(Supplementary material) Acoustics of a partially partitioned narrow slit connected to a half-plane: case study for exponential quasi-bound states in the continuum and their resonant excitation

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1 Introduction

This manuscript describes the mathematical approach taken for the problem considered in the main paper by the authors when no *a priori* assumptions or approximations are made. In addition to considering fields antisymmetric about the geometric centreline of the problem, solutions that are symmetric will also be sought. This allows the full solution of the scattering of an obliquely-incident plane wave to be reconstructed from the solutions described herein.

In the various finite rectangular domains under consideration, solutions are expanded in separation series defined by boundary conditions imposed on opposing parallel surfaces. In the semi-infinite rectangular domain half range Fourier transforms are employed. This allows solutions to the symmetric and antisymmetric problems to be formulated, without approximation, in terms of integral equations for unknown functions. In the symmetric problem we derive a single scalar integral equation to solve for a single function representing the normal derivative of the field at the opening of the rectangular cavity. In the antisymmetric problem the solution is represented by a pair of integral equations involving an additional unknown function representing the normal derivative across a lateral boundary at the partition inside the cavity.

Numerically, solutions are sought by expanding these unknown functions in a series of prescribed functions which incorporate the anticipated singular behaviour in the unknown functions at end points of the intervals over which they are defined. This approximation procedure, characterised by Galerkin's method, reduces the integral equations to finite systems of equations for the expansion coefficients and it is typical of this widely-used approach (e.g. Porter & Evans (1995), Evans & Fernyhough (1995)) that numerical solutions converge rapidly and with high precision as the number of terms in the series increases. For example, in computations performed here, no more than 6 terms were found necessary.

The work described below was performed simultaneously with the analysis in the main paper but, over time, it became necessary to change the notation used in the main paper. We have decided to retain the original notation used for the preparation of this manuscript since it is aligned to the numerical code developed alongside this work[1] However this manuscript is self contained and a full description of the geometry and non-dimensionalisation used in the present document is described at the beginning of the next section. It is also relatively straightforward

[1] see `http://people.maths.bris.ac.uk/~marp/abstracts/dslit.html`

to translate between the set of dimensionless variables used here (left) and those used in the scattering section 4 of the main paper (right):

$$\ell \longleftrightarrow \frac{l}{1+l}, \qquad \varepsilon \longleftrightarrow \frac{\epsilon}{1+l}, \qquad k \longleftrightarrow \kappa(1+l), \qquad \theta_0 \longleftrightarrow \alpha. \tag{1}$$

(noting that k was also used in the main paper to represent the complex eigenvalue.) Additionally, the axes in the two accounts are rotated through 90 degrees and the angles here are measured using θ from the centreline of the channel as opposed to ϕ measured anticlockwise from the wall in the main paper.

In what follows we describe the problem of scattering of plane waves having real dimensionless wavenumber k. The determination of eigenvalues of the unforced problem described in the main paper is acheived by allowing our wavenumber, k, to become complex and seeking non-trivial solutions of the corresponding homogeneous problem. For this, a Newton-Rhapson iteration for complex numbers is used. We must take that the correct branch of the square root function is selected by the numerical scheme when applying it to complex k in order to satisfy the required outgoing condition in the far field. The determination of the coefficient of the resonant dipole strength in the scattering problem involves using the real part of the complex eigenvalue, k, as the input to the scattering problem.

2 Description of the problem

We work in the (x, y) plane with two-dimensional plane acoustic waves incident from $x = \infty$, making an angle θ_0 with respect to the positive x-axis. These are reflected by acoustically-hard walls along $x = 0$ for $|y| > a$ and scattered as circular waves by a rectangular cavity embedded in the wall in $-d < x < 0$, $|y| < a$. The cavity is divided into two half-width cavities along its centreline $y = 0$, from $-d < y < -b$ where $b \geq 0$.

The spatial coordinates are non-dimensionalised by d, the depth of the cavity so that, in dimensionless variables, the walls extend along $x = 0$, $|y| > \varepsilon$ where $\varepsilon = a/d$ and the partition along $y = 0$ extends from $-1 < x < -\ell$ where $\ell = b/d \leq 1$.

The governing equation in the acoustic medium with phase speed c is

$$(\nabla^2 + k^2)\phi(x, y) = 0 \tag{2}$$

where $k = \omega d/c$ is the dimensionless wavenumber. Neumann conditions are placed on ϕ upon all walls.

