TOTAL TRANSMISSION OF WAVES THROUGH NARROW GAPS IN CHANNELS

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Summary

The problem of the transmission of plane waves along a uniform parallel-walled waveguide or channel through a periodically-spaced array of thin screens with gaps placed across the channel is investigated. Of particular interest is the existence of frequencies at which energy it totally transmitted – shown to occur when there are two or more screens – and how they depend upon the size of the gaps in the screen. Attention focuses on small gaps where the phenomenon of total transmission persists and where an approximate analysis of a system of full coupled integrals equations is reduced to simple linear difference equations which can be solved in closed form. Results show comparisons between exact computations, the small-gap computations and corresponding wide-spacing approximations. A discussion of total reflection is also included when the gaps are no longer assumed to be small.

1. Introduction

The phenomenon of 'extraordinary transmission' emerged in the field of optics: see Ebbesen *et al* (1998), a paper which has currently attracted over 6500 citations. Under certain conditions it was found that enhanced transmission of light could occur through a periodic array of sub-wavelength diameter circular holes in a thin metal film when compared with the transmission of light through a single hole (Bethe (1944)). They describe how the array has an essential role in this effect and indicate the dependence of the phenomenon on the array spacing, the film thickness, hole diameter and incident wavelength. The effect was found to be particularly strong for wavelengths much larger than both the holes and the array spacing. An informative recent review of the development of the subject area of enhanced optical transmission is given by Garcia de Abajo (2007).

Although the underlying physics is different, the governing equations describing optics referred to above has analogies in the physical descriptions of acoustics and water waves, particularly when problems are posed in two-dimensions. Accounting for the different physical lengthscales one may therefore draw similarities between results in optics, acoustics and water waves.

In the latter area of surface gravity water waves in a two-dimensional setting, the total transmission of a small-amplitude plane waves at isolated frequencies is a common occurrence in problems involving long submerged obstacles or pairs of identical obstacles. See, for example, Newman (1965), Newman (1974) and Evans (1975). In the latter work, Evans used the work of Packham & Williams (1972) who considered the transmission of surface wave through a small submerged gap in a vertical screen to develop a solution for the transmission through two identical screens each having small gaps. Whilst transmission of waves through a small gap in a single screen was limited it was shown that the addition of a second identical screen resulted in total transmission of waves at isolated frequencies.

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This could be classified as an example of extraordinary transmission albeit in a different physical setting.

When vertical bottom-mounted structures extend with constant cross section through the surface of an incompressible fluid of constant depth, the water wave problems can be posed in the context of two-dimensional acoustics having solutions satisfying the Helmoholtz equation. Here also the phenomenon of total transmission is well known in problems relating to wave transmission past structures that are placed in a parallel-walled waveguide or channel having acoustically-hard (or rigid) walls – some examples are mentioned in the next paragraph. By the method of images this arrangement is identical to normal wave incidence upon a periodic array of structures. Alternatively this geometry can be imagined as a "screen" embedded with a periodic arrangement of gaps. More generally, when normal wave incidence is replaced by oblique wave incidence upon a periodic array of structures with gaps, one may still pose the scattering problem within a single fundamental cell of the array by placing appropriate quasii-periodic boundary conditions on the walls.

For example, Linton & Evans (1993) demonstrate total transmission of oblique wave energy past an infinite periodic row of rigid circular cylinders. When the circular cylinders are replaced by rectangular cylinders Fernyhough & Evans (1995) give extensive results illustrating total transmission for a variety of incident wave angles; also see Yang *et al* (2011) who consider a similar problem in an electromagnetic context. Porter & Evans (1996) considered oblique wave transmission through infinitesimally-thin screens incorporating a periodic arrangement of gaps, a problem with a long history: see, for example, Jones (1986) in the electromagnetic context. Porter & Evans (1996) reaffirmed that total transmission could never occur for a single screen with gaps but that two parallel screens could allow total transmission to occur at isolated frequencies.

Remarkably, if the screens are placed close enough together Porter & Evans (1996) also showed that all the incident wave energy could be reflected. Their work was extended to multiple screens with arbitrary gaps, all sharing the same periodicity, by Biggs & Porter (2005).

