\sin\theta$$

are components in the x and y directions (respectively) of the wavenumber, k, of the incident wave propagating at an angle θ to the positive x-axis. In (2.7), $\psi_0(z)$ is the vertical dependence of the propagating wave in water of constant depth h_0 defined by the first element in the set of depth eigenfunctions $\{\psi_n(z)\}$ $n = 0, 1, 2, \ldots$ given by

$$\psi_n(z) = N_n^{-1/2} \cos k_n(h_0 - z) \qquad \text{with} \qquad N_n = \frac{1}{2}\left(1 + \frac{\sin 2k_n h_0}{2k_n h_0}\right) \tag{2.8}$$

and k_n are defined as the real positive roots of $K = -k_n \tan k_n h_0$ $n \geq 1$ with $k_0 = -\mathrm{i}k$, so that $K = k \tanh kh_0$ is the usual dispersion relation defining k in terms of K and h_0. The functions $\{\psi_n(z)\}$ form a complete orthonormal set in $[0, h_0]$, with

$$\frac{1}{h_0}\int_0^{h_0} \psi_n(z)\psi_m(z)\mathrm{d}z = \delta_{mn}.$$

We define reflection and transmission coefficients for waves of unit amplitude incident from $x = -\infty$ by $R_- = B_-/A_-$ and $T_- = B_+/A_-$. Likewise we define reflection and transmission coefficients for waves of unit amplitude incident from $x = +\infty$ by $R_+ = B_+/A_+$ and $T_+ = B_-/A_+$. It follows from (2.6) that

$$\begin{pmatrix} B_+ \\ \\ B_- \end{pmatrix} = \mathbf{S}\begin{pmatrix} A_- \\ \\ A_+ \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} T_- & R_+ \\ \\ R_- & T_+ \end{pmatrix} \tag{2.9}$$

where $R_\pm$ and $T_\pm$ are the reflection and transmission coefficients for a wave from $x = \pm\infty$. $\mathbf{S}$ is often referred to as the scattering matrix, which is henceforth regarded as the principal unknown in this problem.

3 A standard integral equation formulation

We will obtain integral representations of ϕ by using Green's identity to functions ϕ and G

$$\iint_{\mathcal{D}} (G\nabla^2\phi - \phi\nabla^2 G)\,\mathrm{d}x\,\mathrm{d}z = \int_{\mathcal{S}} \left(\phi\frac{\partial G}{\partial n} - G\frac{\partial \phi}{\partial n}\right)\,\mathrm{d}s \tag{3.10}$$

where $\mathcal{S}$ is the closed boundary of the domain $\mathcal{D} \subset D$ and $\mathrm{d}s$ is a line element on $\mathcal{S}$. In (3.10) $G(x, z; x_0, z_0)$ is chosen to be a Green's function appropriate to this problem, (x, z) and (x_0, z_0) being the source and field points respectively, satisfying

$$(\nabla^2 - l^2)G = -\delta(x - x_0)\delta(z - z_0) \tag{3.11}$$

with

$$\frac{\partial G}{\partial z} + KG = 0 \quad \text{on } z = 0, \qquad \text{and} \qquad \frac{\partial G}{\partial z} = 0 \quad \text{on } z = h_0 \tag{3.12}$$

holding for $-\infty < x, x_0, z, z_0 < \infty$. It is readily shown using standard Green's function techniques that

$$G(x, z; x_0, z_0) = \sum_{n=0}^{\infty} \frac{\psi_n(z)\psi_n(z_0)}{2\alpha_n h_0} \mathrm{e}^{-\alpha_n|x-x_0|}, \qquad (x, z) \neq (x_0, z_0) \tag{3.13}$$

where

$$\alpha_n = \sqrt{k_n^2 + l^2}, \qquad \alpha_0 = -\mathrm{i}\sqrt{l^2 - k^2} = -\mathrm{i}\alpha \tag{3.14}$$

see, for example, Chapman [27] or Heins [28]. Of particular note is the far field form,

$$G \sim \mathrm{i}\frac{\psi_0(z)\psi_0(z_0)}{2\alpha h_0} \mathrm{e}^{\mathrm{i}\alpha|x-x_0|}, \qquad |x - x_0| \to \infty \tag{3.15}$$

whilst we also note the behaviour, $G \sim (-4\pi)^{-1} \ln((x - x_0)^2 + (y - y_0)^2)$ as $(x, y) \to (x_0, y_0)$.

We will find it convenient to decompose G in the form

$$G = G_0 + \widehat{G}, \qquad \text{where} \qquad G_0(x, z; x_0, z_0) = \frac{\mathrm{i}\psi_0(z)\psi_0(z_0)}{2\alpha h_0} \cos\alpha(x - x_0) \tag{3.16}$$

is the separable component of the Green's function previously exposed in (3.13) and

$$\widehat{G}(x, z; x_0, z_0) = -\frac{\psi_0(z)\psi_0(z_0)}{2\alpha h_0} \sin\alpha|x - x_0| + \sum_{n=1}^{\infty} \frac{\psi_n(z)\psi_n(z_0)}{2\alpha_n h_0} \mathrm{e}^{-\alpha_n|x-x_0|} \tag{3.17}$$

is the remainder of G. We decompose the Green's function in this way so that the property $\overline{\widehat{G}(x, z; x_0, z_0)} = \widehat{G}(x, z; x_0, z_0)$ holds, where the overbar denotes complex conjugation. This property turns out to have favourable implications later in terms of formulating self-adjoint integral operators. A similar relation does not hold for G_0, although we note here that the definitions (2.7) can be re-used in (3.16) by exploiting its separable form to give

$$G_0(x, z; x_0, z_0) = \frac{\mathrm{i}}{4\alpha h_0} \left\{ \overline{\phi_0^-}(x, z)\phi_0^-(x_0, z_0) + \overline{\phi_0^+}(x, z)\phi_0^+(x_0, z_0) \right\}. \tag{3.18}$$

Applying Green's identity (3.10) to the functions $\phi(x, z)$ and $G(x, z; x_0, z_0)$ and using the definitions of ϕ and G it may be found, after some routine algebra that

$$\mu\phi(x_0, z_0) = A_-\phi_0^+(x_0, z_0) + A_+\phi_0^-(x_0, z_0) - \int_\Gamma \phi(x, z) \frac{\partial}{\partial n} G(x, z; x_0, z_0) \, \mathrm{d}s. \tag{3.19}$$

Here $\mu = 1$ for $(x_0, z_0) \in D$, $\mu = \frac{1}{2}$ for $(x_0, z_0) \in \partial D$, the boundary of D and $\mu = 0$ for $(x_0, z_0) \notin D \cup \partial D$. The variable s measures the arclength along Γ and implies the parametrisation $(x(s), z(s))$ of Γ. This is the standard 'double-layer' potential representation of the solution, and with $(x_0, z_0) \in \partial D$ represents a second-kind Fredholm integral equation for $\phi(x_0, z_0)$ on Γ. For given values of $A_\pm$, this integral equation may be solved numerically using, for example, a boundary element method. The kernel of the integral operator is proportional to the normal derivative of a logarithm as the field point approaches the source point, and thus appears to be Cauchy singular. However, it is known that this is an apparent difficulty only (see Martin [29] p.160), and the kernel is actually bounded at such points. Nevertheless, this issue would require some attention in the numerical implementation of a boundary element method. There is a huge literature devoted to issues surrounding integral equation formulations for wave scattering problems, and the reader is referred to the comprehensive text of Martin [29].

The purpose of the next section is to use (3.19) as the basis for an alternative integral equation formulation which has certain advantages (which will be discussed in due course) over the standard formulation of (3.19). We believe this to be a new approach to formulating weakly-singular integral equations for wave scattering, although it does build upon the work of [26]. The process of weakening the order of the singularity in integral equations is known as regularisation. The fact that we eventually derive a weakly-singular integral equation is made all the more remarkable in that the procedure we implement will involve taking a further derivative of (3.19).

4 A new integral equation formulation

This is a convenient point to introduce some new notation which will be used extensively in this section. We will use subscripts in x and x_0 and superscripts in z and z_0 to mean differentation and integration respectively, in accordance with the definition

$$f_x^z \equiv \int_{h_0}^{z} \frac{\partial f(x, z')}{\partial x} \, \mathrm{d}z' \tag{4.20}$$

(see Noblesse & Yang [30]). Immediately related to this are the functions

$$\chi_n(z) = -k_n \int_{h_0}^{z} \psi_n(z') \, \mathrm{d}z' = N_n^{-1/2} \sin k_n(h_0 - z), \qquad n = 0, 1, \ldots \tag{4.21}$$

which were also used in [26] and designed so that $\chi_n(h_0) = 0$ which will be used extensively to eliminate free terms from later integration by parts.

In [26], the Cauchy-Riemann equations were used to convert normal derivatives to tangential derivatives, upon which integration by parts was used to move derivatives from the Green's function to the potential. The obvious limitation of this approach is that it can only be applied in two-dimensions. Here, we provide a new generalised version of the two-dimensional result of [26]

$$\frac{\partial G}{\partial n} = -\frac{\partial G_x^z}{\partial s} + \frac{l^2}{\sigma} G^z + \frac{1}{\sigma}\delta(x - x_0)H(z_0 - z) \tag{4.22}$$

which can be confirmed directly from the definitions of $\partial/\partial n$, $\partial/\partial s$ (see (2.5)) and (3.11), noting the use of the definition (4.20). In (4.22) $H(x - x_0)$ is the Heaviside step function which is related to the delta function by

$$H(z_0 - z) = -\delta^z(z - z_0) \tag{4.23}$$

in the notation of (4.20).

It is worth working through the consequences of using the transformation (4.22) in formulations based on Green's identity (3.10). If we now apply (4.22) to (3.19) we find that

$$\begin{aligned} \mu\phi(x_0, z_0) = A_-\phi_0^+(x_0, z_0) + A_+\phi_0^-(x_0, z_0) + \int_\Gamma \phi \left(\frac{\partial}{\partial s} G_x^z - \frac{l^2}{\sigma} G^z \right) \, \mathrm{d}s \\ - \int_\Gamma \phi(x, z)\delta(x - x_0)H(z_0 - z)\frac{\mathrm{d}s}{\sigma} \end{aligned} \tag{4.24}$$

Now to deal with the final term we note that $\mathrm{d}s = \sigma \, \mathrm{d}x$ so

$$\begin{aligned} -\int_\Gamma \phi(x, z)\delta(x - x_0)H(z_0 - z)\frac{\mathrm{d}s}{\sigma} &= -\int_0^a \phi(x, h(x))\delta(x - x_0)H(z_0 - h(x)) \, \mathrm{d}x \\ &= -\phi(x_0, h(x_0))H(z_0 - h(x_0)) \end{aligned} \tag{4.25}$$

which takes the value zero for points $(x_0, z_0) \in D$, $-\frac{1}{2}\phi(x_0, h(x_0))$ for points $(x_0, z_0) \in \partial D$ and $-\phi(x_0, h(x_0))$ for points $(x_0, z_0) \notin D \cup \partial D$. Thus recombining with the left-hand side of (4.24) gives

$$\phi(x_0, z_0) = A_-\phi_0^+(x_0, z_0) + A_+\phi_0^-(x_0, z_0) + \int_\Gamma \phi \left(\frac{\partial}{\partial s} G_x^z - \frac{l^2}{\sigma} G^z \right) \, \mathrm{d}s \tag{4.26}$$

for $(x_0, z_0) \in D \cup \partial D$ which establishes that, unlike the traditional form (3.19), the formulation in (4.26) gives a continuous definition of the fluid potential as the field point moves from the fluid domain to a point on the boundary.

Our approach is to modify (4.26) to put it into a form which allows us to define a self-adjoint integral operator. This crucial step allows us to implement a solution using the Rayleigh-Ritz method (equivalent in this context to Galerkin's method) which, since it is based on a variational method, is known to have excellent convergence properties (see Champan [27] or Porter [11]). With this strategy in mind we decompose the Green's function using (3.16) so that using (3.18) we find

$$\left.\begin{aligned} (G_0)^z &= \frac{\mathrm{i}}{4\alpha h_0}\left(\frac{\mathrm{i}}{k}\right)\left\{\overline{f^-}(x,z)\phi_0^-(x_0,z_0) + \overline{f^+}(x,z)\phi_0^+(x_0,z_0)\right\} \\ (G_0)^z_x &= -\frac{\mathrm{i}}{4\alpha h_0}\left(\frac{\alpha}{k}\right)\left\{\overline{f^-}(x,z)\phi_0^-(x_0,z_0) - \overline{f^+}(x,z)\phi_0^+(x_0,z_0)\right\} \end{aligned}\right\} \tag{4.27}$$

in terms of newly-defined functions

$$f^{\pm}(x,z) = \mathrm{e}^{\pm\mathrm{i}\alpha x}\chi_0(z) \tag{4.28}$$

where (2.7) and (3.18) have been used and we note that $\overline{\chi}_0(z) = -\chi_0(z)$.