We can write

$$\phi(x, y) = \phi_{i,r}(x, y) + \phi_s(x, y) \tag{3}$$

where

$$\phi_{i,r}(x, y) = 2\cos(\alpha_0 x)\mathrm{e}^{-\mathrm{i}\beta_0 y} \tag{4}$$

and $\alpha_0 = k\cos\theta_0$, $\beta_0 = k\sin\theta_0$ represents the incident wave and a plane wave reflected by a wall without a cavity; $\phi_s(x, y)$ therefore takes account of the scattering by the cavity.

Advantage can be taken of the symmetry in the geometry about $y = 0$ to write

$$\phi(x, y) = \phi^s(x, y) + \phi^a(x, y) \tag{5}$$

where

$$\phi^s(x, y) = \phi^s(x, -y), \qquad \phi^a(x, y) = -\phi^a(x, -y). \tag{6}$$

Consequently

$$\frac{\partial \phi^s(x,0)}{\partial y} = \phi^a(x,0) = 0, \qquad -\ell < x, \tag{7}$$

in addition to $\partial \phi^{s,a}(x,0)/\partial y = 0$ for $-1 < x < -\ell$. Then

$$\phi^s_{i,r}(x,y) = 2\cos\alpha_0 x \cos\beta_0 y, \qquad \phi^a_{i,r}(x,y) = -2\mathrm{i}\cos\alpha_0 x \sin\beta_0 y. \tag{8}$$

3 The symmetric problem

The domain over which (2) holds is divided into $x > 0$, $y > 0$ and $-1 < x < 0$, $0 < y < \varepsilon$ in which general solutions of (2) satisfying the boundary conditions will be formulated and then matched across their common interface $x = 0$, $0 < y < \varepsilon$.

Thus, in $-1 < x < 0$, $0 < y < \varepsilon$ we express the general solution as

$$\phi^s(x,y) = b^s_0 \cos k(x+1) + \sum_{n=1}^{\infty} b^s_n \cos p_n y \cosh \alpha_n (x+1) \tag{9}$$

where $p_n = n\pi/\varepsilon$, $\alpha_n = \sqrt{p_n^2 - k^2} \equiv -\mathrm{i}\sqrt{k^2 - p_n^2}$. Note that if $k\varepsilon < \pi$ only the $n = 0$ mode is propagating in the channel.

In $x > 0$, $y < 0$ we let

$$\Phi^s(x,\beta) = \int_0^\infty (\phi^s(x,y) - \phi^s_{i,r}(x,y)) \cos\beta y \, dy \tag{10}$$

be the Fourier cosine transform of the scattered wave component of the solution. It follows that (2) is transformed to

$$\left(\frac{d^2}{dx^2} + \gamma^2\right) \Phi^s(x,\beta) = 0 \tag{11}$$

where $\gamma = \sqrt{\beta^2 - k^2} \equiv -\mathrm{i}\sqrt{k^2 - \beta^2}$ (if $k > |\beta|$ are both real) where the sign is fixed by the radiation condition. Thus

$$\Phi^s(x,\beta) = C(\beta)\mathrm{e}^{-\gamma x}. \tag{12}$$

Now we let $\phi^s_x(0,y) = U^s(y)$ for $0 < y < \varepsilon$ and since $\phi^s_x(0,y) = 0$ for $y > \varepsilon$ and $\phi^s_{i,r}$ satisfies a Neumann condition over all y at $x = 0$, we have

$$\left.\frac{\partial \Phi^s}{\partial x}\right|_{x=0} = -\gamma C(\beta) = \int_0^\varepsilon U^s(y') \cos\beta y' \, dy'. \tag{13}$$

From the inverse transform it follows that

$$\phi^s(x,y) = \phi^s_{i,r}(x,y) - \frac{2}{\pi}\int_0^\infty \frac{\mathrm{e}^{-\gamma x}\cos\beta y}{\gamma} \int_0^\varepsilon U^s(y')\cos\beta y' \, dy' \, d\beta. \tag{14}$$

Returning to (9), the functions $\cos p_n y$ are orthogonal over $0 < y < \varepsilon$ via

$$\frac{1}{\varepsilon}\int_0^\varepsilon \cos p_n y \cos p_m y \, dy = \begin{cases} 1, & m = n = 0, \\ \frac{1}{2}, & m = n \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

