On the existence of embedded surface waves along arrays of parallel plates

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Abstract

The method of residue calculus is used to investigate the existence of embedded surface waves along an infinite periodic comb-like grating of thin plates. Solutions are shown to correspond to two conditions being satisfied. For sufficiently long plates the existence of solutions is proved, whilst results derived from simple formulae based on a long plate approximation are shown to be in excellent agreement with results to the full system of equations.

1 Introduction

In a recent paper by Porter & Evans (2001) numerical examples of 'embedded surface wave modes' were produced in the case of an infinite periodic array of rectangular blocks. Such waves are referred to as Rayleigh-Bloch waves and are characterised by a propagation constant β along the array which characterises the change in phase of the wave as it shifts along one period of the array. Rayleigh-Bloch surface waves are described by solution of the two-dimensional Helmholtz equation for a complex-valued field potential which is localised in the vicinity of the periodic array and does not propagate energy away from the array. Such waves are relevant in the theory of water waves where the two-dimensional array represents the cross-section of uniform vertical cylinders in water of constant depth or in electromagnetic, optical and linearised acoustic theories. When the wavenumber of motion, k, is less than β it is easy to argue, using separation of variables away from the array, that waves (if they exist) will remain localised. The value of $k = \beta$ is referred to the first cut-off and Rayleigh-Bloch surface waves have indeed been shown to exist for a variety of periodic arrays below the first cut-off – i.e. when $k < \beta$. In particular, for arrays of thin plates which is the subject of the present paper, surface modes existing below the first cut-off have been investigated by Hurd (1954) and more recently by Evans & Linton (1993).

In Porter & Evans (2001) strong numerical evidence was given for the existence of Rayleigh-Bloch waves above the first cut-off for arrays of rectangular obstacles – i.e. for values of k satisfying $k > \beta$. Such embedded surface wave modes are of considerable interest since their existence implies the non-uniqueness of the corresponding scattering problem of an incident wave having the appropriate values of k and β. An integral equation formulation of the problem was made and solved using a powerful Galerkin technique involving a series of test functions which were chosen to provide accurate modelling of the expected inverse cube-root singularity at the right-angled corners of the geometry. As a limiting case, Porter & Evans (2001) also produced results for arrays of thin plates (i.e. rectangular obstacles of zero width) where a stronger inverse square-root singularity occurs at the edge of the plate. Nevertheless, convergence of their numerical results was still good and accurate results were claimed.

A powerful alternative approach can be adopted in the particular case of arrays of thin plates. The method of residue calculus illustrated in, for example, Mittra & Lee (1971) has advantages which will be demonstrated in this paper. First, simple approximate formulae can be derived on the assumption that the plate length is large compared to the periodicity of the array, which can be used to provide good approximations to solutions. The residue calculus method can also be used as the basis of a proof of the existence of embedded modes. Previously, the method has been used with success for problems involving thin plates by Evans (1992), Linton & Evans (1993) and McIver *et al* (2001) amongst others.

The paper is arranged as follows. In the following section, the equations governing the motion of Rayleigh-Bloch waves are stated, and the solution is described using the method of residue calculus. In the latter part of section 2 a proof of existence for embedded surface waves is given. Some numerical results are given in section 3 and conclusions are drawn in section 4. Some of the detailed working from the residue calculus method required for the existence proof is relegated to the Appendix. The notation used in this paper has been adopted where appropriate from the closely related work of Porter & Evans (2001) to which the reader is directed for a more detailed description of embedded surface waves, placing it in context with other relevant work in the field.

2 Formulation and solution

The problem is described in terms of a two-dimensional complex-valued field potential $\phi(x, y)$ which satisfies

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0, \qquad \text{in the fluid domain,} \tag{2.1}$$

where k is the wavenumber for wave motion of angular frequency ω. In terms of water waves, k is defined by $\omega^2/g = k \tanh kh$ with g gravitational acceleration and h the fluid depth. In terms of electromagnetic, optical or acoustic wave theories, $k = \omega/c$ where c is the wave speed.

The geometric configuration consists of a periodic array of thin plates occupying $-a < x < 0$, $y = 2md$ where $m \in \mathbb{Z}$ placed perpendicular to a wall at $x = -a$, $-\infty < y < \infty$. The potential satisfies Neumann conditions on these surfaces so that

$$\phi_y = 0, \qquad -a < x < 0, \quad y = 2md, \quad m \in \mathbb{Z} \tag{2.2}$$

and

$$\phi_x = 0, \qquad x = -a, \quad -\infty < y < \infty. \tag{2.3}$$

For a Rayleigh-Bloch surface wave solution, the wave field away from the array vanishes at large distances, which is expressed as

$$\phi \to 0, \qquad \text{as } |x| \to \infty. \tag{2.4}$$

The periodicity of the infinite array implies a Bloch or Floquet-type relation for the solution given by

$$\phi(x, y + 2md) = \mathrm{e}^{2\mathrm{i}m\beta d}\phi(x, y), \qquad m \in \mathbb{Z} \tag{2.5}$$

for some real parameter β, sometimes referred to as the *propagation constant* along the array. On account of the relation in (2.5) we may consider the problem in a semi-infinite rectangular strip of width $2d$, $S_0 = \{-a < x < \infty, 0 \leq y < 2d\}$, by imposing coupled boundary conditions on the lines $y = 0$ and $y = 2d$ for $-a < x < \infty$, thus

$$\phi(x, 2d) = \mathrm{e}^{2\mathrm{i}\beta d}\phi(x, 0), \qquad \phi_y(x, 2d) = \mathrm{e}^{2\mathrm{i}\beta d}\phi_y(x, 0). \tag{2.6}$$

Once $\phi(x, y)$ is determined in S_0, (2.5) can be used to express the potential throughout the remainder of the domain.