At this point, we shall introduce some more notation so we define

$$F^{\pm}(s) = \left(\mp\frac{\alpha}{k}\frac{\partial}{\partial s} + \frac{\mathrm{i}l^2}{k\sigma(x)}\right) f^{\pm}(x,z)\bigg|_{(x,z)\in\Gamma} \tag{4.29}$$

and we also introduce the inner product notation for functions $u(s), v(s) \in \mathcal{H}$ (where $\mathcal{H}$ is the space of functions whose derivatives belong to $L_2(\Gamma)$)

$$\langle u, v\rangle = \int_\Gamma u\overline{v}\ \mathrm{d}s. \tag{4.30}$$

Then, using the decomposition in (3.16) with (4.27) in (4.26) we find, after some algebra, that

$$\begin{aligned} \phi(x_0,z_0) = A_-\phi_0^+(x_0,z_0) + A_+\phi_0^-(x_0,z_0) + \int_\Gamma \phi\left(\frac{\partial}{\partial s}G^z_x - \frac{l^2}{\sigma}G^z\right)\,\mathrm{d}s \\ - \frac{\mathrm{i}}{4\alpha h_0}\Big\{\phi_0^-(x_0,z_0)\langle\phi,F^-\rangle + \phi_0^+(x_0,z_0)\langle\phi,F^+\rangle\Big\}. \end{aligned} \tag{4.31}$$

This is a good point at which to pause for a moment and go back to (4.26) in order to establish relations which arise from consideration of the far field. First, we note that as $|x - x_0| \to \infty$,

$$G^z \sim \frac{\mathrm{i}}{2\alpha h_0}\left(\frac{\mathrm{i}}{k}\right)\overline{\chi}_0(z)\psi_0(z_0)\mathrm{e}^{\mathrm{i}\alpha|x-x_0|} \tag{4.32}$$

and

$$G^z_x \sim -\frac{\mathrm{i}}{2\alpha h_0}\mathrm{sgn}(x-x_0)\left(\frac{\alpha}{k}\right)\overline{\chi}_0(z)\psi_0(z_0)\mathrm{e}^{\mathrm{i}\alpha|x-x_0|} \tag{4.33}$$

which may be determined from (3.15), the far-field form of G. Hence, taking $x_0 \to -\infty$ in (4.26) and using the far-field form of ϕ provided by (2.6) gives, after considerable algebra,

$$B_- = A_+ - \frac{\mathrm{i}}{2\alpha h_0}\langle \phi, F^- \rangle \tag{4.34}$$

where the inner product notation (4.30) has been invoked and F^- is defined by (4.29). In a similar manner, taking the limit $x_0 \to \infty$, we obtain

$$B_+ = A_- - \frac{\mathrm{i}}{2\alpha h_0}\langle \phi, F^+ \rangle. \tag{4.35}$$

These last two equations hold the key to the continued development of the formulation, since now they can used to substitute in the second line of (4.31), resulting in

$$\phi(x_0, z_0) = \tfrac{1}{2}(A_- + B_+)\phi_0^+(x_0, z_0) + \tfrac{1}{2}(A_+ + B_-)\phi_0^-(x_0, z_0) + \int_\Gamma \left(\frac{\partial}{\partial s}\, \widehat{G}_x^z - \frac{l^2}{\sigma}\, \widehat{G}^z \right) \phi(s)\, \mathrm{d}s. \tag{4.36}$$

Equation (4.36) may be regarded as a second-kind integral equation for ϕ for points on Γ, by moving the field point (x_0, z_0) onto Γ. The forcing term is a weighted sum of the two incident wave modes $\phi_0^\pm$. Although this is not our ultimate formulation of the problem, it provides a useful alternative method of solution and can be used to check numerical results. Later in Section 5, we show that the effect of singular behaviour of G is easily taken care of when solved numerically using a simple boundary element scheme.

Therefore if we define an integral operator

$$(\mathcal{K}_1\phi)(s_0) = \phi(s_0) - \int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}_x^z - \frac{l^2}{\sigma}\widehat{G}^z \right) \phi\, \mathrm{d}s \tag{4.37}$$

where $(x_0(s_0), z_0(s_0)) \in \Gamma$ with s_0 measuring the length along Γ and then define a pair of functions $\varphi_1^\pm$ on Γ such that

$$(\mathcal{K}_1\varphi_1^\pm)(s_0) = \phi_0^\pm(s_0), \qquad s_0 \in \Gamma \tag{4.38}$$

then it follows that the solution of (4.36) is given by

$$\phi = \tfrac{1}{2}(A_- + B_+)\varphi_1^+ + \tfrac{1}{2}(A_+ + B_-)\varphi_1^-. \tag{4.39}$$

We note that $\mathcal{K}_1$ is not a self-adjoint integral operator.

We continue in a manner analogous to that of [26], anticipating a self-adjoint structure in the final integral equations that is not enjoyed by an integral equation arising directly from (3.19) or

(4.36). Thus, we first introduce quantities analogous to (2.5) which apply to the field variable (x_0, z_0), namely

$$\left.\begin{aligned}\mathbf{n}_0 &= (-h'(x_0), 1)/\sigma(x_0)\\ \mathbf{s}_0 &= (1, h'(x_0))/\sigma(x_0)\end{aligned}\right\}, \qquad \nabla_0 \equiv \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial z_0}\right) \tag{4.40}$$

and extend the definition of the orthonormal basis, $\{\mathbf{n}_0, \mathbf{s}_0\}$ to points away from the curve Γ.

Application of the operator $\mathbf{n}_0.\nabla_0 \equiv \partial/\partial n_0$ to (4.36) for field points off Γ gives

$$\frac{\partial}{\partial n_0}\phi = \tfrac{1}{2}(A_- + B_+)\frac{\partial}{\partial n_0}\phi_0^+ + \tfrac{1}{2}(A_+ + B_-)\frac{\partial}{\partial n_0}\phi_0^- + \frac{\partial}{\partial n_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}_x^z - \frac{l^2}{\sigma}\widehat{G}^z\right)\phi\,\mathrm{d}s. \tag{4.41}$$

Now, in terms of field variables (4.22) becomes

$$\frac{\partial}{\partial n_0}G = -\frac{\partial}{\partial s_0}G_{x_0}^{z_0} + \frac{l^2}{\sigma}G^{z_0} \tag{4.42}$$

for points away from Γ where $\mathbf{s}_0.\nabla_0 \equiv \partial/\partial s_0$. Using this result to convert normal derivatives to tangential derivatives in (4.41) gives

$$\begin{aligned}\frac{\partial}{\partial n_0}\phi = \tfrac{1}{2}(A_- + B_+)&\frac{\partial}{\partial n_0}\phi_0^+ + \tfrac{1}{2}(A_+ + B_-)\frac{\partial}{\partial n_0}\phi_0^-\\ &- \frac{\partial}{\partial s_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}_{xx_0}^{zz_0} - \frac{l^2}{\sigma}\widehat{G}_{x_0}^{zz_0}\right)\phi\,\mathrm{d}s + \frac{l^2}{\sigma_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}_x^{zz_0} - \frac{l^2}{\sigma}\widehat{G}^{zz_0}\right)\phi\,\mathrm{d}s.\end{aligned} \tag{4.43}$$

This step is critical and therefore worthy of special note; we were able to take the derivatives under the integration sign and then back out again precisely because the field point is not on the bed thus ensuring convergence of the integrals. Now, it is routine to confirm from (2.7), (4.29) and (4.42) that

$$\frac{\partial}{\partial n_0}\phi_0^\pm = F^\pm. \tag{4.44}$$

Using this result we may now let the field point move on to the bed and apply the bed condition (2.4) to give

$$\begin{aligned}0 = \tfrac{1}{2}(A_- + B_+)F^+ &+ \tfrac{1}{2}(A_+ + B_-)F^-\\ &- \frac{\partial}{\partial s_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}_{xx_0}^{zz_0} - \frac{l^2}{\sigma}\widehat{G}_{x_0}^{zz_0}\right)\phi\,\mathrm{d}s + \frac{l^2}{\sigma_0}\int_\Gamma \left(\frac{\partial}{\partial s}\widehat{G}_x^{zz_0} - \frac{l^2}{\sigma}\widehat{G}^{zz_0}\right)\phi\,\mathrm{d}s.\end{aligned} \tag{4.45}$$

We next perform integration by parts to move derivatives away from functions relating to G. Care must be taken to consider the effect of discontinuities in the integrand; therefore in (4.45) we note that $\widehat{G}^{zz_0}_{xx_0}$ is continuous in x whereas $\widehat{G}^{zz_0}_{x}$ is proportional to $\mathrm{sgn}(x - x_0)$ which gives an extra contribution when integrating by parts. Therefore

$$
\begin{aligned}
0 = \tfrac{1}{2}(A_- + B_+)F^+ + \tfrac{1}{2}(A_+ + B_-)F^- + \frac{l^2}{\sigma_0}\left[\phi\widehat{G}^{zz_0}_{x}\right]^{x_0^-}_{x_0^+} \\
+ \frac{\partial}{\partial s_0}\int_\Gamma \left(\widehat{G}^{zz_0}_{xx_0}\frac{\partial\phi}{\partial s} + \frac{l^2}{\sigma}\widehat{G}^{zz_0}_{x_0}\phi\right)\,\mathrm{d}s - \frac{l^2}{\sigma_0}\int_\Gamma\left(\widehat{G}^{zz_0}_{x}\frac{\partial\phi}{\partial s} + \frac{l^2}{\sigma}\widehat{G}^{zz_0}\phi\right)\,\mathrm{d}s
\end{aligned}
\tag{4.46}
$$

where the square brackets denotes the jump in the quantity at $x = x_0$. Thus once again we have obtained a second-kind integral equation for ϕ on Γ although its form and structure is entirely different to (4.36).

We pause for an instant to consider the explicit form of the free term which is found to be

$$
\left[\phi\,\widehat{G}^{zz_0}_{x}\right]^{x_0^-}_{x_0^+} = \phi(x_0, h(x_0))\,\sum_{r=0}^{\infty}\frac{\chi_r^2(h(x_0))}{k_r^2 h_0}
\tag{4.47}
$$

after using (4.21) in (3.13). Now we note from Chapman & Porter [31], or alternatively Chapman [27], that

$$
\sum_{n=0}^{\infty}\frac{\chi_n(h(x_0))\psi_n(z_0)}{k_n h_0} = H(z_0 - h(x_0))
\tag{4.48}
$$

which, when integrated between h_0 and z_0, results in

$$
\sum_{n=0}^{\infty}\frac{\chi_n(h(x_0))\chi_n(z_0)}{k_n^2 h_0} = h_0 - h(x_0)
\tag{4.49}
$$

for all points in the fluid domain. Therefore if we let $z_0 \to h(x_0)$ in (4.49) we find that we can sum (4.47) explicitly to give

$$
\left[\phi\,\widehat{G}^{zz_0}_{x}\right]^{x_0^-}_{x_0^+} = \phi(x_0, h(x_0))\,(h_0 - h(x_0)).
\tag{4.50}
$$

Now we observe that (4.46) is a second-kind integro-differential equation for ϕ defined on the curve Γ. Therefore by defining the integro-differential operator

$$
\begin{aligned}
(\mathcal{K}_2\phi)(s_0) \equiv -\frac{l^2}{\sigma_0}\phi(s_0)(h_0 - h(x_0)) \\
- \frac{\partial}{\partial s_0}\int_\Gamma \left(\widehat{G}^{zz_0}_{xx_0}\frac{\partial\phi}{\partial s} + \frac{l^2}{\sigma}\widehat{G}^{zz_0}_{x_0}\phi\right)\,\mathrm{d}s + \frac{l^2}{\sigma_0}\int_\Gamma\left(\widehat{G}^{zz_0}_{x}\frac{\partial\phi}{\partial s} + \frac{l^2}{\sigma}\widehat{G}^{zz_0}\phi\right)\,\mathrm{d}s
\end{aligned}
\tag{4.51}
$$

and a pair of functions $\varphi_2^{\pm}$ on Γ such that

$$
(\mathcal{K}_2\,\varphi_2^{\pm})(s_0) = F^{\pm}(s_0)
\tag{4.52}
$$

we see that the solution to (4.46) and hence (4.45) is given by

$$\phi(s_0) = \tfrac{1}{2}(A_- + B_+)\,\varphi_2^+ + \tfrac{1}{2}(A_+ + B_-)\,\varphi_2^-. \tag{4.53}$$

It can be readily shown that $\mathcal{K}_2$ is a self-adjoint operator, a fact established by repeated integration by parts and use of the symmetry properties of the integrand which we highlighted in the discussion after (3.17), although the algebra is somewhat protracted and tedious.