It follows from taking an x-derivative of (9) and using (15) that

$$-kb_0^s \sin k = \frac{1}{\varepsilon}\int_0^\varepsilon U^s(y)\,dy, \qquad b_n^s \alpha_n \sinh\alpha_n = \frac{2}{\varepsilon}\int_0^\varepsilon U^s(y)\cos p_n y\,dy, \quad (n \geq 1) \tag{16}$$

and so

$$\phi^s(x,y) = -\frac{\cos k(x+1)}{k\varepsilon\sin k}\int_0^\varepsilon U^s(y)\,dy + \sum_{n=1}^{\infty}\frac{2\cosh\alpha_n(x+1)\cos p_n y}{\alpha_n\varepsilon\sinh\alpha_n}\int_0^\varepsilon U^s(y')\cos p_n y'\,dy'. \tag{17}$$

An integral equation for the unknown function $U^s(y)$ is now formulated by matching (17) with (14) over $x = 0$, $0 < y < \varepsilon$:

$$\frac{2}{\pi}\int_0^\infty \frac{\cos\beta y}{\gamma}\int_0^\varepsilon U^s(y')\cos\beta y'\,dy'\,d\beta - \frac{\cot k}{k\varepsilon}\int_0^\varepsilon U^s(y)\,dy \\ + \sum_{n=1}^{\infty}\frac{2\cos p_n y}{\alpha_n\varepsilon\tanh\alpha_n}\int_0^\varepsilon U^s(y')\cos p_n y'\,dy' = 2\cos\beta_0 y \tag{18}$$

after using (8).

3.1 Numerical approximation

We approximate the solution to the integral equation using a Galerkin method in which the unknown function $U^s(y)$ is approximated by an expansion in a finite set of basis functions

$$U^s(y) \approx \frac{1}{\varepsilon}\sum_{p=0}^{P} a_p^s u_p^s(y/\varepsilon) \tag{19}$$

where we have chosen

$$u_p^s(t) = \frac{(-1)^p(2p)!\Gamma(\frac{1}{6})2^{1/6}}{\pi\Gamma(2p+\frac{1}{6})(1-t^2)^{1/3}}C_{2p}^{(1/6)}(t), \qquad p = 0, 1, \ldots \tag{20}$$

and $C_n^{(\nu)}(\cdot)$ are Gegenbauer polynomials and $\Gamma(\cdot)$ is the Gamma function. The choice (20) incorporates the known inverse cubic-root behaviour in the gradient of the field variable at the sharp corner of the opening of the rectangular cavity and even polynomials are used to reflect the Neumann condition on $y = 0$. The remaining multiplicative factors are used to simplify subsequent algebra.

In particular, it is known (see Evans & Fernyhough (1995) for example) that

$$\frac{1}{\varepsilon}\int_0^\varepsilon u_p^s(y/\varepsilon)\cos p_n y\,dy = \begin{cases} \dfrac{J_{2p+1/6}(n\pi)}{(n\pi)^{1/6}} & n \geq 1, \\ 6/(2^{1/6}\Gamma(\frac{1}{6}))\delta_{p0}, & n = 0. \end{cases} \tag{21}$$

Similarly,

$$\frac{1}{\varepsilon}\int_0^\varepsilon u_p^s(y/\varepsilon)\cos\beta y\,dy = \begin{cases} \dfrac{J_{2p+1/6}(\beta\varepsilon)}{(\beta\varepsilon)^{1/6}}, & \beta \neq 0, \\ 6/(2^{1/6}\Gamma(\frac{1}{6}))\delta_{p0}, & \beta_0 = 0. \end{cases} \tag{22}$$

Using (19) in (18) and multiplying through by $u^s_q(y/\varepsilon)/\varepsilon$, for $q = 0, 1, \dots, P$ and integrating over $0 < y < \varepsilon$ characterises the Galerkin method and results in the system of algebraic equations

$$\sum_{p=0}^{P} a^s_p K^s_{p,q} = F^s_q, \qquad q = 0, 1, \dots P \tag{23}$$

where

$$K^s_{p,q} = \frac{2}{\pi}\int_0^\infty \frac{J_{2p+1/6}(\beta\varepsilon)J_{2q+1/6}(\beta\varepsilon)}{\gamma(\beta\varepsilon)^{1/3}}\,d\beta - \left(\frac{6}{2^{1/6}\Gamma(\frac{1}{6})}\right)^2 \frac{\cot k}{k\varepsilon}\delta_{p0}\delta_{q0} + \sum_{n=1}^{\infty} \frac{2\coth\alpha_n}{\alpha_n\varepsilon}\frac{J_{2p+1/6}(n\pi)J_{2q+1/6}(n\pi)}{(n\pi)^{1/3}} \tag{24}$$

and

$$F^s_q = \begin{cases} 2\dfrac{J_{2q+1/6}(\beta_0\varepsilon)}{(\beta_0\varepsilon)^{1/6}}, & \beta_0 \neq 0, \\ 12/(2^{1/6}\Gamma(\frac{1}{6}))\delta_{p0}, & \beta_0 = 0. \end{cases} \tag{25}$$