Rayleigh-Bloch surface waves are therefore determined by values of k for prescribed values of β for which (2.1)–(2.4) and (2.6) are satisfied by some non-trivial function $\phi(x, y)$.

In fact we need only consider values of β in the interval $0 \le \beta \le \pi/2d$, since it can easily be verified that

$$k(\beta) = k(\pi/d - \beta), \qquad k(\beta) = k(\beta + j\pi/d), \quad j \in \mathbb{Z}.$$

The first cut-off for this problem is at $k = \beta$ and the purpose of the present paper is to determine solutions above the first cut-off and below the second cut-off which occurs at $k = \pi/d - \beta$. Specifically, we shall confine attention to values of k occurring in $\pi/2d < k < \pi/d - \beta$. The lower bound of $k = \pi/2d$ is introduced to ensure that two wave-like modes exist between any two adjacent plates in the array. This plays an important role in determining solutions as will be shown.

The governing equations are now in place. To summarise, we seek non-trivial functions $\phi(x, y)$ in the semi-infinite strip $-a \le x < \infty$, $0 \le y \le 2d$ satisfying (2.1)–(2.4) and (2.6) for a given β in the interval $0 \le \beta \le \pi/2d$ and a corresponding value of k in $\pi/2d < k < \pi/d - \beta$. The solution in two rectangular sub-domains of the strip S_0 is constructed easily using separation of variables. For this purpose we adopt the notation used by Porter & Evans (2001).

First, in the inner region, $-a \le x \le 0$, bounded by plates on $y = 0$ and $y = 2d$ the potential is represented by writing

$$\phi(x, y) = \sum_{n=0}^{\infty} A_n \cosh \alpha_n (x + a) \psi_n(y) \tag{2.7}$$

where A_n are coefficients to be determined,

$$\psi_n(y) = \epsilon_n^{\frac{1}{2}} \mathrm{i}^n \cos p_n y, \qquad p_n = \frac{n\pi}{2d}, \qquad \epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n \ge 1 \end{cases} \tag{2.8}$$

and

$$\begin{aligned} \alpha_n &= (p_n^2 - k^2)^{\frac{1}{2}}, \qquad n = 2, 3, \ldots \\ \alpha_n &= -\mathrm{i}k_n, \qquad k_n = (k^2 - p_n^2)^{\frac{1}{2}}, \qquad n = 0, 1, \end{aligned} \tag{2.9}$$

such that $k_0 \equiv k$. The first two terms in the summation in (2.7) represent standing wave-like modes between the plates.

In the region, $x \ge 0$, extending from the edge of the plates to infinity we write

$$\phi(x, y) = \sum_{n=-\infty}^{\infty}{}' B_n \mathrm{e}^{-\gamma_n x} \Psi_n(y) \tag{2.10}$$

where B_n are also coefficients to be determined, the prime refers to the omission of the $n = 0$ term in the summation and

$$\Psi_n(y) = \mathrm{e}^{\mathrm{i}\beta_n (y-c)}, \qquad \text{with } \beta_n = \beta + n\pi/d \tag{2.11}$$

such that (2.6) is satisfied whilst

$$\gamma_n = (\beta_n^2 - k^2)^{\frac{1}{2}}, \qquad n = \pm 1, \pm 2, \ldots \tag{2.12}$$

The exclusion of incoming and outgoing wave-likes terms in the expansion in (2.10) ensures that (2.4) is satisfied since all the terms in the summation are exponentially decreasing with increasing x. Unlike Porter & Evans (2001), it is not necessary here to introduce an additional condition to ensure the amplitudes of wave-like modes in $x > 0$ are zero. The residue calculus method is self-consistent in this regard and the solution that is derived automatically ensures (2.4) is satisfied.

Note also the orthogonality relations

$$\frac{1}{2d} \int_0^{2d} \Psi_n(y) \overline{\Psi_m(y)} \, \mathrm{d}y = \frac{1}{2d} \int_0^{2d} \psi_n(y) \overline{\psi_m(y)} \, \mathrm{d}y = \delta_{mn}, \tag{2.13}$$

where δ_{mn} is the Kronecker delta.

Matching expressions for ϕ and ϕ_x across $x = 0$, $0 \le y \le 2d$, multiplying through by $\overline{\psi_m(y)}$ and using orthogonality (2.13) gives the two equations

$$2dA_m \cosh \alpha_m a = \epsilon_m^{\frac{1}{2}} \mathrm{i}^{m-1} c_m \sum_{n=-\infty}^{\infty}{}' \frac{B_n \beta_n}{\gamma_n^2 - \alpha_m^2}, \qquad m = 0, 1, 2, \ldots \tag{2.14}$$

and

$$2dA_m \alpha_m \sinh \alpha_m a = -\epsilon_m^{\frac{1}{2}} \mathrm{i}^{m-1} c_m \sum_{n=-\infty}^{\infty}{}' \frac{B_n \gamma_n \beta_n}{\gamma_n^2 - \alpha_m^2}, \qquad m = 0, 1, 2, \ldots \tag{2.15}$$

for the unknown coefficients A_n and B_n, where we have used the result

$$\int_0^{2d} \cos p_m y \mathrm{e}^{\mathrm{i}\beta_n y} \, \mathrm{d}y = -\mathrm{i} \frac{\beta_n c_m}{\gamma_n^2 - \alpha_m^2}, \qquad c_m = (-1)^m \mathrm{e}^{2\mathrm{i}\beta d} - 1, \qquad m = 0, 1, 2, \ldots. \tag{2.16}$$