We note that the form of (4.46) is entirely different from (4.36) therefore each method provides an independent check on the other. However, if we compare (4.39) with (4.53) we must have $\varphi_1^\pm = \varphi_2^\pm$ in which case, as the approach to determining the scattering matrix is identical whichever method is used, we drop the suffix on $\varphi^\pm$.

Finally, to recover the scattering matrix we use (4.39) or (4.53) with (4.34) and (4.35) to deduce that

$$\left.\begin{aligned} B_- &= A_+ - \lambda\Big((A_- + B_+)P^{(+,-)} + (A_+ + B_-)P^{(-,-)}\Big) \\ B_+ &= A_- - \lambda\Big((A_- + B_+)P^{(+,+)} + (A_+ + B_-)P^{(-,+)}\Big) \end{aligned}\right\} \tag{4.54}$$

where we have defined

$$P^{(\pm,\pm)} = \langle \varphi^\pm, F^\pm \rangle, \qquad \text{and} \qquad \lambda = \frac{\mathrm{i}}{4\alpha h_0} \tag{4.55}$$

and order of the superscripts on the left-hand side match the order in which they appear on the right-hand side. Then rearranging these equations we find

$$(\mathbf{I} + \lambda\mathbf{P})\begin{pmatrix} B_+ \\ B_- \end{pmatrix} = (\mathbf{I} - \lambda\mathbf{P})\begin{pmatrix} A_- \\ A_+ \end{pmatrix}, \qquad \text{where} \qquad \mathbf{P} = \begin{pmatrix} P^{(+,+)} & P^{(-,+)} \\ P^{(+,-)} & P^{(-,-)} \end{pmatrix} \tag{4.56}$$

and $\mathbf{I}$ is the 2×2 identity matrix. Finally, comparison with (2.9) shows that

$$\mathbf{S} = (\mathbf{I} + \lambda\mathbf{P})^{-1}(\mathbf{I} - \lambda\mathbf{P}). \tag{4.57}$$

Edge waves

It is well-known that edge waves can be supported by uniform horizontal submerged structures (see Evans & Kuznetsov [32]). Edge waves are waves that are trapped above the topography, their energy propagating in the direction of the ridge but not away from it. They exist in the absence of incident wave forcing. In the present context, the problem of determining edge waves along submerged ridges amounts to specifying k and l (the 'longshore' wavenumber), related previously by $l = k\sin\theta$ in the scattering problem, as independent parameters. Zero wave radiation away from the ridge is ensured by choosing $l > k$ (when it is common terminology to say that the longshore wavenumber is 'above the cut-off'). We then try to find an edge wave dispersion relation $l = l(k)$, between the longshore wavenumber and frequency. Much of the preceding analysis carries across to this case and the reader is referred to Chapman [27] for full details. Briefly, the 'edge wave' condition that $\phi \to 0$ as $|x| \to \infty$, ensured by the choice $l > k$, implies a change to the Green's function in which the first $n = 0$ term is converted from a wave-like mode into an evanescent mode, such as those defined by $n \geq 1$. With this small change, the analysis goes through as before, but no free terms due to the incident wave are present, nor are any thrown out during the formulation (since now $G = \widehat{G}$). Thus, edge waves correspond to the non-trivial solutions to the homogeneous versions of either (4.38) or (4.52).

5 Approximation and numerical method

The problem of determining the reflection and transmission coefficients has been reduced to one in which we need to determine the four matrix elements $P_{\pm}^{\pm}$ of $\mathbf{P}$ which are defined in (4.55) in terms of inner products involving functions $\varphi^{\pm}$ which are the solution of the integral equations in (4.38) or the integro-differential equations in (4.52). These alternative formulations of the problem call for different styles of approach depending critically upon whether or not the integral operator is self-adjoint.

5.1 Boundary element approach

The operator $\mathcal{K}_1$ is not self-adjoint and furthermore the physical nature of the adjoint problem is unclear. Therefore it is not evident how to solve integral equations based on this operator by the Rayleigh-Ritz method. In fact we find that this formulation is particularly amenable to a boundary element approach. In what follows we assume that the bed profile $z = h(x)$ is single-valued in x, that is to say there are no overhangs. This is not unduly restrictive, our main reason for requiring it is simplification of the parameterisation of the curve by projecting the curve Γ onto the x axis. In any case this is purely a numerical issue in that it affects how we choose to parameterise the surface. If $h(x)$ was no longer single valued the method would not fail, we would choose an alternative parameterisation defined along the curve rather than using projection and parameterising along the x axis.

Projecting (4.38) onto the x axis using $\mathrm{d}s = \sigma\, \mathrm{d}x$ and $\partial/\partial s = \sigma^{-1}\partial/\partial x$ results in

$$(\mathcal{K}_1\phi)(x_0, h(x_0)) = \phi(x_0, h(x_0)) - \int_0^a \left(\frac{\partial}{\partial x}\widehat{G}^z_x - l^2\widehat{G}^z\right)\phi\, \mathrm{d}x \tag{5.1}$$

where the argument of the Green's function terms $\widehat{G}^z$ and $\widehat{G}^z_x$ is $(x, h(x); x_0, h(x_0))$. Therefore defining a pair of functions $\varphi_1^{\pm}(x)$ for $x \in [0, a]$, the integral equation (4.38) becomes

$$(\mathcal{K}_1\varphi^{\pm})(x_0) = \phi_0^{\pm}(x_0, h(x_0)), \qquad x_0 \in [0, a]. \tag{5.2}$$

We now solve (5.2) by a boundary element approach, namely subdividing the x axis into N equal length elements and assuming ϕ takes a specific form on each element. Often in boundary integral approaches increasingly sophisticated choices of the form of ϕ are taken. For example piecewise polynomials, defined on each element so that ϕ and one or more of its derivatives are continuous at the ends of adjacent panels, might be chosen in the hope that they will better approximate the exact solution. In fact we find that, in this formulation, the simplest approximation that ϕ is a constant on each panel is extremely effective. Thus writing (5.2) in full using (5.1) and assuming φ is a constant on each panel results in the equation

$$\varphi_n^{\pm}(x_0, h(x_0)) - \sum_{n=1}^{N}\varphi_n^{\pm}\left(\int_{x_n-\delta}^{x_n+\delta}\left(\frac{\mathrm{d}}{\mathrm{d}x}\widehat{G}^z_x - l^2\widehat{G}^z\right)\mathrm{d}x\right) = \phi_0^{\pm}(x_0, h(x_0)) \tag{5.3}$$

where $x_n = (2n-1)\delta$ with $\delta = a/2N$, defines the position of the midpoint of the $n'th$ panel. In this form it is clear that the main advantage of choosing such a simple form of φ on each panel enables us to integrate out the potentially most singular term in the equation explicitly. Therefore we may write

$$\varphi^{\pm}(x_0, h(x_0)) - \sum_{n=1}^{N}\varphi_n^{\pm}\left(\left[\widehat{G}^z_x\right]_{x_n-\delta}^{x_n+\delta} - l^2\int_{x_n-\delta}^{x_n+\delta}\widehat{G}^z\, \mathrm{d}x\right) = \phi_0^{\pm}(x_0, h(x_0)). \tag{5.4}$$

We proceed by collocating, i.e. requiring (5.4) to hold exactly at the centre point of each panel in

x_0, therefore (5.4) becomes

$$\varphi_m^{\pm} - \sum_{n=1}^{N} \varphi_n^{\pm} \left(A_{mn} - l^2 B_{mn}\right) = \phi_{0,m}^{\pm}, \qquad m = 1, \ldots, N \tag{5.5}$$

where $\varphi_m^{\pm} = \varphi^{\pm}(x_m, h(x_m))$, $\phi_{0,m}^{\pm} = \phi_0^{\pm}(x_m, h(x_m))$,

$$A_{mn} = \left[\widehat{G}_x^z(x, h(x); x_m, h(x_m))\right]_{x_n-\delta}^{x_n+\delta} \tag{5.6}$$

and

$$B_{mn} = \int_{x_n-\delta}^{x_n+\delta} \widehat{G}^z(x, h(x); x_m, h(x_m))\,\mathrm{d}x. \tag{5.7}$$

We now simply solve (5.5) which is a system of two straight forward $N \times N$ matrix equations differing only by the forcing terms on the right hand side.

Finally, using (4.39) in (4.55) we deduce that

$$P^{(\pm,\pm)} = \sum_{n=1}^{N} \varphi_n^{\pm} F_n^{\pm} \tag{5.8}$$

where $F_n^{\pm}$ is given by

$$F_n^{\pm} = \mp\frac{\alpha}{k}\left[f^{\pm}(x, h(x))\right]_{x_n-\delta}^{x_n+\delta} - \frac{\mathrm{i}l^2}{k}\int_{x_n-\delta}^{x_n+\delta} f^{\pm}(x, h(x))\,\mathrm{d}x \tag{5.9}$$

thus giving us all the information we require to calculate the scattering matrix. It should be stressed that there are no numerical difficulties with any of the calculated quantities. The only quantity which might have been expected to cause difficulties, that is A_{mn} for $m = n$, is integrated out explicitly. Although deceptively simple, this approach is extremely powerful and efficient, when it is compared with the amount of effort required to implement the traditional approach using normal derivatives rather than our switch to tangential derivatives (see, for example, Fenton [4]).

5.2 Rayleigh-Ritz approach

Crucially to what follows the integro-differential operator $\mathcal{K}_2$ is self-adjoint. Therefore, in order to solve the integro-differential equation, we use a standard variational principle applicable to self-adjoint operators. The formal details of this approach can be found in [26] or Chapman [27], where it is shown to be equivalent to Galerkin's method in which we make the approximation

$$\varphi^{\pm} \approx \widetilde{\varphi}^{\pm} = \sum_{n=0}^{N} a_n^{\pm} P_n(s) \tag{5.10}$$

where $a_n^{\pm}$, $n = 0, 1, \ldots, N$ are coefficients determined by the solution of the system of equations

$$\sum_{n=0}^{N} a_n^{\pm} \langle \mathcal{K}_2 P_n, P_m \rangle = \langle F^{\pm}, P_m \rangle \tag{5.11}$$

and where the set of functions $P_n(s)$ is specified to span an $(N+1)$-dimensional subspace $\mathcal{H}_{N+1}$ of $\mathcal{H}$. The functions $P_n(s)$ will be prescribed later, as will a discussion of the value of N, with reference to physical considerations particular to the geometry and wave parameters being considered.

The resulting approximations to $P^{(\pm,\pm)}$ are

$$\widetilde{P}^{(\pm,\pm)} = \sum_{n=0}^{N} a_n^{\pm} \langle P_n, F^{\pm} \rangle \tag{5.12}$$

where the convention holds that the order of the superscripts on the left-hand side match the order in which they appear on the right-hand side. As a consequence of the variational principle underpinning the approximation, we are provided with the estimates

$$|P^{(\pm,\pm)} - \widetilde{P}^{(\pm,\pm)}| = O(||\varphi^{\pm} - \widetilde{\varphi}^{\pm}|| \, ||\varphi^{\pm} - \widetilde{\varphi}^{\pm}||) \tag{5.13}$$

using the definition of the norm $||u||^2 = \langle u, u \rangle$. That is, the approximations to the quantities of interest are second-order accurate with respect to first-order approximations to the exact solutions of the integral equations.