For numerical purposes we write the integral in (24) as the sum of three separate integrals, defined by changes of integration variables, as

$$\frac{2\mathrm{i}}{\pi}\int_0^{\pi/2} \frac{J_{2p+1/6}(k\varepsilon\sin u)J_{2q+1/6}(k\varepsilon\sin u)}{(k\varepsilon\sin u)^{1/3}}\,du + \frac{2}{\pi}\int_0^{\cosh^{-1}(2)} \frac{J_{2p+1/6}(k\varepsilon\cosh v)J_{2q+1/6}(k\varepsilon\cosh v)}{(k\varepsilon\cosh v)^{1/3}}\,dv + \frac{2}{\pi}\int_{2k\varepsilon}^\infty \frac{J_{2p+1/6}(t)J_{2q+1/6}(t)}{t^{1/3}\sqrt{t^2-(k\varepsilon)^2}}\,dt. \tag{26}$$

The remaining infinite integral can be treated by removing the large-t asymptotics from the integrand and adding back on a term that can be evaluated explicitly using special functions.

3.2 Properties of the solution

One of the key outputs from the problem is the strength of the scattered waves from the mouth of the cavity. Starting with (14), written as

$$\phi^s(x,y) - \phi^s_{i,r}(x,y) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{e}^{-\gamma x}}{\gamma}(\mathrm{e}^{\mathrm{i}\beta y} + \mathrm{e}^{-\mathrm{i}\beta y})\int_0^\varepsilon U^s(y')\cos\beta y'\,dy'\,d\beta \tag{27}$$

and we consider $kr = k\sqrt{x^2+y^2} \to \infty$ by letting $x = r\cos\theta$, $y = r\sin\theta$ $(0 < \theta < \pi/2)$ and by letting $\beta = k\sin u$ for $0 < \beta < k$ – the range of values of β which give rise to propagating, as opposed to evanescent, waves – gives us

$$\phi^s(x,y) - \phi^s_{i,r}(x,y) \sim \frac{\mathrm{i}}{\pi}\int_0^{\pi/2} \left(\mathrm{e}^{\mathrm{i}kr\cos(\theta-u)} + \mathrm{e}^{\mathrm{i}kr\cos(\theta+u)}\right)\int_0^\varepsilon U^s(y')\cos(ky'\sin u)\,dy'\,du. \tag{28}$$

Since this is being assessed in the limit $kr \to \infty$ we pick out the dominant behaviour using the method of stationary phase so that

$$\phi^s(x,y) - \phi^s_{i,r}(x,y) \sim \mathrm{e}^{\mathrm{i}kr-\mathrm{i}\pi/4}\left(\frac{2}{\pi kr}\right)^{1/2} A^s(\theta) \tag{29}$$

where the diffraction coefficient $A^s(\theta)$ is defined to be

$$A^s(\theta) = \mathrm{i} \int_0^\varepsilon U^s(y') \cos(ky' \sin\theta)\, dy'. \tag{30}$$

Under the approximation (19) and using the result (22) we find

$$A^s(\theta) \approx \mathrm{i} \sum_{p=0}^{P} a_p^s \frac{J_{2p+1/6}(k\varepsilon \sin\theta)}{(k\varepsilon\sin\theta)^{1/6}} \tag{31}$$

with a suitable adjustment for $\theta = 0$. To determine the strength of the monopole contribution to $A^s(\theta)$ and can expand the Bessel function in (31) in its power series and then find that

$$A^s(\theta) = \sum_{n=0}^{\infty} (-1)^n A_n^s \cos 2n\theta. \tag{32}$$

where the first coefficient in (32) representing the monopole strength is given, after significant algebra, by

$$A_0^s = \frac{\mathrm{i}}{2^{1/6}} \sum_{p=0}^{P} a_p^s \sum_{m=0}^{\infty} \frac{(-1)^m (k\varepsilon/4)^{2m+2p}(2m+2p)!}{m! 4^m [(m+p)!]^2 \Gamma(m+2p+\frac{7}{6})} \tag{33}$$

and which requires use of the result

$$\int_{-\pi/2}^{\pi/2} (\sin\theta)^{2n}\, d\theta = \frac{\pi (2n)!}{2^{2n}(n!)^2}. \tag{34}$$