Note here that when $\beta = 0$, c_m is zero for m even. This in turn implies from (2.14) that $A_m = 0$ for m even, and this information can be reintroduced into the original expansion for $\phi(x, y)$ to modify the resulting equations in the case $\beta = 0$. The outcome is that, in what follows, it is to be understood that only the *odd* m occur when considering $\beta = 0$.

Eliminating A_m from (2.14) and (2.15) and rearranging gives

$$\sum_{n=-\infty}^{\infty}{}' B_n \beta_n \left(\frac{1}{\gamma_n - \alpha_m} + \frac{\mathrm{e}^{-2\alpha_m a}}{\gamma_n + \alpha_m} \right) = 0, \qquad m = 0, 1, 2, \ldots \tag{2.17}$$

which is precisely what is found in Evans & Linton (1993) apart from the missing $n = 0$ term in the summation since we are above the first cut-off in the present work.

We now closely follow Evans & Linton (1993). A good approximation to the solution can be derived for $a/d \gg 1$ by discarding the terms involving $\mathrm{e}^{-2\alpha_m a}$ for $m \ge 2$. Then we have

$$\sum_{n=-\infty}^{\infty}{}' \frac{B_n \beta_n}{\gamma_n - \alpha_m} = -C_0 \mathrm{e}^{2\mathrm{i}k_0 a} \delta_{m0} - C_1 \mathrm{e}^{2\mathrm{i}k_1 a} \delta_{m1} \tag{2.18}$$

where

$$C_m = \sum_{n=-\infty}^{\infty}{}' \frac{B_n \beta_n}{\gamma_n - \mathrm{i}k_m}, \qquad m = 0, 1. \tag{2.19}$$

Now consider a function $f(z)$ of complex argument which has simple poles at $z = \gamma_n$, $n = \pm 1, \pm 2, \ldots$ and simple zeros at $z = \alpha_n$ for $n = 2, 3, \ldots$, but not at $n = 0, 1$. We also require that $f(z) = O(|z|^{-\frac{1}{2}})$ as $|z| \to \infty$. Then consider the integrals defined by

$$I_m = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_N} \frac{f(z)}{z - \alpha_m} \, \mathrm{d}z, \qquad m = 0, 1, 2, \ldots$$

and

$$J_m = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_N} \frac{f(z)}{z + \alpha_m} \, \mathrm{d}z, \qquad m = 0, 1, 2, \ldots.$$

where $\mathcal{C}_N$ are circles of radius $R_N = (N - \frac{1}{4})\pi$. Then $I_m \to 0$ and $J_m \to 0$ as $N \to \infty$ so that

$$\sum_{n=-\infty}^{\infty}{}' \frac{R(f : \gamma_n)}{\gamma_n - \alpha_m} + f(-\mathrm{i}k_0)\delta_{m0} + f(-\mathrm{i}k_1)\delta_{m1} = 0, \qquad m = 0, 1, 2, \ldots \tag{2.20}$$

and

$$\sum_{n=-\infty}^{\infty}{}' \frac{R(f:\gamma_n)}{\gamma_n+\alpha_m} + f(-\alpha_m) = 0, \qquad m = 0,1,2,\ldots \tag{2.21}$$

where $R(f:w)$ denotes the residue of $f(z)$ at $z=w$. Given that such a function $f(z)$ can be constructed, comparison of (2.20) with (2.18) and (2.21) with (2.19) allows us to identify the unknown coefficients in terms of properties of the function $f(z)$. Therefore

$$B_n\beta_n = R(f:\gamma_n) \quad \text{and} \quad C_m \mathrm{e}^{2\mathrm{i}k_m a} = f(-\mathrm{i}k_m), \quad C_m = -f(\mathrm{i}k_m), \qquad m=0,1$$

so that

$$\mathrm{e}^{-2\mathrm{i}k_m a} = -\frac{f(\mathrm{i}k_m)}{f(-\mathrm{i}k_m)}, \qquad m=0,1 \tag{2.22}$$

and these two equations for $m=0,1$ provide the necessary conditions for the existence of embedded surface waves under the assumption $a/d \gg 1$. Note that in the special case $\beta=0$ only the condition (2.22) with $m=1$ need be taken.