Things now become a little complicated as we sift through the details of what (5.11) and (5.12) imply. We write the matrix of elements

$$\langle \mathcal{K}_2 P_n, P_m \rangle \equiv K_{mn} = K_{mn}^{(1)} + K_{mn}^{(2)} + K_{mn}^{(3)} + K_{mn}^{(4)} + K_{mn}^{(5)} \tag{5.14}$$

where

$$K_{mn}^{(1)} = -\int_\Gamma P_m(s_0) \frac{\mathrm{d}}{\mathrm{d}s_0} \int_\Gamma \widehat{G}_{xx_0}^{zz_0} \frac{\mathrm{d}P_n(s)}{\mathrm{d}s} \,\mathrm{d}s \,\mathrm{d}s_0, \tag{5.15}$$

$$K_{mn}^{(2)} = -l^2 \int_\Gamma P_m(s_0) \frac{\mathrm{d}}{\mathrm{d}s_0} \int_\Gamma \widehat{G}_{x_0}^{zz_0} P_n(s) \,\mathrm{d}s \,\mathrm{d}s_0, \tag{5.16}$$

$$K_{mn}^{(3)} = l^2 \int_\Gamma P_m(s_0) \int_\Gamma \widehat{G}_{x}^{zz_0} \frac{\mathrm{d}P_n(s)}{\mathrm{d}s} \,\mathrm{d}s \frac{\mathrm{d}s_0}{\sigma_0}, \tag{5.17}$$

$$K_{mn}^{(4)} = l^4 \int_\Gamma P_m(s_0) \int_\Gamma \widehat{G}^{zz_0} P_n(s) \,\mathrm{d}s \frac{\mathrm{d}s_0}{\sigma_0} \tag{5.18}$$

and

$$K_{mn}^{(5)} = -l^2 \int_\Gamma (h_0 - h(x_0)) P_m(s_0) P_n(s_0) \frac{\mathrm{d}s_0}{\sigma_0}. \tag{5.19}$$

To simplify matters further, it is assumed that $x(s)$ is monotonic increasing with s (i.e. $h(x)$ is a single-valued function and so the ridge has no overhangs) so that we may project the integration from the curve Γ onto the interval $x \in [0,a]$. Furthermore it allows us to write $s = s(x)$, $\mathrm{d}s = \sigma(x)\,\mathrm{d}s$ whence $P_n(s) \equiv p_n(x)$ which we may assume is real without loss of generality. Then after integrating (5.15) and (5.16) by parts (noting that there are no discontinuities which affect the integration and that the free terms are zero by construction) we obtain the simplified form

$$K^{(1)}_{mn} = \int_0^a \int_0^a \widehat{G}^{zz_0}_{xx_0}(x, h(x); x_0, h(x_0)) p'_n(x) p'_m(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 \tag{5.20}$$

$$K^{(2)}_{mn} = l^2 \int_0^a \int_0^a \widehat{G}^{zz_0}_{x_0}(x, h(x); x_0, h(x_0)) p_n(x) p'_m(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 \tag{5.21}$$

$$K^{(3)}_{mn} = l^2 \int_0^a \int_0^a \widehat{G}^{zz_0}_{x}(x, h(x); x_0, h(x_0)) p'_n(x) p_m(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 \tag{5.22}$$

$$K^{(4)}_{mn} = l^4 \int_0^a \int_0^a \widehat{G}^{zz_0}(x, h(x); x_0, h(x_0)) p_n(x) p_m(x_0)\,\mathrm{d}x\,\mathrm{d}x_0 \tag{5.23}$$

and

$$K^{(5)}_{mn} = -l^2 \int_0^a (h_0 - h(x_0)) p_n(x_0) p_m(x_0)\,\mathrm{d}x_0. \tag{5.24}$$

This latter step, which transforms (5.15)-(5.16) into (5.20)-(5.21) via integration by parts implies that the remaining integral kernels are no more than weakly-singular.

Since $p_n(x)$ is real each of the kernels in (5.20)-(5.24) is real. Furthermore the individual kernels have the following symmetry properties $K^{(1)}_{mn} = K^{(1)}_{nm}$, $K^{(2)}_{mn} = K^{(3)}_{nm}$, $K^{(4)}_{mn} = K^{(4)}_{nm}$, and $K^{(5)}_{mn} = K^{(5)}_{nm}$, which are easily deduced from (5.19) to (5.23) by switching variables. Hence it follows that K_{mn} is a real symmetric matrix (useful for numerical efficiency), a property which is a direct consequence of the self-adjointness of $\mathcal{K}_2$. We have not explicitly written out the definitions of $\widehat{G}^{zz_0}_{xx_0}$ etc., occurring in (5.20) to (5.23), but simply comment that they can be easily derived from the definition (3.17) using (4.20) with (4.21).

Let us now consider the right-hand side terms in (5.11) and develop them in a similar manner. Thus from the definition of $F^{\pm}$ in (4.49), and integrating by parts,

$$\langle F^{\pm}, P_m \rangle \equiv F^{\pm}_m = \int_\Gamma \left\{ \pm \left(\frac{\alpha}{k}\right) \frac{\partial \overline{P}_m}{\partial s} - \left(\frac{\mathrm{i}l^2}{k\sigma}\right) \overline{P}_m \right\} f^{\pm}(x,z)|_{(x,z)\in\Gamma}\ \mathrm{d}s. \tag{5.25}$$

Projecting onto the x axis and, as $p_n(x)$ is real, we find

$$F^{\pm}_m = \int_0^a \left\{ \pm \frac{\alpha}{k} p'_m(x) - \frac{\mathrm{i}l^2}{k} p_m(x) \right\} \mathrm{e}^{\pm \mathrm{i}\alpha x} \chi_0(h(x))\ \mathrm{d}x \tag{5.26}$$

using (4.28). It follows that $F_m^{\mp} = \overline{F_m^{\pm}}$ and, on account of K_{mn} being real, $a_n^{\mp} = \overline{a_n^{\pm}}$. Thus only one complex set of equations in (5.11) needs to be solved. Alternatively, a pair of systems of *real* equations for the real and imaginary parts of the right-hand side terms, $F_m^{\pm}$ could be solved to generate $a_n^{\pm}$.

All that is required of the functions $p_n(x)$ is that they define a complete set in the interval $(0, a)$. However, functions which incorporate the local fluid behaviour at the end points of Γ will, in general, provide better results. For topographies where the join with the region of constant depth is not smooth, the flow at the join is locally like potential flow within a wedge and hence the bed flux must vanish at the join. In such cases therefore we choose $p_n(x) = \cos n\pi x/a$ which gives the required behaviour at the join. In the case of smooth joins Legendre polynomials, or a full Fourier series, would be appropriate.

The case of normal incidence requires more careful thought because, when $l = 0$ and $p_0(x) = 1$ then the matrix K_{mn} becomes degenerate with zero entries in the first row and column. This is because, in this case, $K_{mn} = K_{mn}^{(1)}$ which depends only upon $p_n'(x)$, which are functions modelling $\partial\phi/\partial s$, the flux along the bed. In this case, one should expand the functions $p_n'(x)$ in a complete orthogonal set, recovering the method of Staziker *et al.* [25].

Let us briefly discuss some of the issues surrounding the computation of the solution. There are five separate elements, $K_{mn}^{(i)}$, $i = 1, 5$ which make up K_{mn}. Although symmetry implies only half the elements need to be computed, and that they are real, each of the $i = 1, 4$ factors requires the evaluation of a double-integral in which the kernel is not separable. The terms in the infinite sums in the kernels decrease exponentially away from the line $0 < x = x_0 < a$ in the square domain of integration, where convergence is only algebraic. This is only an issue with the factor $K_{mn}^{(1)}$ which inherits a logarithmic singularity from the original Green's function, and so the series defining the kernel $\widehat{G}_{xx_0}^{zz_0}$ is divergent along $0 < x = x_0 < a$. However, this factor is identical to that considered in [26] (it is the normally-incident component of K_{mn}) where it is treated as follows. We write

$$\widehat{G}_{xx_0}^{zz_0}(x, z; x_0, z_0) = \widetilde{G}_{xx_0}^{zz_0}(x, z; x_0, z_0) - \frac{1}{4\pi} \ln L(x, z; x_0, z_0) \tag{5.27}$$

where

$$
\begin{aligned}
\widetilde{G}^{zz_0}_{xx_0}(x,z;x_0,z_0) &= \frac{\alpha\chi_0(z)\chi_0(z_0)}{2k^2h_0}\sin\alpha|x-x_0| \\
&\quad - \sum_{r=1}^{\infty}\left\{\frac{\alpha_r\chi_r(z)\chi_r(z_0)}{2k_r^2h_0}\mathrm{e}^{-\alpha_r|x-x_0|} - \frac{\sin(r\pi z/h_0)\sin(r\pi z_0/h_0)}{r\pi}\mathrm{e}^{-r\pi|x-x_0|/h_0}\right\} \qquad (5.28)
\end{aligned}
$$

and

$$
L(x,z;x,z_0) = \frac{\sin^2\{\frac{1}{2}\pi(z+z_0)/h_0\} + \sinh^2\{\frac{1}{2}\pi(x-x_0)/h_0\}}{\sin^2\{\frac{1}{2}\pi(z-z_0)/h_0\} + \sinh^2\{\frac{1}{2}\pi(x-x_0)/h_0\}}. \qquad (5.29)
$$

The sum in (5.28) above is now convergent for all x,x_0 and Gradshteyn & Ryzhik [33] has been used to sum the series explicitly. See [26] for details on how the terms proportional to $\ln T$ is integrated out analytically.

6 Oblique scattering by a step of arbitrary profile

We now outline how the techniques introduced in the previous part of this paper may be used to tackle the substantially more demanding problem of oblique scattering by a section of varying bed, uniform in cross-section, connecting two semi-infinite regions of constant, but different, depths so that $h(x) = h_1$ for $x < 0$ and $h(x) = h_2 > h_1$ for $x > a$, as shown in figure A.2. The new transformation (4.22) is the key to providing a full generalisation to oblique wave incidence of the paper of [26] on normal incidence for scattering by a step of arbitrary profile. We can therefore follow the formulation set out in the work of [26], but must be careful to follow the implementation of the transformation given by (4.22) described in Section 4.

We align our notation in this section as closely as possible with the previous part of the paper, but also incorporate as much of the notation and structure of [26] as possible, anticipating that the reader ought to be able to replicate the work presented below with reference to that paper.

We proceed as before having reduced the three-dimensional problem to a quasi two-dimensional problem. The boundary-value problem for $\phi(x,z)$ is

$$(\nabla^2 - l^2)\phi = 0, \qquad \text{in } D : \{-\infty < x < \infty,\ 0 < z < h(x)\}, \tag{6.1}$$

$$\frac{\partial \phi}{\partial z} + K\phi = 0, \qquad \text{on } z = 0, \tag{6.2}$$

$$\frac{\partial \phi}{\partial z} = 0, \qquad \text{on } \begin{cases} z = h_1, & x < 0, \\ z = h_2, & x > a, \end{cases} \tag{6.3}$$

and

$$\frac{\partial \phi}{\partial n} = 0, \qquad \text{on } \Gamma : \{0 < x < a,\ z = h(x)\}. \tag{6.4}$$

We shall assume that the simplest of the two cases considered in [26] holds, namely that $h(x) \leq h_2$, assumed without loss of generality to be the greater of the two depths.

6.1 Notation and the scattering matrix

We modify our notation from the previous sections in recognition that there are two principal regions of differing constant depth h_1 and h_2. Thus, we shall use an additional subscript, $i = 1$ or $i = 2$ to quantities which relate to the depths h_i, $i = 1, 2$. For instance, k_i, $i = 1, 2$ will replace k and represent the wavenumber of propagating waves over the constant depth h_i, satisfying the dispersion relation appropriate to that depth namely, $K = k_i \tanh k_i h_i$. Similarly the quantities $k_{i,n}$, α_i, $\alpha_{i,n}$, $N_{i,n}$, $\psi_{i,n}(z)$, $\chi_{i,n}(z)$, $G_i(x, z; x_0, z_0)$, $\phi^{\pm}_{i,0}(x, z)$ will replace, in order, k_n, α, α_n, N_n, $\psi_n(z)$, $\chi_n(z)$, $G(x, z; x_0, z_0)$, $\phi^{\pm}_0(x, z)$ with, in each case, h_i $(i = 1, 2)$ replacing h_0. The angles θ_1, θ_2 at which waves propagate, either side of the varying bed, are related by Snell's Law

$$l = k_1 \sin\theta_1 = k_2 \sin\theta_2.$$

In the new notation, the radiation conditions are

$$\phi(x,z) \sim \begin{cases} A_-\phi^{+}_{1,0}(x,z) + B_-\phi^{-}_{1,0}(x,z), & x \to -\infty \\ A_+\phi^{-}_{2,0}(x,z) + B_+\phi^{+}_{2,0}(x,z), & x \to \infty \end{cases} \tag{6.5}$$

(the translation $A_- = -A_1$, $B_- = -B_1$, $A_+ = A_2$, $B_+ = B_2$ is needed to connect with [26]), where $A_\pm$ and $B_\pm$ are incoming and outgoing wave amplitudes respectively. We find it convenient to change the definition of the scattering matrix $\mathbf{S}$ in this section, so that

$$\begin{pmatrix} B_+ \\ B_- \end{pmatrix} = \mathbf{S} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} R_+ & T_- \\ T_+ & R_- \end{pmatrix} \tag{6.6}$$

and, as before, $R_\pm$, $T_\pm$ denote the reflection and transmission coefficients for an incident wave from $x = \pm\infty$.