In none of the work described in the paragraph above does attention ever focus on characterising the range of geometrical configurations which support total transmission of waves. Nor is there any detailed discussion on the the case when the gaps between adjacent structures is very small where the connection to enhanced optical transmission could be made. The purpose of this paper is to consider both these aspects using a particular extension of the problems considered in Porter & Evans (1996) and Biggs & Porter (2005) for screens with equal centreline gaps spanning a channel. We will show that 100% transmission can occur at a given frequency for an infinite sequence of spacings, when one or more extra identical screens with small gaps are introduced.

The paper is organised as follows. In Section 2 the general solution to the wave equation in a rigid-walled channel accommodating an arbitrary arrangement of thin screens with equal gaps on the centreline is formulated in terms of coupled integral equations. The special case of a single screen is considered in Section 3 where a small-gap approximation is made and compared to exact results. Section 4 applies the small-gap approximation to the integral equation formulation for N screens which are now also assumed to be equally separated. These steps allow the reflection and transmission coefficients for the complete arrangement to be expressed in closed form in terms of coefficients defined by a linear difference equation. It is important to note that the only approximation made is that of a small gap and all

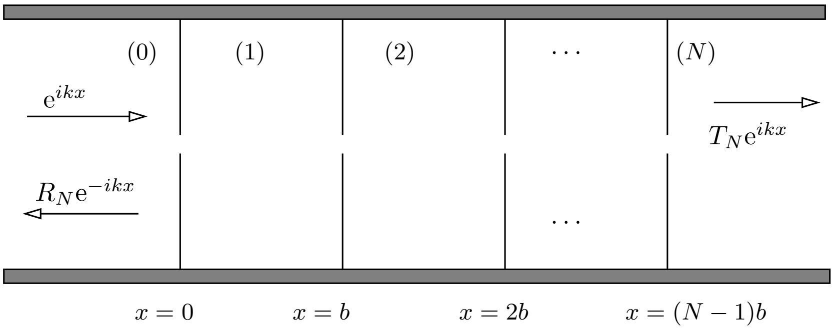

Fig. 1 The configuration of a channel of width $2d$ with N identical periodically-spaced screens with centrally-placed gaps of width $2a$.

the evanescent mode effects between neighbouring screens are captured in the solution given. Results are illustrated in Section 5 which also includes a discussion of the additional algebraic simplifications that arise when evanescent wave interactions are neglected. This is the so-called wide-spacing approximation and is similar the approach used from the outset in Porto *et al* (1998) and Lu *et al* (2007) to analyse enhanced transmission through slits in optically-thick metallic gratings. In addition, we derive an explicit expression for reflection by a semi-infinite arrangement of equal screens and we also discuss the range of thin screen geometries for which total reflection occurs.

2. Formulation of the N-screen problem

Cartesian coordinates (x, y) are used with the origin placed in the centre of an open-ended channel with walls $y = \pm d$, $-\infty < x < \infty$. A series of N thin screens or screens perpendicular to the channel walls are positioned at $x = b_n = nb$, $n = 0, 1, 2, \ldots, N-1$. The screens are identical and contain gaps in them between $-a < y < a$ such that $a/d < 1$.

Within the channel a complex-valued function $\phi(x, y)$, say, satisfies the two-dimensional wave equation

$$(\nabla^2 + k^2)\phi = 0 \tag{2.1}$$

and homogeneous Neumann conditions apply to ϕ on the channel walls and screens. In (2.1) $k = \omega/c$ where ω is an assumed angular frequency of motion and c is the phase speed. The problem can therefore be interpreted in many physical settings such as linear acoustics and TM-polarised electromagnetics. In the context of linear water waves on a fluid of constant depth h, the problem is one in which the geometry described above has been extended into a perpendicular (depth) direction and the depth dependence has been factorised from the three-dimensional velocity potential (in addition to the $\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence). In this case k is related to the fluid depth via

$$\omega^2 = gk \tanh kh$$

where g is gravitational acceleration.

A wave is incident from $x = -\infty$ upon the arrangement of screens and partially reflected and partially transmitted. We assume that the frequency of the incident wave is below the first channel cut-off ($kd < \pi$) and so only one wave mode can propagate along the channel. Thus we write

$$\phi(x,y) \sim \begin{cases} e^{ikx} + R_N e^{-ikx}, & x \to -\infty \\ T_N e^{ikx}, & x \to \infty \end{cases} \tag{2.2}$$

where R_N and T_N define the reflection and transmission coefficients for N screens.