It remains to determine the function $f(z)$. We write

$$f(z) = f_0(z)g(z) \tag{2.23}$$

with

$$g(z) = g_1(z)/g_2(z), \qquad \text{where} \quad \left\{ \begin{array}{l} g_1(z) = \displaystyle\prod_{n=2}^{\infty}(1-z/\alpha_n)\mathrm{e}^{2zd/n\pi} \\ g_2(z) = \displaystyle\prod_{n=1}^{\infty}(1-z/\gamma_n)(1-z/\gamma_{-n})\mathrm{e}^{2zd/n\pi} \end{array} \right. \tag{2.24}$$

and the exponential factors ensure convergence of the infinite products. Also

$$f_0(z) = \mathrm{e}^{(2zd/\pi)(1-\ln 2)} \tag{2.25}$$

is included to ensure the correct algebraic behaviour for $f(z)$ as $|z|\to\infty$, which can be verified using techniques similar to those in Evans (1992). Thus, from (2.22) we have, for $m=0,1$,

$$\mathrm{e}^{-2\mathrm{i}k_m a} = -\mathrm{e}^{2\mathrm{i}(\chi_m - (2k_m d/\pi)\ln 2)} \tag{2.26}$$

where

$$\chi_m(\beta,k) = \sum_{n=2}^{\infty}\left\{\tan^{-1}\left(\frac{k_m}{\gamma_n}\right) + \tan^{-1}\left(\frac{k_m}{\gamma_{-n}}\right) - \tan^{-1}\left(\frac{k_m}{\alpha_n}\right)\right\} + \tan^{-1}\left(\frac{k_m}{\gamma_1}\right) + \tan^{-1}\left(\frac{k_m}{\gamma_{-1}}\right) \tag{2.27}$$

whence

$$k_m a = (n_m + \tfrac{1}{2})\pi + \frac{2k_m d}{\pi}\ln 2 - \chi_m(\beta,k) \tag{2.28}$$

for $n_m \in \mathbb{Z}$, $m=0,1$.

Note once again, that when $\beta=0$, only $m=1$ applies since the equation with $m=0$ is derived from an earlier equation which becomes redundant for the value $\beta=0$.

The special case of $\beta=0$ has been treated separately by Evans & Linton (1994). Known as Dirichlet trapped modes or Parker γ-modes (see Parker & Stoneman (1989)), the expression for approximating the wavenumber k for a given plate length a given in Evans & Linton (1994) is in fact wrong and should be

$$k_1 a = (n_1 + \tfrac{1}{2})\pi - \sum_{n=1}^{\infty} \left\{ \tan^{-1}\left(\frac{k_1}{\alpha_{2n}}\right) - \tan^{-1}\left(\frac{k_1}{\alpha_{2n+1}}\right) \right\} \tag{2.29}$$

for $n_1 \in \mathbb{Z}$. It is possible to show that taking $\beta = 0$ in (2.28), (2.27) with $m = 1$ results in (2.29) as required although care must be taken over the convergence of the series involved.

In order to provide a rigorous proof of existence of embedded surface waves, we need a complete representation of the solution, rather than the approximate formulae derived above. We therefore return to (2.17), now retaining terms proportional to $\mathrm{e}^{-2\alpha_m a}$ for all $m \geq 0$. Again, the analysis closely follows Evans & Linton (1993). We consider the integrals defined by

$$I'_m = \lim_{N\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}_N} F(z) \left(\frac{1}{z - \alpha_m} + \frac{\mathrm{e}^{-2\alpha_m a}}{z + \alpha_m} \right) \mathrm{d}z, \qquad m = 0, 1, 2, \ldots$$

where the contours $\mathcal{C}_N$ are as before and now define

$$F(z) = f(z)h(z) = f_0(z)g(z)h(z) \tag{2.30}$$

where $f(z)$ is given by (2.23) and

$$h(z) = 1 + \sum_{n=2}^{\infty} \frac{H_n}{2(z - \alpha_n)d}. \tag{2.31}$$

Then it follows that

$$\sum_{n=-\infty}^{\infty}{}' R(F : \gamma_n) \left(\frac{1}{\gamma_n - \alpha_m} + \frac{\mathrm{e}^{-2\alpha_m a}}{\gamma_n + \alpha_m} \right) + F(\alpha_m) + \mathrm{e}^{-2\alpha_m a} F(-\alpha_m) = 0, \qquad m = 0, 1, 2, \ldots \tag{2.32}$$

which can matched with (2.17) given the relations $B_n \beta_n = R(F : \gamma_n)$ and

$$F(\alpha_m) + \mathrm{e}^{-2\alpha_m a} F(-\alpha_m) = 0, \qquad m = 0, 1, 2, \ldots \tag{2.33}$$

For $m \geq 2$ this reduces, using (A.1) and (A.2), to the system

$$H_m + D_m \sum_{n=2}^{\infty} \frac{H_n}{2(\alpha_m + \alpha_n)d} = D_m, \qquad m = 2, 3, \ldots \tag{2.34}$$

where D_m is given by (A.3) and we note that

$$D_m \sim \text{const} \times \mathrm{e}^{-2\alpha_m a}, \qquad \text{as } a/d \to \infty.$$

The conditions that guarantee that (2.34) has a unique square-summable solution are

$$\sum_{n=2}^{\infty} D_n^2 < \infty, \qquad \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{D_m^2}{4(\alpha_m + \alpha_n)^2 d^2} < 1$$

(see for example, Hutson & Pym (1980)) which are easily seen to hold for a/d sufficiently large.