To recover the transmission coefficients for the free surface displacements we must take into account the different depths into which the transmitted waves propagate so that

$$\tilde{T}_- = T_- \psi_{2,0}(0)/\psi_{1,0}(0) \qquad \tilde{T}_+ = T_+ \psi_{1,0}(0)/\psi_{2,0}(0),$$

see Miles [8].

6.2 Construction of integral equations for the step

We start by defining a Green's function G_- where

$$G_- = G_1(x, z; x_0, z_0) + G_1(-x, z; x_0, z_0)$$

which satisfies (6.1) in $x < 0, 0 < z < h_1$, (6.2) on $z = 0, x < 0$, and (6.3) on $z = h_1, x < 0$. It is constructed so that

$$G_- = 0 \qquad \text{on } x = 0,\ 0 \le z \le h_1. \tag{6.7}$$

We now apply Green's identity, (3.10), to the domain $-X < x_0, x < 0$, $0 < z < h_1$ and evaluate the limit $X \to \infty$ from (6.5) and (3.15) to give

$$\phi(x_0, z_0) = A_-(\phi_{1,0}^+(x_0, z_0) + \phi_{1,0}^-(x_0, z_0)) + \int_0^{h_1} G_-(0, z; x_0, z_0)\phi_x(0, z)\,\mathrm{d}z \tag{6.8}$$

for $x_0 < 0, 0 < z < h_1$. Then using the far-field forms of ϕ from (6.5) and G_1 implied by (3.15) in (6.8) we deduce that A_- and B_- are related by

$$B_- = A_- + \frac{\mathrm{i}}{\alpha_1 h_1} \int_0^{h_1} \psi_{1,0}(z)\phi_x(0, z)\,\mathrm{d}z. \tag{6.9}$$

Similarly we define a Green's function G_+ where

$$G_+ = G_2(x, z; x_0, z_0) + G_2(-x, z; x_0, z_0)$$

which satisfies (6.1) in $x > 0$, $0 < z < h_2$, (6.2) on $z = 0, x > 0$, and (6.3) on $z = h_2$, $0 < x$. In addition $\partial G_+/\partial x = 0$ on $x = 0$ for $0 < z < h_2$. Again we apply Green's identity, this time to the domain $0 < x_0, x < X$, $0 < z < h_2$ and evaluate the limit $X \to \infty$ to give

$$\mu\phi(x_0, z_0) = A_+(\phi^+_{2,0}(x_0, z_0) + \phi^-_{2,0}(x_0, z_0)) - \int_0^{h_1} G_+(0, z; x_0, z_0)\phi_x(0, z)\,\mathrm{d}z - \int_\Gamma \phi(x, z)\frac{\partial G_+}{\partial n}\,\mathrm{d}s, \tag{6.10}$$

for $0 < x_0, 0 < z \leq h(x_0)$ where μ is 1 or $\frac{1}{2}$ depending upon whether (x_0, z_0) is inside, on the boundary of, the fluid domain D.

Then using the far-field forms of ϕ from (6.5) and G_2 from (3.15) in (6.10) we deduce that A_+ and B_+ are related by

$$B_+ = A_+ - \frac{\mathrm{i}}{\alpha_2 h_2}\int_0^{h_1} \psi_{2,0}(z)\phi_x(0, z)\,\mathrm{d}z - \frac{\mathrm{i}}{\alpha_2 h_2}\int_\Gamma \phi(x, z)\frac{\partial}{\partial n}(\cos(\alpha_2 x)\psi_{2,0}(z))\,\mathrm{d}s. \tag{6.11}$$

Analogously to (3.16) we decompose the Green's functions so that

$$G_i = G_{i,0} + \widehat{G}_i \qquad \text{implying} \qquad G_\pm = G_{\pm,0} + \widehat{G}_\pm \tag{6.12}$$

where

$$G_{i,0} = \mathrm{i}\frac{\psi_{i,0}(z)\psi_{i,0}(z_0)}{2\alpha_i h_i}\cos\alpha_i(x - x_0). \tag{6.13}$$

Using (6.12) and (6.13) in (6.8) and (6.10), we are able to deduce, after some work, that

$$\phi(x_0, z_0) = (A_- + B_-)\cos(\alpha_1 x_0)\psi_{1,0}(z_0) + \int_0^{h_1} \widehat{G}_-(0, z; x_0, z_0)\phi_x(0, z)\,\mathrm{d}z \tag{6.14}$$

for $x_0 < 0, 0 < z_0 < h_1$ and

$$\mu\phi(x_0, z_0) = (A_+ + B_+)\cos(\alpha_2 x_0)\psi_{2,0}(z_0) - \int_0^{h_1} \widehat{G}_+(0, z; x_0, z_0)\phi_x(0, z)\,\mathrm{d}z - \int_\Gamma \phi(x, z)\frac{\partial \widehat{G}_+}{\partial n}\,\mathrm{d}s, \tag{6.15}$$

for $0 < x_0,\ 0 < z \leq h(x_0)$.

We convert the normal derivative in the integral along Γ in (6.15) to tangential derivatives using (4.22) to give

$$\phi(x_0, z_0) = (A_+ + B_+)\cos(\alpha_2 x_0)\psi_{2,0}(z_0) - \int_0^{h_1} \widehat{G}_+(0, z; x_0, z_0)\phi_x(0, z)\,\mathrm{d}z$$
$$- \int_\Gamma \phi(x, z)\left(-\frac{\partial}{\partial s}(\widehat{G}_+)^z_x + \frac{l^2}{\sigma}(\widehat{G}_+)^z\right)\mathrm{d}s \tag{6.16}$$

for $0 < x_0,\ 0 < z \le h(x_0)$. combining the contribution from the final term in (4.22) with the prefactor μ so that (6.16) now provides a definition of ϕ which is continuous across the boundary of D.

We obtain our first integral equation by matching the expressions for $\phi(0, z_0)$ from (6.14) and (6.16) for $0 \le z_0 \le h_1$. The first step is to take the limit $x_0 \to 0$ in (6.16), integrate by parts and then match the resulting expression with (6.14) evaluated at $x_0 = 0$ to find, after some rearrangement

$$\int_0^{h_1} \{\widehat{G}_-(0, z; 0, z_0) + \widehat{G}_+(0, z; 0, z_0)\}\phi_x(0, z)\,\mathrm{d}z + \int_\Gamma \frac{\partial\phi}{\partial s}(\widehat{G}_+)^z_x(x, z; 0, z_0) + \frac{l^2}{\sigma}(\widehat{G}_+)^z(x, z; 0, z_0)\phi(x, z)\,\mathrm{d}s = -(A_- + B_-)\psi_{1,0}(z_0) + (A_+ + B_+)\psi_{2,0}(z_0) \quad (6.17)$$

for $0 \le z_0 \le h_1$. A second integral equation is obtained by applying the operator $\partial/\partial n_0$ to (6.16) for points $(x_0, z_0) \notin \Gamma$. We use relations similar to that expressed by (4.42), with $\widehat{G}$ replaced by $\widehat{G}_+$ whilst using (4.44) with (4.28) and (4.29) gives

$$\begin{aligned} \frac{\partial}{\partial n_0}(\cos(\alpha_2 x_0)\psi_{2,0}(z_0)) &= F_2^+(x_0, z_0) \\ &\equiv -\frac{\mathrm{i}\alpha_2}{k_2}\frac{\partial}{\partial s_0}(\chi_{2,0}(z_0)\sin(\alpha_2 x_0)) - \frac{\mathrm{i}l^2}{k_2\sigma_0}(\chi_{2,0}(z_0)\cos(\alpha_2 x_0)). \end{aligned} \quad (6.18)$$

The outcome of the above procedure has been to convert all derivatives with respect to n_0 into a combination of terms which involve derivatives with respect to s_0. Then, moving the (x_0, z_0) onto Γ and applying the bed condition (4.21) leads eventually to the integro-differential equation

$$\begin{aligned} 0 = {} & (A_+ + B_+)F_2^+(x_0, z_0) \\ & + \frac{\partial}{\partial s_0}\int_0^{h_1} (\widehat{G}_+)^{z_0}_{x_0}(0, z; x_0, z_0)\phi_x(0, z)\,\mathrm{d}z - \frac{l^2}{\sigma_0}\int_0^{h_1} (\widehat{G}_+)^{z_0}(0, z; x_0, z_0)\phi_x(0, z)\,\mathrm{d}z \\ & + \frac{\partial}{\partial s_0}\int_\Gamma \left((\widehat{G}_+)^{zz_0}_{xx_0}\frac{\partial\phi}{\partial s} + \frac{l^2}{\sigma}(\widehat{G}_+)^{zz_0}_{x_0}\phi\right)\mathrm{d}s - \frac{l^2}{\sigma_0}\int_\Gamma \left((\widehat{G}_+)^{zz_0}_x\frac{\partial\phi}{\partial s} + \frac{l^2}{\sigma}(\widehat{G}_+)^{zz_0}\phi\right)\mathrm{d}s \\ & + \frac{l^2}{\sigma_0}\phi(x_0, z_0)(h_2 - h(x_0)) \qquad (x_0, z_0) \in \Gamma \end{aligned} \quad (6.19)$$

in which integration by parts is performed to move tangential derivatives onto ϕ. In the process the final term in (6.19) appears (see the discussion surrounding equations (4.47)–(4.50)).

Thus, (6.17) and (6.19) constitute a coupled system of integral equations for the unknown functions $\phi_x(0, z)$, $0 < z < h_1$ and $\phi(s)$, $s \in \Gamma$.

Introducing a more compact notation to expose the structure of the integral equations (6.17) and (6.19) can be written

$$\left.\begin{aligned} &\mathcal{K}_{11}q + \mathcal{K}_{12}\phi = (A_- + B_-)F_1^- + (A_+ + B_+)F_1^+, \qquad z_0 \in (0, h_1) \\ &\mathcal{K}_{21}q + \mathcal{K}_{22}\phi = (A_- + B_-)F_2^- + (A_+ + B_+)F_2^+, \qquad s_0 \in \Gamma \end{aligned}\right\} \tag{6.20}$$

where we have written $q(z) \equiv \phi_x(0, z)$ and defined the operators $\mathcal{K}_{ij}$ by

$$(\mathcal{K}_{11}q)(z_0) = \int_0^{h_1} \{\widehat{G}_-(0, z; 0, z_0) + \widehat{G}_+(0, z; 0, z_0)\} q(z)\, \mathrm{d}z, \tag{6.21}$$

$$(\mathcal{K}_{12}\phi)(z_0) = \int_\Gamma \frac{\partial\phi}{\partial s}(\widehat{G}_+)^z_x(x, z; 0, z_0) + \frac{l^2}{\sigma}(\widehat{G}_+)^z(x, z; 0, z_0)\phi(x, z)\, \mathrm{d}s \tag{6.22}$$

$$(\mathcal{K}_{21}q)(s_0) = \frac{\partial}{\partial s_0}\int_0^{h_1} (\widehat{G}_+)^{z_0}_{x_0}(0, z; x_0, z_0)q(z)\, \mathrm{d}z - \frac{l^2}{\sigma_0}\int_0^{h_1} (\widehat{G}_+)^{z_0}(0, z; x_0, z_0)q(z)\, \mathrm{d}z \tag{6.23}$$

$$\begin{aligned}(\mathcal{K}_{22}\phi)(s_0) = {}& \frac{\partial}{\partial s_0}\int_\Gamma \left((\widehat{G}_+)^{zz_0}_{xx_0}\frac{\partial\phi}{\partial s} + \frac{l^2}{\sigma}(\widehat{G}_+)^{zz_0}_{x_0}\phi \right) \mathrm{d}s \\ & - \frac{l^2}{\sigma_0}\int_\Gamma \left((\widehat{G}_+)^{zz_0}_{x}\frac{\partial\phi}{\partial s} + \frac{l^2}{\sigma}(\widehat{G}_+)^{zz_0}\phi \right) \mathrm{d}s + \frac{l^2}{\sigma_0}\phi(s_0)(h_2 - h(x_0)) \end{aligned} \tag{6.24}$$

with the definition of F_2^+ in (6.18) combined with

$$\begin{aligned} F_1^-(z) &= -\psi_{1,0}(z) \quad \text{and} \quad F_1^+(z) = \psi_{2,0}(z), \qquad z \in (0, h_1), \\ F_2^-(s) &= 0, \qquad s \in \Gamma. \end{aligned}$$