Since there is symmetry about $y = 0$ in both the geometry and the forcing (the incident wave) it must follow that $\phi(x,-y) = \phi(x,y)$ and we can impose, in addition to the condition $\phi_y(x,d) = 0$, that $\phi_y(x,0) = 0$, for all $-\infty < x < \infty$ and only need solve for $\phi(x,y)$ in $0 < y < d$.

In addition to these lateral boundary conditions, on the nth screen ($n = 0, 1, 2, \ldots, N-1$) we have

$$\phi_x(b_n^{\pm}, y) = 0, \quad a < y < d. \tag{2.3}$$

The method of solution starts in obvious fashion by writing separation solutions in each region of the domain separated by screens – see Fig. 1. For $x < 0$, general solutions of (2.1) satisfying (2.2) are

$$\phi(x,y) = e^{ikx} + R_N e^{-ikx} + \sum_{r=1}^{\infty} \frac{kF_r^{(0)} e^{\alpha_r x} \cos p_r y}{\alpha_r} \tag{2.4}$$

where $p_r = r\pi/d$, $\alpha_r = (p_r^2 - k^2)^{1/2}$, $\alpha_0 = -ik$, and we have already noted that $k < \pi/d$. For $x > (N-1)b$, we write

$$\phi(x,y) = T_N e^{ik(x-b_{N-1})} - \sum_{r=1}^{\infty} \frac{kF_r^{(N-1)} e^{-\alpha_r(x-b_{N-1})} \cos p_r y}{\alpha_r}. \tag{2.5}$$

Finally, in each finite region $(n-1)b < x < nb$, $n = 1, 2, \ldots, N-1$, we write

$$\phi(x,y) = \sum_{r=0}^{\infty} \frac{k(F_r^{(n)} \cosh \alpha_r(x - b_{n-1}) - F_r^{(n-1)} \cosh \alpha_r(b_n - x)) \cos p_r y}{\alpha_r \sinh \alpha_r b}. \tag{2.6}$$

In the above $F_r^{(n)}$ are undetermined coefficients and these definitions ensure that $\phi_x(b_n^+, y) = \phi_x(b_n^-, y)$, $0 < y < d$, $n = 0, 1, 2, ..., N-1$. It follows that

$$\phi_x(b_n, y) \equiv F^{(n)}(y) = \sum_{r=0}^{\infty} kF_r^{(n)} \cos p_r y$$

once the definition of the coefficients are extended to include

$$F_0^{(0)} = i(1 - R_N), \qquad F_0^{(N-1)} = iT_N. \tag{2.7}$$

Now (2.3) is used to give

$$F_r^{(n)} = \frac{\epsilon_r}{2kd} \int_0^a F^{(n)}(t) \cos p_r t \, dt \tag{2.8}$$

with $\epsilon_0 = 1$, $\epsilon_r = 2$, for $r > 0$. Continuity across the gap in the nth screen demands that $\phi(b_n^+, y) = \phi(b_n^-, y)$, for $0 < y < a$ and gives, after using (2.4), (2.5), (2.6) and (2.8),

$$\int_0^a \left(F^{(n+1)}(t)K_1(y,t) - 2F^{(n)}(t)K_2(y,t) + F^{(n-1)}(t)K_1(y,t)\right) \mathrm{d}t = 0, \tag{2.9}$$

for $n = 1, 2, \ldots, N-2$, with

$$\int_0^a \left(F^{(1)}(t)K_1(y,t) - F^{(0)}(t)K_3(y,t)\right) \mathrm{d}t = 2, \tag{2.10}$$

and

$$\int_0^a \left(F^{(N-1)}(t)K_3(y,t) - F^{(N-2)}(t)K_1(y,t)\right) \mathrm{d}t = 0, \tag{2.11}$$

all three equations applying over $0 < y < a$, and in which

$$K_1(y,t) = \sum_{r=0}^{\infty} \frac{\epsilon_r \mathrm{cosech} \alpha_r b}{2\alpha_r d} \cos p_r y \cos p_r t, \tag{2.12}$$

$$K_2(y,t) = \sum_{r=0}^{\infty} \frac{\epsilon_r \coth \alpha_r b}{2\alpha_r d} \cos p_r y \cos p_r t, \tag{2.13}$$

and

$$K_3(y,t) = \sum_{r=0}^{\infty} \frac{\epsilon_r (1 + \coth \alpha_r b)}{2\alpha_r d} \cos p_r y \cos p_r t. \tag{2.14}$$

These coupled integral equations can, in principle, be solved numerically without further approximation or simplification. However, since our primary focus in this paper centres on small gaps we can make further analytic progress, first by considering wave interaction with a single screen.