For $k\varepsilon \ll 1$ we have

$$A_0^s \approx \frac{6\mathrm{i}}{2^{1/6}\Gamma(\frac{1}{6})} a_0^s + O((k\varepsilon)^2). \tag{35}$$

On account of the form of the far field assumed in (29), the coefficient in (33) matches the coefficient of multiplying the first-kind Hankel function $H_0^{(1)}(kr)$ in a Fourier-Bessel series solution of the far field.

Apart from the monopole strength, we will be interested in the amplitude of the response in the cavity and, provided $k\varepsilon < \pi$ (as we anticipate) this is characterised by b_0^s, which from (16) can be computed with

$$b_0^s = -\frac{\pi}{k\varepsilon \sin k} \frac{6}{2^{1/6}\Gamma(\frac{1}{6})} a_0^s. \tag{36}$$

3.3 Remark

A separate asymptotic analysis conducted for a narrow channel shows that, as $\varepsilon \to 0$ and when resonant conditions hold,

$$\phi^s(x,y) - \phi_{i,r}^s(x,y) \sim -2H_0^{(1)}(kr); \tag{37}$$

this implies that we expect $A_0^s \approx -2$ at symmetric resonance. This information is useful in validating numerical results.

4 The antisymmetric problem

There is some overlap with the method of solution for the symmetric problem, but this component of the solution is complicated by the boundary condition switching from Dirichlet to Neumann at $x = -\ell$ part of the way along the cavity.

Following the symmetric problem we have, in $-1 < x < -\ell$,

$$\phi^a(x, y) = -\frac{\cos k(x+1)}{k\varepsilon \sin k(1-\ell)} \int_0^\varepsilon V^a(y)\, dy + \sum_{n=1}^{\infty} \frac{2\cosh \alpha_n(x+1) \cos p_n y}{\alpha_n \varepsilon \sinh \alpha_n(1-\ell)} \int_0^\varepsilon V^a(y') \cos p_n y'\, dy' \tag{38}$$

where we have introduced

$$V^a(y) = \phi^a_x(-\ell, y), \qquad 0 < y < \varepsilon \tag{39}$$

as the unknown horizontal velocity across $x = -\ell$, distinct from $U^a(y)$ which will be used to represent $\phi^a_x(0, y)$ over $0 < y < \varepsilon$.

In $x > 0$ we can also follow the approach in the symmetric problem by taking Fourier sine transforms in y rather than cosine transforms. This leads directly to

$$\phi^a(x, y) = \phi^a_{i,r}(x, y) - \frac{2}{\pi} \int_0^\infty \frac{\mathrm{e}^{-\gamma x} \sin \beta y}{\gamma} \int_0^\varepsilon U^a(y') \sin \beta y'\, dy'\, d\beta \tag{40}$$

as the analogue of (10).

The final part of the solution relates to the domain $-\ell < x < 0$, $0 < y < \varepsilon$ which connects the solutions (38) and (40) at $x = -1$ and $x = 0$. On $y = 0$, $\phi^a(x, 0) = 0$, and so the general solution is written

$$\phi^a(x, y) = \sum_{n=1}^{\infty} (c^a_n \mathrm{e}^{-\kappa_n(x+\ell)} + d^a_n \mathrm{e}^{\kappa_n x}) \sin q_n y \tag{41}$$

where $q_n = (n - \frac{1}{2})\pi/\varepsilon$ and $\kappa_n = \sqrt{q_n^2 - k^2} \equiv -\mathrm{i}\sqrt{k^2 - q_n^2}$ if $k > q_n$. The values of κ_n are all real provided $k\varepsilon < \pi/2$ implying no propagating modes in $-\ell < x < 0$ for low enough frequencies. The eigenfunctions $\sin q_n y$ are orthogonal across $0 < y < \varepsilon$:

$$\frac{2}{\varepsilon} \int_0^\varepsilon \sin q_n y \sin q_m y\, dy = \delta_{m,n}, \qquad m, n = 1, 2, \ldots \tag{42}$$

and it follows from (39) that

$$d^a_n - c^a_n \mathrm{e}^{-\kappa_n \ell} = \frac{2}{\kappa_n \varepsilon} \int_0^\varepsilon U^a(y) \sin q_n y\, dy \tag{43}$$

and

$$d^a_n \mathrm{e}^{-\kappa_n \ell} - c^a_n = \frac{2}{\kappa_n \varepsilon} \int_0^\varepsilon V^a(y) \sin q_n y\, dy. \tag{44}$$