Using a compact matrix/vector, (6.20) can be expressed as

$$\boldsymbol{\mathcal{K}}\boldsymbol{\varphi} = (A_- + B_-)\mathbf{F}^- + (A_+ + B_+)\mathbf{F}^+ \tag{6.25}$$

where

$$\boldsymbol{\mathcal{K}} = \begin{pmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{pmatrix}, \qquad \boldsymbol{\varphi} = \begin{pmatrix} q \\ \phi \end{pmatrix}, \qquad \mathbf{F}^\pm = \begin{pmatrix} F_1^\pm \\ F_2^\pm \end{pmatrix}. \tag{6.26}$$

The solution of (6.25) is given by

$$\boldsymbol{\varphi} = (A_- + B_-)\boldsymbol{\varphi}^- + (A_+ + B_+)\boldsymbol{\varphi}^+ \tag{6.27}$$

where

$$\boldsymbol{\mathcal{K}}\boldsymbol{\varphi}^\pm = \mathbf{F}^\pm. \tag{6.28}$$

By defining the composite inner product

$$\langle\!\langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \rangle\!\rangle = \langle q_1, q_2 \rangle_1 + \langle \phi_1, \phi_2 \rangle_2 \tag{6.29}$$

in terms of the inner products

$$\langle q_1, q_2 \rangle_1 = \int_0^{h_1} q_1(z)\overline{q_2(z)}\,\mathrm{d}z, \qquad \langle p_1, p_2 \rangle_2 = \int_\Gamma p_1(s)\overline{p_2(s)}\,\mathrm{d}s \tag{6.30}$$

the equations (6.9) and (6.11) can now be written, with reference to (6.18), as

$$B_- = A_- - \frac{\mathrm{i}}{\alpha_1 h_1}\langle\!\langle \boldsymbol{\varphi}, \mathbf{F}^- \rangle\!\rangle, \qquad B_+ = A_+ - \frac{\mathrm{i}}{\alpha_2 h_2}\langle\!\langle \boldsymbol{\varphi}, \mathbf{F}^+ \rangle\!\rangle. \tag{6.31}$$

Now, inserting (6.27) into the above expressions and rearranging the resulting pair of equations into the form suggested by (6.6) we find that the scattering matrix is given by

$$\mathbf{S} = (\mathbf{D} + \mathrm{i}\mathbf{P})^{-1}(\mathbf{D} - \mathrm{i}\mathbf{P}) \tag{6.32}$$

in which

$$\mathbf{D} = \begin{pmatrix} \alpha_2 h_2 & 0 \\ 0 & \alpha_1 h_1 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} \langle\!\langle \boldsymbol{\varphi}^+, \mathbf{F}^+ \rangle\!\rangle & \langle\!\langle \boldsymbol{\varphi}^-, \mathbf{F}^+ \rangle\!\rangle \\ \langle\!\langle \boldsymbol{\varphi}^+, \mathbf{F}^- \rangle\!\rangle & \langle\!\langle \boldsymbol{\varphi}^-, \mathbf{F}^- \rangle\!\rangle \end{pmatrix}. \tag{6.33}$$

In summary, whilst the details for the step problem are significantly more complicated than for the ridge, the same ideas can be applied and the final structure of the equations is essentially the same.

6.3 Approximation and numerical method

As in the ridge problem, this problem is amenable to numerical solution either by a Rayleigh-Ritz method or a boundary element method. We shall only give a brief outline of the implementation of these methods here.

Rayleigh-Ritz

The matrix system of integral equations in (6.28) we have derived can be viewed as a generalisation of those obtained by [26] for normal incidence (confirmed by putting $l = 0$) to oblique incidence. Indeed, the operator $\mathcal{K}_{11}$ in (6.21) is barely unchanged from the normal incidence case, whilst the most complicated element of the integral equations, $\mathcal{K}_{22}$ in (6.24) is a slightly modified version of the operator $\mathcal{K}_2$ for the ridge. We can therefore combine the methods from [26] and the preceding sections to perform the Rayleigh-Ritz approximation for the current problem. Briefly, the two sets of unknown functions representing $q(z) = \phi_x(0, z)$ and $\phi(s)$, $s \in \Gamma$ are expanded in a finite series (including $N_1 + 1$ and $N_2 + 1$ terms respectively), of test functions over the two intervals. The resulting equations are then made orthogonal to each test function in the set, in the manner encapsulated in (5.11) earlier to form a finite coupled system of algebraic equations for the coefficients represented each set of unknown functions. This is the Galerkin method, and allows a final integration by parts to be made in which the differential operators $\partial/\partial s_0$ are moved onto the test functions. For those integrals evaluated over Γ, the bed profile is projected onto the x axis over the interval $0 < x < a$ to form matrix elements reminiscent of those written out in equations (5.20)–(5.24). The choice of test function is dependent on how the topography joins at the top of the slope (this can result in a discontinuous gradient and a singularity in the gradient of ϕ). These issues are discussed in detail in [26].

Boundary Element

A boundary element approach has been implemented to provide an independent assessment of the accuracy of the results obtained by the Rayleigh-Ritz method. A numerical approximation to the solution of the coupled integral equations, (6.14) and (6.16) in terms of the unknown functions $q(z) = \phi_x(0, z)$ and $\phi(s)$, $s \in \Gamma$ (which maps onto $\phi(x, h(x))$, $0 < x < a$) is performed in the general manner outlined in section 5.1. The added complication arises because there are two functions to be determined over two intervals and we therefore subdivide the vertical boundary $x = 0$, $z \in [0, h_1]$ into N_1 panels and the horizontal axis defined by $x \in [0, a]$ into N_2 panels. We then collocate assuming that the unknown functions are constant on each interval between the collocation points. More details of this approach can be found in Chapman [27].

7 Results

Let us first assess the convergence properties of the Boundary Element (BE) method and the Rayleigh-Ritz (RR) method by comparing them with 'exact' results obtained for the specific geometry of a semi-circular ridge or diameter a. We know of no other exact results published for oblique incidence upon ridges and steps. The semi-circular ridge is chosen since it can be used to give highly accurate estimates to the reflection and transmission coefficients using a powerful multipole method, as outlined in the Appendix. In the experiment described below, the semi-circular ridge is given a diameter to depth ratio of $a/h_0 = 1$ and truncated at $N = 16$ terms in (A.11) and (A.12), sufficient to confidently predict six decimal places.

The number of panels, N, in the BE method was varied for a range of incident-wave angles. In table A.1, results are presented for $\theta = 45°$. Note that for ridges, unlike steps, $|R| = |R_-| = |R_+|$.

We anticipate the semi-circular ridge geometry to be a severe test of our numerical approximation for both the BE and RR methods. This is because during the implementation of the numerical schemes, integrals with respect to arclength have been projected onto integrals with respect to x, which is singular at the endpoints $x = 0$ and $x = a$ where $h'(x)$ is infinite. An alternative parametrisation should ideally be applied to cope with such examples, though the aim here has been to address the simplest and most realistic topographies without complicating matters unnecessarily. The approximation of integrals has been performed using a 10-point Gaussian quadrature scheme.

We find that normal incidence provides the most severe test of convergence generally requiring $N = 200$ panels to gain two to three significant figure accuracy. Obliqueness appears assist convergence so that, referring to table A.1, for $\theta = 45°$ and $ka > 3$, only $N = 140$ panels suffice for four significant figure accuracy. These results are impressive for such a simple collocation scheme when tested against such a demanding profile.

Table A.2 presents equivalent convergence results for the RR method, where N now indicates the number of functions used in the approximation in (5.10). We choose $p_n(x) = \cos n\pi x/a$, a half-range cosine series on $0 < x < a$ so that the tangential bed velocity, modelled by $p_n'(x)$ is zero at the end points $x = 0, a$. It is worth noting that for all values of N used above the RR method is noticeably quicker than the BE method when taken with 100 panels or more. We see that convergence is usually rapid in this severe test of convergence, typically achieving three significant figures with just $N = 10$ functions. Further numerical experimentation confirms this result that the RR method is characterised by rapid convergence with modest truncation sizes. Indeed, convergence is significantly improved once more moderate topography with $h'(x)$ bounded throughout the interval including $x = 0, a$ is chosen.

Having two independent methods of solution allows us to compare the BE and RR methods for consistency across all other examples of oblique wave scattering.

In figure (A.3) the modulus of the reflection coefficient is plotted as a function of non-dimensional wavenumber, ka, for incident waves making an angle $\theta = 0°$, $30°$, $45°$ and $60°$ to the x axis by a ridge whose profile, $h(x) = h_0(1 - \frac{1}{2}\sin(\pi x/a))$, $0 < x < a$, is a represented by a half-period of a sine function representing a ridge which protrudes to half the water depth. The ridge width to depth ratio is fixed at $a/h_0 = 1$.

Results for the RR method were produced with a truncation size of $N = 10$ to ensure accurate results and the results for the BE method were produced with a modest number ($N = 100$) of panels. It is evident from the plots that the two sets of results are visibly indistinguishable. In fact for almost all of the range of values the results agree to at least five decimal places and in many cases to six decimal places.

As expected, for longer waves (as $ka \to 0$), the reflection tends to zero as the ridge becomes transparent to the waves. Similarly, $|R| \to 0$ at shorter wavelengths ($ka \gg 1$) where the wave energy is located closer to the free surface. As θ is reduced from $\theta = 0$ (normal incidence) to $\theta \approx 45°$, the general trend is for $|R|$ to become smaller. For increasingly oblique waves where the distance over the ridge in the direction of the waves increases, the reflection also generally increases.

Finally we turn to the problem of determining the dispersion relation for edge waves over a submerged ridge. Results are presented in figure A.4 for edge waves over a ridge for which $h(x) = h_0 - 0.475h_0(1 - \cos 2\pi x/a)$, in which the ridge is represented by one period of the cosine function, smoothly joining the flat bed at $x = 0, a$ and rising to fill 95% of the water depth at its peak. The numerical problem involves finding frequencies, or values of k such that $k < l$ for a given value of longshore wavenumber l such that the determinant of a real symmetric matrix vanishes. This can be done numerically using any basic root-finding algorithm. In figure A.4 we plot the ratio k/l against the non-dimensional wavenumber lh_0 and through the series (a) to (d) we vary the ratio of the bed width to water depth from $a/h_0 = 1$ to $a/h_0 = 4$. Thus the number of edge waves increases as the width of the ridge increases (the number being roughly in proportion to a). This is not unexpected and similar observations have been made by Evans & McIver [34] for edge waves over rectangular protrusions.

We now turn to the step problem, where it proves convenient to non-dimensionalise the profile of the step using the transformation

$$\hat{h}(\hat{x}) = (h_2 - h(a\hat{x}))/b, \qquad 0 \le \hat{x} \le 1, \qquad b = h_2 - h_1, \tag{7.1}$$

so that $\hat{h}(0) = 1$ and $\hat{h}(1) = 0$. In the results that follow we use the Legendre polynomials to model the bed flux along the varying slope. This is in contrast to [26] who suggest the Jacobi polynomials for profiles which do not join the domains of constant depth smoothly at the end points. The discussion in [26] concerning convergence of the system according to the choice of trial functions carries forward to the oblique problem unchanged, therefore we will not labour this point further. We would anticipate slower convergence rates with the choice of Legendre functions for non-smooth joins rather than a set which incorporated the singular behaviour at the end points and our numerical experiments confirmed this.

Figures A.5 and A.6 show the results for a smoothly varying bed form defined by

$$\hat{h}(\hat{x}) = 1 - 3\hat{x}^2 + 2\hat{x}^3 \tag{7.2}$$

which joins the domains of constant depth smoothly. In this case the choice of Legendre polynomials (as would any other complete set such as a Fourier series) is appropriate and, now there is a smooth join, has rapid convergence with truncation size. Figures (A.5) and (A.6) plot the reflection coefficient $|R_+|$ for oblique scattering of waves incident from the deeper domain for the depth ratio $h_1/h_2 = \frac{1}{4}$ and $a/b = \frac{1}{2}, 1, 2, 4, 8$ for angles of incidence of 30° and 60° respectively. For most of the data points truncation sizes of $N_1 = 8$, $N_2 = 12$ in the expansion of the unknown functions $q(z)$ and $\phi(s)$ were sufficient to give at least four significant figure accuracy. However, for larger transient bed widths ($a/b = 8$) and higher wavenumbers ($k_1 h_1 > 3$) we found that taking $N_2 = 15$ was necessary to obtain the same degree of accuracy.

Of more interest is the scattering of waves travelling from shallow to deeper water where total reflection is possible. This occurs if $k_2/k_1 < \sin\theta_1 < 1$ in which case there is no transmission. In this case we obtain a solution in which a wave travels along the step and decays exponentially away from the step. Figure (A.7) show the reflection and transmission coefficients for oblique scattering of waves incident from the shallower domain when $a/b = 2$.