The 'exact' conditions for an embedded Rayleigh-Bloch wave are now given by using $m = 0, 1$ in (2.33), which gives

$$\mathrm{e}^{-2\mathrm{i}k_m a} = -\frac{F(\mathrm{i}k_m)}{F(-\mathrm{i}k_m)} = -\mathrm{e}^{2\mathrm{i}(\chi_m + \delta_m - (2k_m d/\pi)\ln 2)}, \qquad m = 0, 1$$

and so

$$k_m a = (n_m + \tfrac{1}{2})\pi + \frac{2k_m d}{\pi}\ln 2 - \chi_m(\beta, k) - \delta_m(\beta, k, a) \tag{2.35}$$

$n_m \in \mathbb{Z}$ with χ_m defined in as before in (2.27). Here

$$e^{2i\delta_m} = \frac{h(ik_m)}{h(-ik_m)}, \qquad m = 0, 1$$

which implies

$$\delta_m(\beta, k, a) = \arg\left\{1 - \sum_{n=2}^{\infty}\frac{H_n}{2(ik_m + \alpha_n)d}\right\}, \tag{2.36}$$

since the H_n are real.

Now, as $a/d \to \infty$, $D_m \to 0$ ($m \geq 2$) and hence $H_m \to 0$ ($m \geq 2$) from (2.34) also. Then from (2.36), $\delta_m \to 0$ for $m = 0, 1$ and we recover the approximate conditions (2.28) for an embedded Rayleigh-Bloch mode. Further, for any given $\epsilon > 0$ we can choose a/d to be large enough such that $\max\{|\delta_0|, |\delta_1|\} < \epsilon$.

For $0 < \beta < \pi/2d$, it is easily confirmed that $\gamma_{-n} < \alpha_{2n} < \gamma_n < \alpha_{2n+1} < \gamma_{-(n+1)} < \ldots$ and this allows us place bounds on χ_m in (2.27), thus

$$0 < \chi_m(\beta, k) < \{\tan^{-1}(k_m/\gamma_{-1}) + \tan^{-1}(k_m/\gamma_1)\} < \pi, \qquad m = 0, 1.$$

Proof of the existence of embedded Rayleigh-Bloch modes follows using arguments similar to those in McIver *et al* (2001). Thus, defining $c_m = \frac{1}{2}\pi - \chi_m - \delta_m$, $m = 0, 1$, we can choose a/d sufficiently large such that $0 < |c_m| < \frac{1}{2}\pi$. Substituting c_m into (2.35) and eliminating a between the equations for $m = 0$ and $m = 1$ gives

$$(k_1/k_0)n_0\pi - n_1\pi = c_1 - (k_1/k_0)c_0 \tag{2.37}$$

and notice from (2.35) that the condition of a/d large implies n_0 large since $k_0 \equiv k$ is bounded in the interval $\pi/2d < k_0 < \pi/d - \beta$, $0 < \beta < \pi/2d$.

For any fixed β in the interval $0 < \beta < \pi/2d$, (k_1/k_0) varies between 0 and σ where

$$0 < \sigma = (1 - \tfrac{1}{4}(1 - (\beta d/\pi))^{-2})^{\frac{1}{2}} < 1$$

and the left-hand side of (2.37) varies continuously between $-n_1\pi$ and $(\sigma n_0 - n_1)\pi$. Provided n_0 is sufficiently large the right-hand side of (2.37) varies continuously and always lies in the interval $(-\pi, \pi)$. It follows that for any β in $(0, \pi/2d)$ and for any $n_1 \geq 1$ we can choose n_0 large enough such that (i) equation (2.34) has a unique solution (ii) both c_0 and c_1 lie in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and (iii) $\sigma n_0 - n_1 > 1$. It then follows that there must be at least one value of k in the interval $(\pi/2d, \pi/d - \beta)$ for which (2.37) is satisfied. The corresponding value of a/d is given by (2.35).

3 Results

The results shown in figure 1 were calculated by simultaneously satisfying the two conditions (2.28) with $m = 0, 1$. These conditions were derived under the assumption that $a/d \gg 1$, but comparison with the solution of the full system using (2.35) with $m = 0, 1$ reveals that the approximate conditions gave values of a/d and kd that are correct to at least three decimal places in all cases. Agreement was also found between the results calculated using the present method and those in Porter & Evans (2001) using a Galerkin approximation to the solution of certain integral equations. Indeed, there is no observable difference between figure 1 here and figure 3 in Porter & Evans (2001), apart from the labelling of the curves.