We can find c^a_n and d^a_n by eliminating between these two equations and subsequently it can be shown that

$$\phi^a(0, y) = \sum_{n=1}^{\infty} \frac{2 \sin q_n y}{\kappa_n \varepsilon} \left\{ \coth \kappa_n \ell \int_0^\varepsilon U^a(y') \sin q_n y'\, dy' - \operatorname{cosech} \kappa_n \ell \int_0^\varepsilon V^a(y') \sin q_n y'\, dy' \right\} \tag{45}$$

and

$$\phi^a(-\ell,y)=\sum_{n=1}^{\infty}\frac{2\sin q_n y}{\kappa_n\varepsilon}\left\{\operatorname{cosech}\kappa_n\ell\int_0^{\varepsilon}U^a(y')\sin q_n y'\,dy'-\coth\kappa_n\ell\int_0^{\varepsilon}V^a(y')\sin q_n y'\,dy'\right\}. \tag{46}$$

We can now match (45) with (40) on $x=0$ to get

$$\begin{aligned}&\frac{2}{\pi}\int_0^{\infty}\frac{\sin\beta y}{\gamma}\int_0^{\varepsilon}U^a(y')\sin\beta y'\,dy'\,d\beta\\&+\sum_{n=1}^{\infty}\frac{2\sin q_n y}{\kappa_n\varepsilon}\left\{\coth\kappa_n\ell\int_0^{\varepsilon}U^a(y')\sin q_n y'\,dy'-\operatorname{cosech}\kappa_n\ell\int_0^{\varepsilon}V^a(y')\sin q_n y'\,dy'\right\}=-2\mathrm{i}\sin\beta_0 y\end{aligned} \tag{47}$$

and with (38) on $x=-\ell$ to get

$$\begin{aligned}&\frac{\cot k(1-\ell)}{k\varepsilon}\int_0^{\varepsilon}V^a(y')\cos p_n y'\,dy'-\sum_{n=1}^{\infty}\frac{2\coth\alpha_n(1-\ell)\cos p_n y}{\alpha_n\varepsilon}\int_0^{\varepsilon}V^a(y')\cos p_n y'\,dy'\\&+\sum_{n=1}^{\infty}\frac{2\sin q_n y}{\kappa_n\varepsilon}\left\{\operatorname{cosech}\kappa_n\ell\int_0^{\varepsilon}U^a(y')\sin q_n y'\,dy'-\coth\kappa_n\ell\int_0^{\varepsilon}V^a(y')\sin q_n y'\,dy'\right\}=0\end{aligned} \tag{48}$$

both for $0<y<\varepsilon$.

4.1 Numerical approximation

We have a coupled pair of integral equations to solve for the two functions $V^a(y)$ and $U^a(y)$. Whilst this is more algebraically demanding, the method used in the simpler symmetric problem is unchanged. That is we expand each unknown in a finite basis and apply the Galerkin method to determine a (coupled) system of algebraic equations for the expansion coefficients. Thus, we write

$$U^a(y)\approx\frac{1}{\varepsilon}\sum_{p=0}^{P}a^a_{2p+1}u^a_{2p+1}(y/\varepsilon),\qquad V^a(y)\approx\frac{1}{\varepsilon}\sum_{p=0}^{P}a^a_{2p}u^a_{2p}(y/\varepsilon), \tag{49}$$

where

$$u^a_{2p+1}(t)=\frac{(-1)^p(2p+1)!\Gamma(\frac{1}{6})2^{1/6}}{\pi\Gamma(2p+\frac{4}{3})(1-t^2)^{1/3}}C^{(1/6)}_{2p+1}(t) \tag{50}$$

serves the same purpose of incorporating the anticipated inverse cube-root behaviour of the solution at the corner of the mouth, whilst now being zero on $t=0$ to satisfy the Dirichlet condition on the centreline of the cavity. Also,

$$u^a_{2p}(t)=\frac{2(-1)^p}{\pi\sqrt{t(2-t)}}T_{2p}(1-t) \tag{51}$$

where $T_{2p}(\cdot)$ is a Chebychev polynomial and the order being even ensures the Neumann condition on the wall $y=\varepsilon$ of the cavity is satisfied. Additionally, the basis functions incorportate the anticipated inverse *square-root* singularity in the solution at the end of the dividing barrer on the channel centreline.