Finally, we present results for Booij's test problem using the same scalings on the vertical and horizontal axes as Booij [20] but, for the first time, for oblique angles. The linear ramp profile of Booij is given, in our notation, by $\hat{h}(\hat{x}) = 1 - \hat{x}$, $0 < \hat{x} < 1$ and the depth ratio and wave frequency are fixed by $h_1/h_2 = \frac{1}{3}$ (so that $b/h_2 = \frac{2}{3}$) and $Kh_2 = \frac{3}{5}$. The variation of $|R_-|$, for a wave from the deeper water is measured against Ka, which therefore is a measure of the gradient of the ramp.

8 Conclusions

We have considered the interaction of oblique surface waves with submerged topographical features, including ridges and steps. A novel integral equation formulation has been developed which generalises the two-dimensional methods of Staziker *et al.* [25] and [26], based on the Cauchy-Riemann relations. The central idea has been to regularise the standard integral equations derived from a direct application of Green's identity, rendering them only weakly-singular. Essentially, integration by parts is used to transfer two derivatives off the logarithmically-singular Green's function, G, by expressing normal derivatives of G in terms of particular combinations of other functions involving tangential derivatives. Intrinsic to this process is the application of the Rayleigh-Ritz approximation, which itself takes advantage of the self-adjointness of the integro-differential operator at the heart of the formulation.

The symmetry of the resulting formulation implied by the self-adjointness, the fact that it is only weakly-singular, and that the second-order accurate Rayleigh-Ritz method is used to approximate solutions are all desirable elements of the present approach, not enjoyed the more traditional approaches to this type of problem. Although the algebraic details of our formulation may appear somewhat complicated, the system to be ultimately solved is remarkably simple and elegant (encapsulated by equations (5.20)–(5.24)).

We have also provided an alternative integral equation formulation (which we also believe to be new) and which lends itself to approximation by the boundary element method as it allows the logarithmic singularity in G to be treated analytically and in a simple manner. The boundary element method, which is straightforward to implement numerically, provides an engineering accuracy (three to four significant figures) by typically solving a 100×100 system of equations. The Rayleigh-Ritz method is more demanding to implement numerically but, in contrast, achieves much improved accuracy by solving a much smaller system which has the benefit of being real and symmetric.

Acknowledgements

Work is currently underway to extend the present methods to more complicated three-dimensional topographical structures, such as axisymmetric seamounts and more arbitrary bed profiles.

One of us, GJDC, would like to acknowledge the UK Natural Environment Research Council's kind support under grant no. NER/S/A/2002/1033 for this research.

A Solution for wave interaction with a semi-circular ridge using multipole potentials

Multipole potential methods are often used for wave scattering problems involving geometries with a combination of circular and rectangular geometries (see Linton & McIver [35], or Martin [29] for example). They provide an efficient and accurate method at determining scattering characteristics. We use this technique to provide accurate independent results for a particular ridge cross-section, being that of a semi-circular protrusion of radius $\frac{1}{2}a$, in an ocean of otherwise constant depth h_0. In the results quoted below we use a polar coordinate system (r, θ) with origin at $(x, z) = (\frac{1}{2}a, h_0)$, the center of the ridge with $\theta = \frac{1}{2}\pi$ aligned with the positive x-axis. We only present a summary of the method; the full details can be found in Champan [27].

The boundary-value problem is given by (2.1), (2.2), (2.3) with (2.4) specifically written as

$$\frac{\partial \phi}{\partial r} = 0, \qquad \text{on } r = \tfrac{1}{2}a,\ \theta \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi) \tag{A.1}$$

We write $\phi = \phi_0^+ + \phi_s$ where ϕ_0^+ represents the incident wave travelling in the positive x-direction and the scattered component is written as a sum over all possible multipoles,

$$\phi_s(x,z) = N_0^{-1/2}\sum_{n=0}^{\infty}\epsilon_n a_{2n}\sigma_{2n}(lr,\theta) + 2\mathrm{i}N_0^{-1/2}\sum_{n=1}^{\infty}a_{2n-1}\sigma_{2n-1}(lr,\theta) \tag{A.2}$$

where $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n \geq 1$ and where a_n, $n = 0, 1, \ldots$ are undetermined coefficients and other terms provide simplification in the final expressions. Also, $l = k\sin\theta$.

The multipole potentials are designed, using the method of Thorne [36] to satisfy (2.1), (2.2), (2.3) and are outgoing as $|x| \to \infty$. Those symmetric about $\theta = 0$, are given by

$$\sigma_{2n}(lr,\theta) = K_{2n}(lr)\cos 2n\theta + \sum_{m=0}^{\infty}C_{mn}I_{2m}(lr)\cos 2m\theta \tag{A.3}$$

where I_m and K_m are modified Bessel functions and

$$C_{mn} = \epsilon_m \fint_0^{\infty} A(t)\cosh 2nt\cosh 2mt\,\mathrm{d}t + \frac{\mathrm{i}\epsilon_m\pi}{2\alpha h_0 N_0}\cosh(2nt_0)\cosh(2mt_0) \tag{A.4}$$

for $m, n = 0, 1, \ldots$ and $t_0 = \cosh^{-1}(k/l)$. The integral is of principal-value type. For $n = 1, 2, \ldots$, the antisymmetric multipoles are defined by

$$\sigma_{2n-1}(lr,\theta) = K_{2n-1}(lr)\sin(2n-1)\theta + \sum_{m=1}^{\infty}D_{mn}I_{2m-1}(lr)\sin(2m-1)\theta \tag{A.5}$$

where

$$D_{mn} = 2\fint_0^{\infty} A(t)\sinh(2n-1)t\sinh(2m-1)t\,\mathrm{d}t + \frac{\mathrm{i}\pi}{\alpha h_0 N_0}\sinh((2n-1)t_0)\sinh((2m-1)t_0). \tag{A.6}$$

for $m, n = 1, 2, \ldots$.

The reflection and transmission coefficients, R_- and T_-, are given by the far-field behaviour of the multipole potentials (see Chapman [27] for details)

$$R_- = \frac{\pi\mathrm{i}}{2\alpha h_0 N_0}\left(\sum_{n=0}^{\infty}\epsilon_n c_{2n}\cosh(2nt_0) - 2\sum_{n=1}^{\infty}c_{2n-1}\sinh((2n-1)t_0)\right) \tag{A.7}$$

and

$$T_- = 1 + \frac{\pi\mathrm{i}}{2\alpha h_0 N_0}\left(\sum_{n=0}^{\infty}\epsilon_n c_{2n}\cosh(2nt_0) + 2\sum_{n=1}^{\infty}c_{2n-1}\sinh((2n-1)t_0)\right) \tag{A.8}$$

The coefficients c_n are determined by the application of the no-flow condition (A.1) which implies

$$\frac{\partial\phi_s}{\partial r} = -\frac{\partial\phi_0^+}{\partial r}, \qquad \text{on } r = \tfrac{1}{2}a,\ -\tfrac{1}{2}\pi \leq \theta \leq \tfrac{1}{2}\pi. \tag{A.9}$$

Expanding the incident wave potential (see Chapman [27]) with

$$\phi_0^+ \equiv \mathrm{e}^{i\alpha x}\psi_0(z) = N_0^{-1/2}\sum_{m=0}^{\infty}\epsilon_m \cosh(2mt_0) I_{2m}(lr)\cos 2m\theta$$
$$+ 2\mathrm{i}N_0^{-1/2}\sum_{m=1}^{\infty}\sinh((2m-1)t_0) I_{2m-1}(lr)\sin(2m-1)\theta. \quad (A.10)$$

and imposing (A.9) and equating coefficients of $\cos 2m\theta$, $\sin(2m-1)\theta$ yields the decoupled systems of equations

$$c_{2m} + Z_{2m}\sum_{n=0}^{\infty} c_{2n}\frac{\epsilon_n}{\epsilon_m}C_{mn} = -Z_{2m}\cosh(2mt_0), \qquad m = 0, 1, 2, \dots \quad (A.11)$$

and

$$c_{2m-1} + Z_{2m-1}\sum_{n=1}^{\infty} c_{2n-1}D_{mn} = -Z_{2m-1}\sinh((2m-1)t_0), \qquad m = 1, 2, \dots \quad (A.12)$$

where we have written $Z_m = I_m'(\frac{1}{2}la)/K_m'(\frac{1}{2}la)$.

Numerically, this pair of linear system of equations is truncated at a finite value, N, say so that coefficients c_n, for $n = 0, 1, \dots, 2N+1$ are found and then the reflection and transmission coefficients can be found from (A.7) and (A.8). The convergence characteristics of the systems (A.11), (A.12) are excellent and a high degree of accuracy is attained for small values of N.

In the case of edge waves along a semi-circular ridge, one relaxes the scattering relationship $l = k\sin\theta$ and look for a relationship $l = l(k) > k$, above the cut-off. In doing so, the incident wave is discarded and so edge waves are determined by finding non-trivial solutions to the homogeneous versions of (A.11), (A.12).

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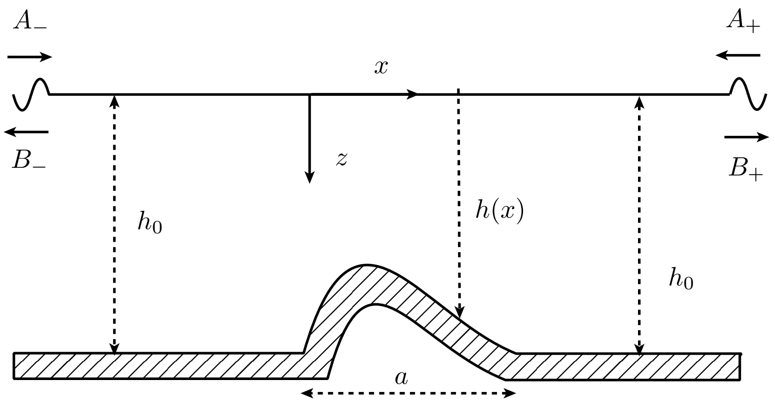


Figure A.1: Geometrical description of the ridge

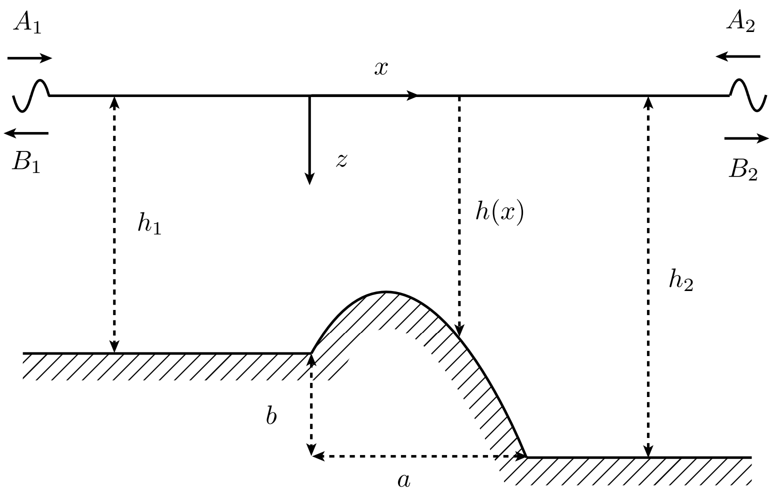


Figure A.2: Geometrical description for a varying bed joining two semi-infinite domains of constant depth