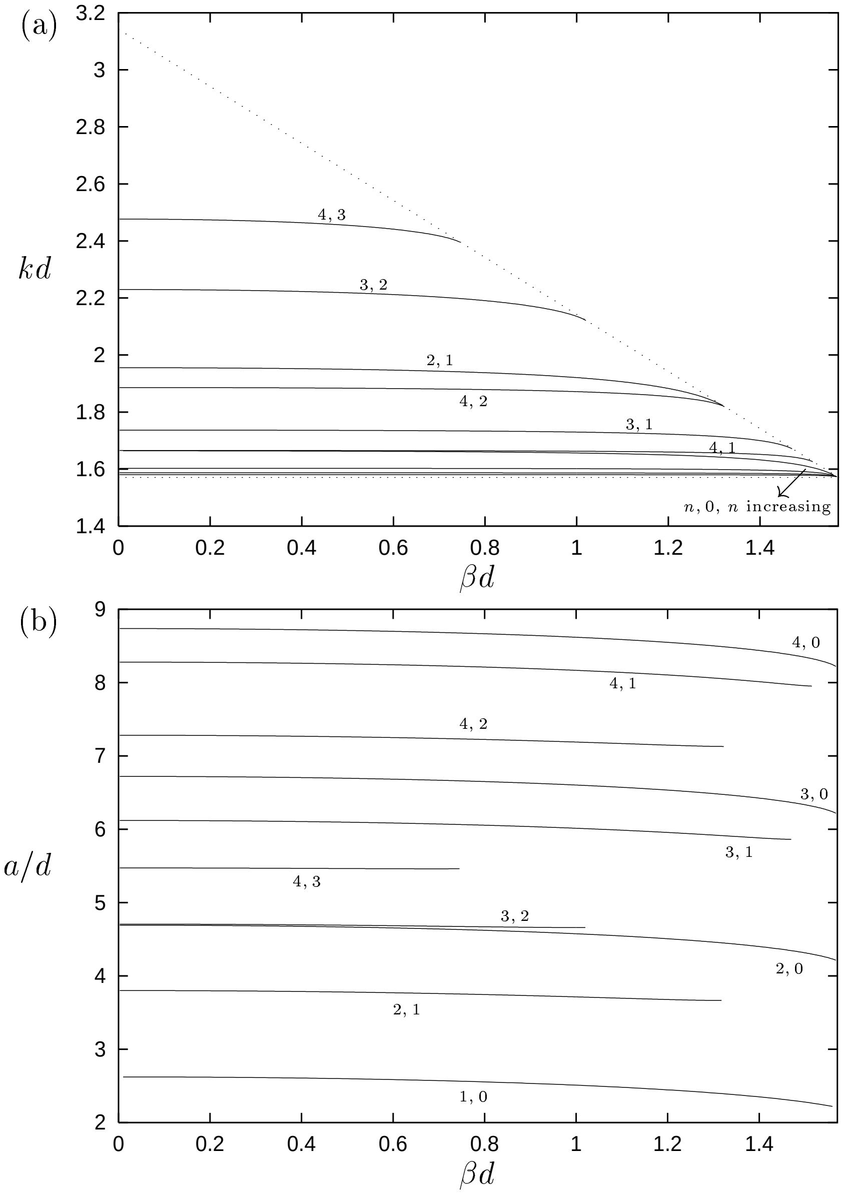

Figure 1: Variation of (a) kd and (b) a/d with βd for symmetric modes along arrays of thin plates. The curves are labelled by values of n_0, n_1 used to generate them.

In figure 1 the pair of numbers against each of the curves in figure 3 represent the values of n_0 and n_1 respectively used in (2.28), $m = 0, 1$. For each value of β in $(0, \pi/2d)$ there exists an infinite sequence of solutions. Only a selection of those for which $a/d < 9$ are shown in figure 1. Interestingly, the curves labelled $n, 0$ for $n = 1, 2, \ldots$, referring to the fact that the value $n_1 = 0$ was used to generate the solutions, are not included in the existence proof at the end of the previous section despite the fact that they are the only solutions that exist for all β in the interval $(0, \pi/2d)$. All other curves pass through the second cut-off for some value of β less than $\pi/2d$. However, for each value of $n_1 \geq 1$ we observe that, as n_0 increases, the interval of β for which solutions exist tends to $(0, \pi/2d)$ in accordance with the existence proof.

As $\beta \to 0$ each of the curves shown in figure 1 tend to a particular value of kd in the interval $\pi/2 < kd < \pi$ and to a corresponding value of a/d. These particular values lie on curves for $\beta = 0$ which describe Dirichlet trapped modes (or Parker γ-modes). The value of $\beta = 0$ is a special case of Rayleigh-Bloch mode when the problem can be decomposed into two potentials are symmetric and antisymmetric about the line containing the plate (see Porter & Evans (2001). By considering the antisymmetric component of the problem shows that it is equivalent to a problem involving a