Again, using Evans & Fernyhough (1995) (for example), we have that that

$$\frac{1}{\varepsilon}\int_0^\varepsilon u^a_{2p+1}(y/\varepsilon)\sin q_n y\,dy = \frac{J_{2p+7/6}((n-\frac{1}{2})\pi)}{((n-\frac{1}{2})\pi)^{1/6}} \tag{52}$$

for $n \geq 1$. Similarly,

$$\frac{1}{\varepsilon}\int_0^\varepsilon u^a_{2p+1}(y/\varepsilon)\sin \beta y\,dy = \frac{J_{2p+7/6}(\beta\varepsilon)}{(\beta\varepsilon)^{1/6}} \tag{53}$$

which tends to 0 as $\beta \to 0$.

Also, we find, after the substitution $v = 1 - t$ and use of Erdélyi *et al.* (1954, 10.1.2), that

$$\frac{1}{\varepsilon}\int_0^\varepsilon u^a_{2p}(y/\varepsilon)\sin q_n y\,dy = (-1)^{n-1}J_{2p}((n-\tfrac{1}{2})\pi) \tag{54}$$

for $n = 1, 2, \ldots$ and

$$\frac{1}{\varepsilon}\int_0^\varepsilon u^a_{2p}(y/\varepsilon)\cos p_n y\,dy = \begin{cases} (-1)^n J_{2p}(n\pi), & n \neq 0, \\ \delta_{p0}, & n = 0. \end{cases} \tag{55}$$

The result of applying the Galerkin approximation to the coupled integral equations is the coupled system of equations given by

$$\sum_{p=0}^{P} a^a_{2p+1}K^a_{2p+1,2q+1} + a^a_{2p}K^a_{2p,2q+1} = F^a_{2q+1} \tag{56}$$

and

$$\sum_{p=0}^{P} a^a_{2p+1}K^a_{2p+1,2q} + a^a_{2p}K^a_{2p,2q} = 0 \tag{57}$$

for $q = 0, \ldots, P$, where

$$K^a_{2p+1,2q+1} = \frac{2}{\pi}\int_0^\infty \frac{J_{2p+7/6}(\beta\varepsilon)J_{2q+7/6}(\beta\varepsilon)}{\gamma(\beta\varepsilon)^{1/3}}\,d\beta + \sum_{n=1}^{\infty}\frac{2\coth\kappa_n\ell}{\kappa_n\varepsilon}\frac{J_{2p+7/6}((n-\frac{1}{2})\pi)J_{2q+7/6}((n-\frac{1}{2})\pi)}{((n-\frac{1}{2})\pi)^{1/3}} \tag{58}$$

and

$$K^a_{2p,2q+1} = \sum_{n=1}^{\infty}\frac{2(-1)^n\operatorname{cosech}\kappa_n\ell}{\kappa_n\varepsilon}\frac{J_{2p}((n-\frac{1}{2})\pi)J_{2q+7/6}((n-\frac{1}{2})\pi)}{((n-\frac{1}{2})\pi)^{1/6}} \tag{59}$$

with

$$K^a_{2p+1,2q} = \sum_{n=1}^{\infty}\frac{2(-1)^n\operatorname{cosech}\kappa_n\ell}{\kappa_n\varepsilon}\frac{J_{2p+7/6}((n-\frac{1}{2})\pi)J_{2q}((n-\frac{1}{2})\pi)}{((n-\frac{1}{2})\pi)^{1/6}} \tag{60}$$

$$K^a_{2p,2q} = -\sum_{n=1}^{\infty}\frac{2\coth\kappa_n\ell}{\kappa_n\varepsilon}J_{2p}((n-\tfrac{1}{2})\pi)J_{2q}((n-\tfrac{1}{2})\pi) + \frac{\cot k(1-\ell)}{k\varepsilon}\delta_{p0}\delta_{q0} - \sum_{n=1}^{\infty}\frac{2\coth\alpha_n(1-\ell)}{\alpha_n\varepsilon}J_{2p}(n\pi)J_{2q}(n\pi) \tag{61}$$

and

$$F^a_{2q+1} = -2\mathrm{i}J_{2q+7/6}(\beta_0\varepsilon)/(\beta_0\varepsilon)^{1/6} \tag{62}$$

provided $\beta_0 \neq 0$ with $F^a_{2q+1} = 0$ in the case that $\beta_0 = 0$. Obviously in this case there is no forcing in the system of equations and the solution is zero everywhere. This just reaffirms that normal incidence generates no antisymmetry in the response.