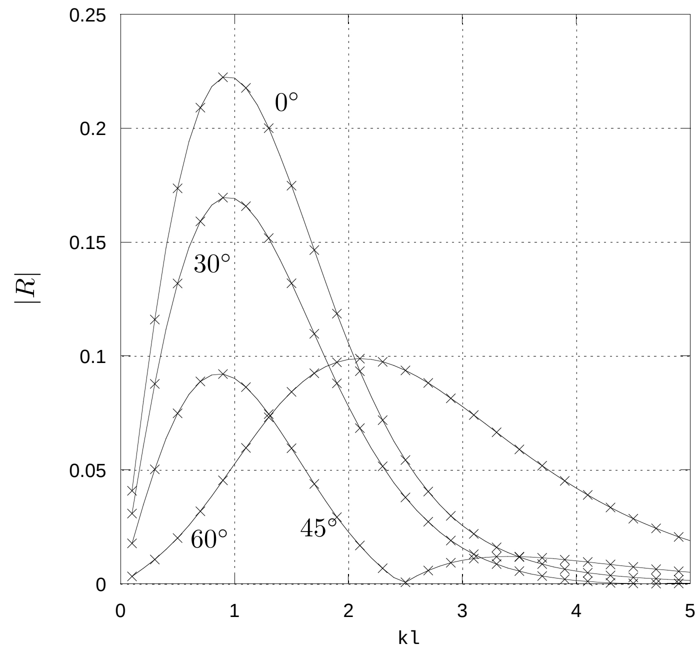
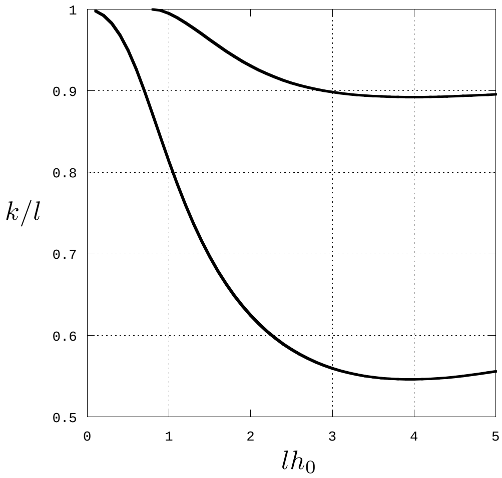
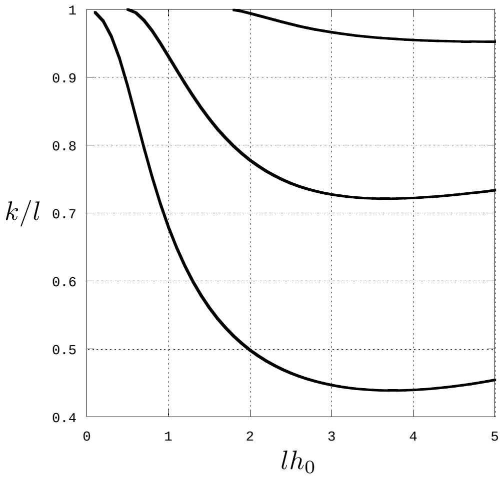


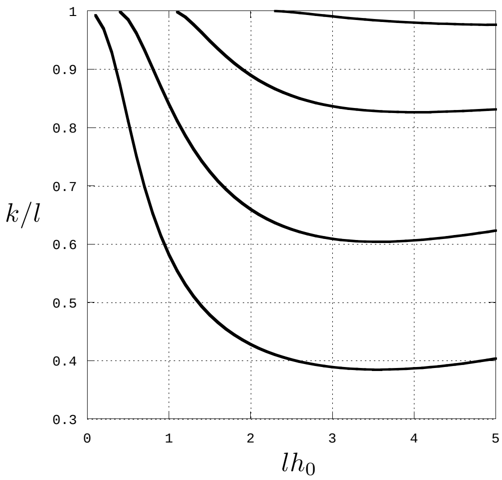
Figure A.3: Comparison of BE and RR methods for oblique scattering by a ridge where $h(x) = h_0(1 - \frac{1}{2}\sin(\pi x/a))$ and $a/h_0 = 1$. (RR solid line, BE $\times$)



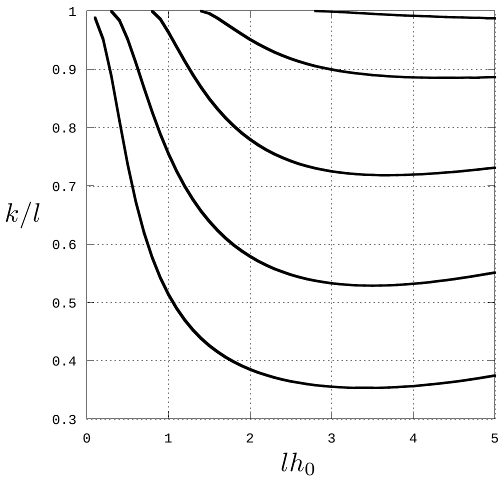
(a) $a/h_0 = 2$



(b) $a/h_0 = 3$



(c) $a/h_0 = 4$



(d) $a/h_0 = 5$

Figure A.4: Edge wave dispersion relations for a ridge where $h(x) = h_0 - 0.475h_0(1 - \cos 2\pi x/a)$.

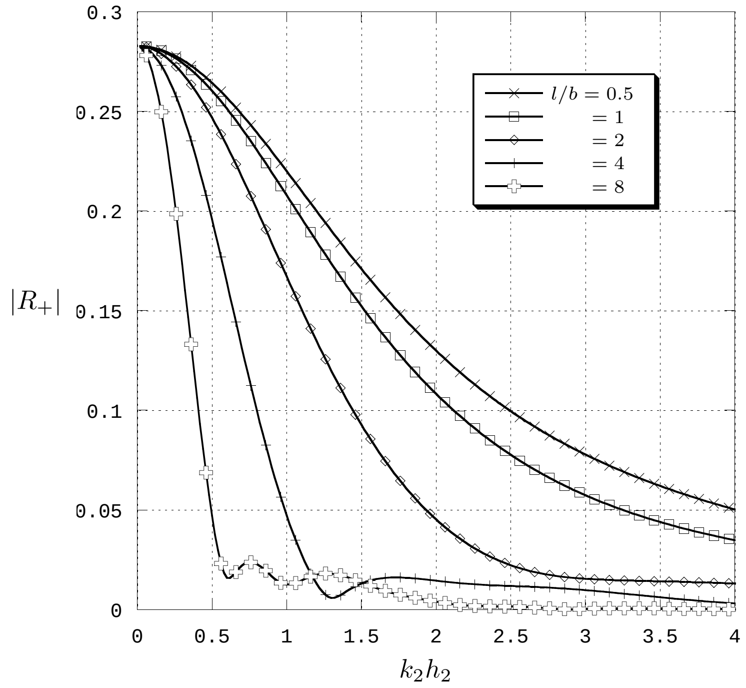


Figure A.5: Reflection coefficients for oblique waves incident from deeper water at 30° to the step $\hat{h}(\hat{x}) = 1 - 3\hat{x}^2 + 2\hat{x}^3$ where $h_1/h_2 = 0.25$.

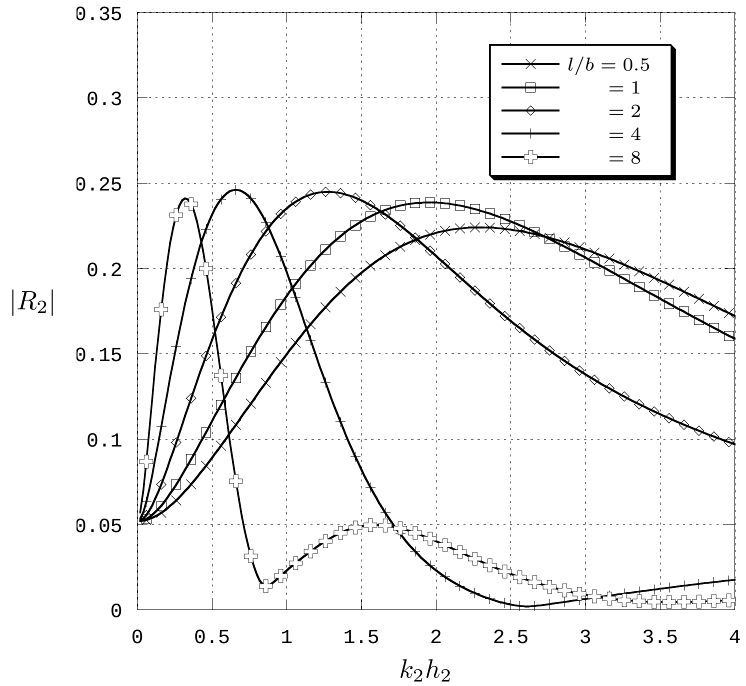


Figure A.6: Reflection coefficients for oblique waves incident from deeper water at 60° to the step $\hat{h}(x) = 1 - 3x^2 + 2x^3$ where $h_1/h_2 = \frac{1}{4}$.

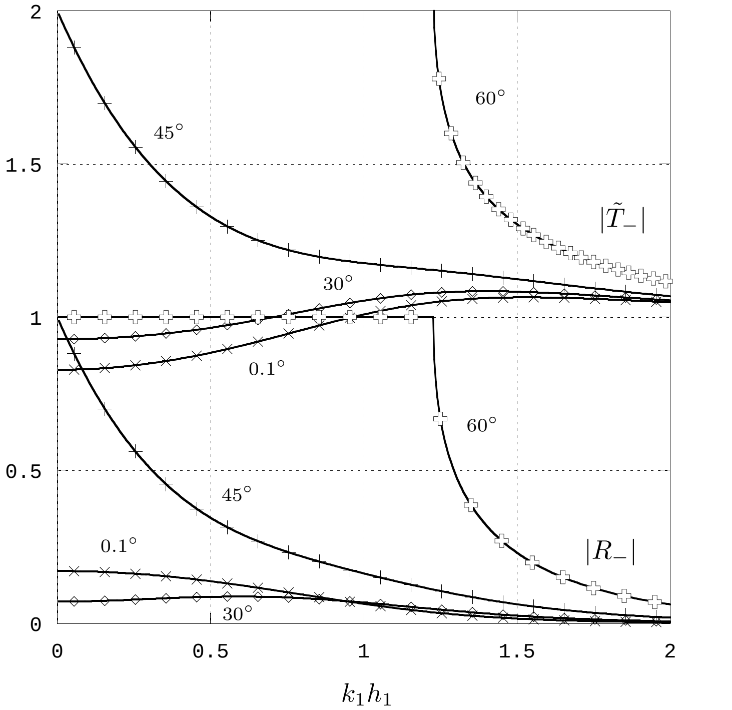
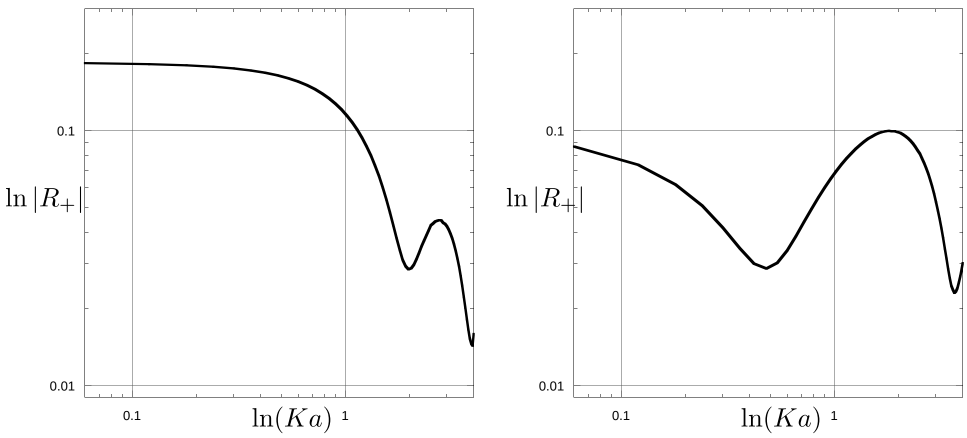


Figure A.7: Reflection and transmission coefficients for oblique waves incident from the shallower domain onto the step $\hat{h}(x) = 1 - 3x^2 + 2x^3$, $h_1/h_2 = \frac{1}{2}$, $a/b = 2$.



(a) $\theta = 30°$

(b) $\theta = 60°$

Figure A.8: Reflection coefficients for Booij's test problem for oblique incidence of $\theta = 30°$ and $\theta = 60°$.

N	$ka = 1$	$ka = 2$	$ka = 3$	$ka = 4$	$ka = 5$
40	0.091391	0.002509	0.024899	0.015723	0.006492
80	0.091468	0.002014	0.025130	0.015750	0.006482
120	0.091485	0.001889	0.025187	0.015756	0.006479
160	0.091491	0.001836	0.025211	0.015759	0.006478
200	0.091495	0.001808	0.025224	0.015760	0.006477
Exact	0.091503	0.001738	0.025256	0.015764	0.006476

Table A.1: Convergence of $|R| = |R_{\pm}|$ by the Boundary Element method for scattering, $\theta = 45°$, by a semi-circular ridge where $a/h_0 = 1$.

N	$ka = 1$	$ka = 2$	$ka = 3$	$ka = 4$	$ka = 5$
5	0.092749	0.004323	0.024976	0.016205	0.006680
10	0.091693	0.002239	0.025300	0.015925	0.006545
15	0.091620	0.002007	0.025249	0.015824	0.006503
20	0.091549	0.001867	0.025265	0.015804	0.006493
25	0.091535	0.001826	0.025258	0.015788	0.006486
30	0.091517	0.001790	0.025262	0.015783	0.006484
Exact	0.091503	0.001738	0.025256	0.015764	0.006476

Table A.2: Convergence of $|R| = |R_{\pm}|$ by the Rayleigh-Ritz method for oblique scattering, $\theta = 45°$, by a semi-circular ridge, $a/h_0 = 1$.