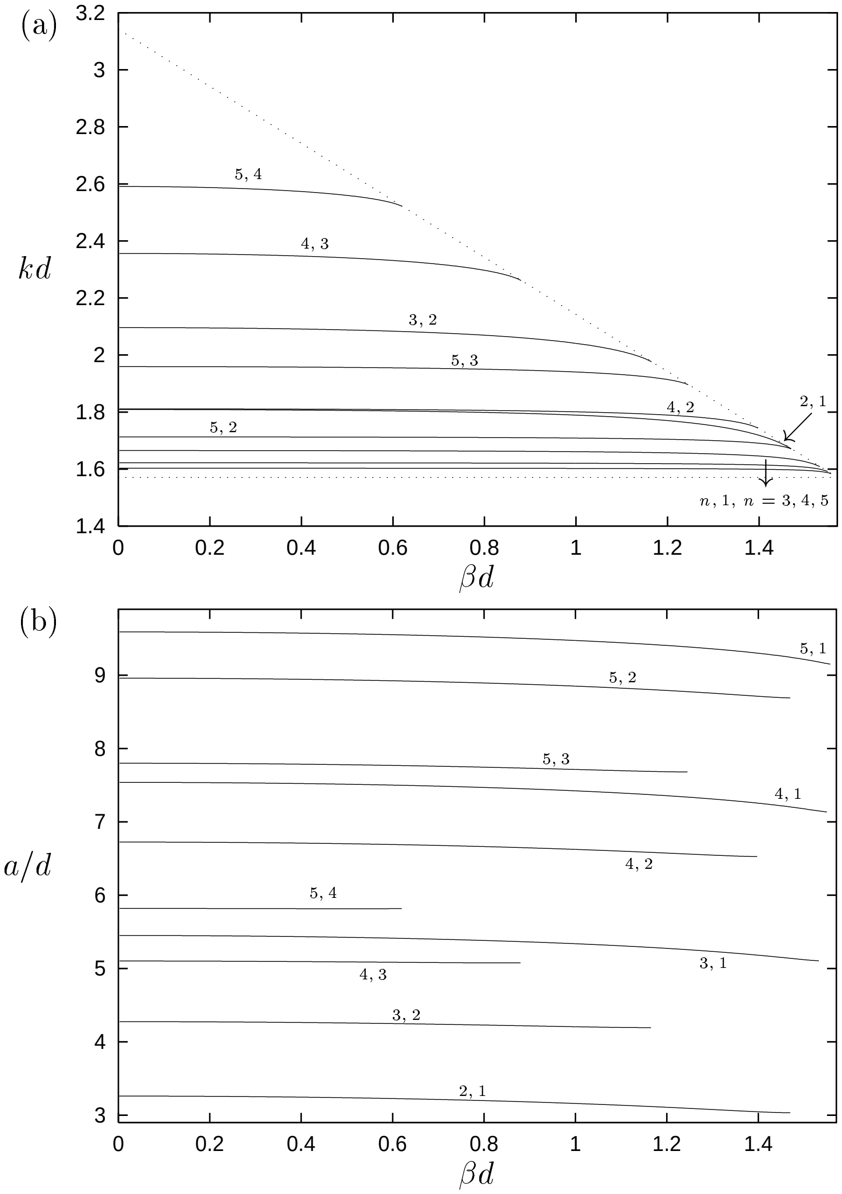

Figure 2: Variation of (a) kd and (b) a/d with βd for antisymmetric modes along arrays of thin plates. The curves are labelled by values of n_0, n_1 used to generate them.

plate on the centreline of a guide of width $2d$ having Dirichlet conditions on both the walls and the centreline of the guide. For this so-called Dirichlet trapped mode problem, we are below the cut-off provided $kd < \pi$, whilst modes are known not to exist for $kd < \pi/2$ by McIver & Linton (1995). In fact it can be shown (see, for example, Evans & Porter (1998)) that Dirichlet trapped modes exist for all plates lengths and curves showing the variation of wavenumber $\pi/2 < kd < \pi$ with a/d are given in Evans & Linton (1994) (although their curves are quantatively incorrect due to the wrong approximate formula being used – the approximate formula is corrected in (2.29)) and in Porter & Evans (2001). Thus along each of these curves for $\beta = 0$ to which the previous discussion refers, there are certain particular values of a/d and kd which indicate the start of a branch of embedded surface waves modes into $\beta > 0$.

The problem that has been considered so far this paper has involved satsifying a Neumann condition on the line $x = -a$ (see equation (2.3)). Hence, the problem may be regarded as a surface wave travelling along a 'cliff' with an infinite periodic array of plates of length a protruding from the cliff. This problem is equivalent, by symmetry, to waves travelling along a periodic array of thin plates of length $2a$ where the waves are symmetric about the centreline of the array. We will refer to these solutions as symmetric modes. If, instead, solutions which are antisymmetric about the centreline of an array of plates of length $2a$ are sought, the Neumann condition on $x = -a$ in (2.3) must be replaced by a Dirichlet condition, thus

$$\phi(-a, y) = 0, \qquad -\infty < y < \infty \tag{3.1}$$

Only minor changes to the formulation are needed to consider solutions satisfying (3.1), referred to as antisymmetric modes, and these are outlined briefly now. First, in (2.7) the $\cosh \alpha_m(x+a)$ must be changed to $\sinh \alpha_m(x+a)$ to satisfy (3.1). Working through with this change results in the addition of the two terms in brackets in (2.17) being changed to a subtraction. Following on from this results in the same formula in (2.22) apart from the minus sign multiplying the right-hand side of the equation is changed to a plus sign. This in turn implies that the approximate condition (2.28) is replaced with

$$k_m a = n_m \pi + \frac{2k_m d}{\pi} \ln 2 - \chi_m(\beta, k), \qquad n_m \in \mathbb{Z}, \quad m = 0, 1 \tag{3.2}$$

where $\chi_m(\beta, k)$ defined in (2.27) as in the symmetric case. Further, the exact condition is also only changed by the factor of $\frac{1}{2}\pi$ from the symmetric case to give

$$k_m a = n_m \pi + \frac{2k_m d}{\pi} \ln 2 - \chi_m(\beta, k) - \delta_m(\beta, k, a), \qquad n_m \in \mathbb{Z}, \quad m = 0, 1. \tag{3.3}$$

A reworking of the proof of existence can be made in an identical manner, where the only change from the symmetric case is in the definition of c_m, where now $c_m = -\chi_m - \delta_m$ such that $-\pi < c_m < 0$. Thereafter none of the arguments used for proving the existence of symmetric modes needs altering for the antisymmetric case.

Results for antisymmetric modes are shown in figure 2, computed using (3.2) where the same labelling mechanism referred to in the symmetric case is used. Again, the values of kd and a/d are correct to at least 3 decimal places when compared with the exact values obtained from (3.3). The curves in figure 2 show the same type of behaviour as in the case for symmetric modes. The only significant difference is that none of the curves shown in figure 2 exist throughout the whole interval $0 < \beta < \pi/2d$.