4.2 Properties of the solution

We can follow the analysis of the symmetric far field diffraction coefficient and quickly get to

$$A^a(\theta) = -\int_0^{\varepsilon} U^a(y')\sin(ky'\sin\theta)\,dy' \tag{63}$$

from which (40) gives us

$$A^a(\theta) = -\sum_{p=0}^{P} a^a_{2p+1}\frac{J_{2p+7/6}(k\varepsilon\sin\theta)}{(k\varepsilon\sin\theta)^{1/6}}. \tag{64}$$

As in the symmetric case, we can use the power series expansion of the Bessel function to express (64) as

$$A^a(\theta) = \sum_{n=0}^{\infty}(-\mathrm{i})^{2n+1}A^a_n\sin(2n+1)\theta \tag{65}$$

and find, using (34), that the dipole coefficient in this expansion is

$$A^a_0 = \mathrm{i}\frac{k\epsilon}{2^{13/6}}\sum_{p=0}^{P} a^a_{2p+1}\sum_{m=0}^{\infty}\frac{(-1)^m(k\varepsilon/4)^{2m+2p}(2m+2p+2)!}{m!4^m[(m+p+1)!]^2\Gamma(m+2p+\frac{13}{6})}. \tag{66}$$

Assuming $k\varepsilon \ll 1$ we have

$$A^a_0 \sim \mathrm{i}\frac{k\varepsilon}{2^{7/6}\Gamma(\frac{13}{6})}a^a_1 + O((k\varepsilon)^3). \tag{67}$$

That is, A^a_0 is the coefficient associated with the term $H^{(1)}_1(kr)\sin\theta$ in the Fourier-Bessel expansion of the field in $x > 0$.

The amplitude of the dominant mode in the channel $-1 < x < -\ell$ is

$$-\frac{a^a_0}{k\varepsilon\sin k(1-\ell)}. \tag{68}$$

4.3 Remark

According to result quoted in the main paper, the scattered field at resonance in the antisymmetric problem for small openings is approximately

$$-4\mathrm{i}\sin\theta_0 H^{(1)}_1(kr)\sin\theta \tag{69}$$

implying a resonant dipole strength of $-4\mathrm{i}\sin\theta_0$ which can be used for numerical comparison with the result (66) derived above as $k\varepsilon \to 0$.

5 Trapped modes in an infinite waveguide with a dividing plate

There are several different methods one can used to determine the frequencies at which trapped modes occur. Evans, Linton & Ursell (1993) used a modified residue calculus approach, based on eigenfunction expansions to determine a wide spacing approximation

$$k(1-\ell) = (n+1/2)\pi - \frac{k\varepsilon}{\pi}\ln 4 + \chi \tag{70}$$

where

$$\chi = \sum_{n=1}^{\infty}\big(2\sin^{-1}(k\varepsilon/n\pi) - \sin^{-1}(2k\varepsilon/n\pi)\big) \tag{71}$$

which is $O((k\varepsilon)^3)$.

Numerically, we can use the method described in §4 but assuming just two regions in the waveguide and, in the expansion (41), letting $d_n^a = 0$. This way, we determine trapped modes as the vanishing of the real determinant of the matrix with elements

$$\begin{aligned} K_{p,q} = -\sum_{n=1}^{\infty}\frac{2}{\kappa_n\varepsilon}J_{2p}((n-\tfrac{1}{2})\pi)J_{2q}((n-\tfrac{1}{2})\pi) + \frac{\cot k(1-\ell)}{k\varepsilon}\delta_{p0}\delta_{q0} \\ -\sum_{n=1}^{\infty}\frac{2\coth\alpha_n(1-\ell)}{\alpha_n\varepsilon}J_{2p}(n\pi)J_{2q}(n\pi) \end{aligned} \tag{72}$$

which is just $K^a_{2p,2q}$ defined by (61) with the $\coth\kappa_n\ell$ term set to unity. There are other methods that can be used, including taking Fourier transforms in the direction of the guide.

References

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