4 Conclusions

In this paper the existence of Rayleigh-Bloch surface waves occurring at wavenumbers *above* the first cut-off ($k > \beta$) along periodic arrays of thin parallel plates of sufficient length has been proved. This has been done using an adaption of Evans & Linton (1993) who considered the existence of Rayleigh-Bloch waves *below* the first cut-off ($k < \beta$). Solutions above the cut-off correspond to two conditions being met simultaneously. Numerical results based on simple approximate formulae derived under a long-plate assumptions are shown to be in excellent agreement with the solution of the full problem and also confirm the calculations of Porter & Evans (2001). They show that for each value of β in the range $(0, \pi/2d)$ which characterises the propagation of Rayleigh-Bloch surface waves along the array, there exists a sequence of discrete values of wavenumber, kd, and plate length a/d corresponding to solutions.

The existence of surface waves above the first cut-off may play an important role in determining solutions to scattering problems. Since $k > \beta$ we may write the relation between them as $k \cos\theta = \beta$ where θ represents the angle of incidence with respect to the axis of the array of an incoming wave. Thus, for particular geometries and wavenumber, there would exist an incident wave angle for which the scattering problem is non-unique since any multiple of the corresponding Rayleigh-Bloch surface wave solution may be added to the scattering problem.

The scattering problem for arrays of thin plates has been investigated by Linton & Evans (1993) using the residue calculus method described here. Unfortunately, Linton & Evans (1993) had to restrict the parameter regime to $k < \pi/2d$ so that only one wave-like mode existed in between the plates, whilst here we have considered the regime $k > \pi/2d$. It should be possible to adapt Linton & Evans (1993) to this new regime although the expressions for the reflection and transmission coefficients would almost certainly become obscured by the added algebraic complexity when $k > \pi/2d$. On the other hand, it may well be possible to determine explicitly the behaviour of the reflection and transmission coefficients close to non-uniqueness. See, for example equations (2.59), (2.60) and (2.65) in Linton & Evans (1993) and compare with (2.27) and (2.28) in the present paper. This work is currently begin undertaken.

Appendix A: The coefficients D_m

From (2.24) we have that

$$g_1(z)g_1(-z) = \prod_{n=2}^{\infty}(1 - z^2/\alpha_m^2) = \prod_{n=2}^{\infty} \frac{((n\pi)^2 - (2\zeta d)^2)}{((n\pi)^2 - (2kd)^2)} = \frac{\pi^2 - (2kd)^2}{\pi^2 - (2\zeta d)^2}\left(\frac{k\sin(2\zeta d)}{\zeta\sin(2kd)}\right)$$

where $\zeta^2 = z^2 + k^2$. Also,

$$g_2(z)g_2(-z) = \prod_{n=1}^{\infty}(1 - z^2/\gamma_n^2)(1 - z^2/\gamma_{-n}^2) = \frac{(k^2 - \beta^2)(\cos(2\beta d) - \cos(2\zeta d))}{(\zeta^2 - \beta^2)(\cos(2\beta d) - \cos(2kd))}$$

(see equation (A.2) in Evans & Linton (1993)). Thus, we have

$$g(z)g(-z) = \frac{k(\pi^2 - (2kd)^2)(\zeta^2 - \beta^2)(\cos(2\beta d) - \cos(2kd))\sin(2\zeta d)}{\zeta(\pi^2 - (2\zeta d)^2)(k^2 - \beta^2)(\cos(2\beta d) - \cos(2\zeta d))\sin(2kd)}.$$

Now $g(z)$ has simple poles at $z = \alpha_m$ for $m \geq 2$, so that from (2.30), (2.31)

$$F(\alpha_m) = f_0(\alpha_m)H_m \lim_{z\to\alpha_m} \frac{g(z)}{2(z - \alpha_m)d} = f_0(\alpha_m)H_m\widetilde{D}_m/g(-\alpha_m) \tag{A.1}$$

using the fact that $z \to \alpha_m$, $\zeta \sim (m\pi/2d) + (2\alpha_m d/m\pi)(z - \alpha_m)$ and where

$$\widetilde{D}_m = \frac{(2kd)(\pi^2 - (2kd)^2)((m\pi)^2 - (2\beta d)^2)(\cos(2kd) - \cos(2\beta d))(2\alpha_m d)}{(m\pi)^2(\pi^2 - (m\pi)^2)((2kd)^2 - (2\beta d)^2)(1 - (-1)^m\cos(2\beta d))\sin(2kd)}$$

Also from (2.30), (2.31)

$$F(-\alpha_m) = f_0(-\alpha_m)g(-\alpha_m)\left(1 - \sum_{n=2}^{\infty}\frac{H_n}{2(\alpha_m + \alpha_n)d}\right) \tag{A.2}$$

It follows from (2.33), (2.34) that

$$D_m = -\mathrm{e}^{-2\alpha_m a} f_0^2(-\alpha_m)g^2(-\alpha_m)/\widetilde{D}_m \tag{A.3}$$

where

$$g(-\alpha_m) = \frac{\mathrm{e}^{2\alpha_m d/\pi}}{(1 + \alpha_m/\gamma_1)(1 + \alpha_m/\gamma_{-1})}\prod_{n=2}^{\infty}\frac{(1 + \alpha_m/\alpha_n)}{(1 + \alpha_m/\gamma_n)(1 + \alpha_m/\gamma_{-n})}$$

and

$$f_0(-\alpha_m) = \mathrm{e}^{-(2\alpha_m d/\pi)(1 - \ln 2)}$$

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