3. The single screen and a small-gap approximation

This generic problem involving the scattering of a plane wave by a single screen at the origin forms the basis of the approach to the solution we shall adopt for the N-screen problem. Details of the solution can be found in Porter & Evans (1996) for example. Thus it is straightforward to show in this case that $R_1 + T_1 = 1$ and that the horizontal velocity $F^{(0)}(y)$ across $0 < y < a$ satisfies

$$\int_0^a F^{(0)}(t)K(y,t)\,\mathrm{d}t = -R_1, \qquad 0 < y < a, \tag{3.1}$$

with

$$\int_0^a F^{(0)}(t)\,\mathrm{d}t = 2\mathrm{i}kdT_1 \tag{3.2}$$

where

$$K(y,t) = \sum_{r=1}^{\infty} \frac{\cos p_r y \cos p_r t}{\alpha_r d}. \tag{3.3}$$

For future reference, note that the term $r = 0$ does not appear in the infinite series above. In general, the integral equation (3.1) is not invertible. Porter & Evans (1996) describe a numerical approximation based on Galerkin's method for calculating R_1 and T_1 efficiently.

However, from this point on we make the assumption that $a/d \ll 1$ and exploit the fact that the kernel $K(y,t)$ is dominated by a logarithmic term. Specifically, we can write (e.g. Jones (1986) eqn. (16.1))

$$K(y,t) = -\frac{1}{2\pi}\ln(2|\cos(\pi y/d) - \cos(\pi t/d)|) + \sum_{r=1}^{\infty}\left(\frac{1}{\alpha_r d} - \frac{1}{r\pi}\right)\cos p_r y \cos p_r t$$

so that for $y, t \to 0$

$$K(y,t) \sim -\frac{1}{2\pi}\ln|y^2 - t^2| + S(kd) - \frac{1}{\pi}\ln(\pi/d) \tag{3.4}$$

where

$$S(\kappa) = \sum_{r=1}^{\infty}\Big(\frac{1}{\sqrt{r^2\pi^2 - \kappa^2}} - \frac{1}{r\pi}\Big).$$

Substituting (3.4) into (3.1) and using the fact that $F^{(0)}(y)$ can be extended as an even function in y, allows us to approximate the integral equation (3.1) by

$$\int_{-a}^{a} F^{(0)}(t)\ln|y - t|\,dt = A, \qquad -a < y < a, \tag{3.5}$$

where

$$A = 2\pi R_1 + (\pi S(kd) - \ln(\pi/d))\int_{-a}^{a} F^{(0)}(t)\,\mathrm{d}t. \tag{3.6}$$

The singular integral equation (3.5), where the $F^{(0)}(y)(a^2 - y^2)^{1/2}$ is bounded as $|y| \to a$ as is the case here, has an explicit solution but all we require is the result

$$\int_{-a}^{a} F^{(0)}(t)\,\mathrm{d}t = A/\ln(a/2). \tag{3.7}$$

For a proof see Cooke (1970), and, for applications in water waves, Evans (1975), and Packham & Williams (1972) who also describe a three-dimensional version. It follows from (3.2), (3.5) (3.6) and (3.7) that we can express the reflection and transmission coefficients in the form

$$\left.\begin{aligned} T_1 &= \cos\delta\mathrm{e}^{\mathrm{i}\delta} \\ R_1 &= -\mathrm{i}\sin\delta\mathrm{e}^{\mathrm{i}\delta} \end{aligned}\right\} \qquad \text{where} \qquad \tan\delta = 2kd\left(S(kd) - \frac{1}{\pi}\ln\left(\frac{\pi a}{2d}\right)\right) > 0. \tag{3.8}$$

The phase of T_1 will play a key role in the more general N-screen case where we will we make use of the result which follows from the above that if

$$\int_{0}^{a} F^{(n)}(t)K(y,t)\,dt = C, \quad 0 < y < a, \tag{3.9}$$

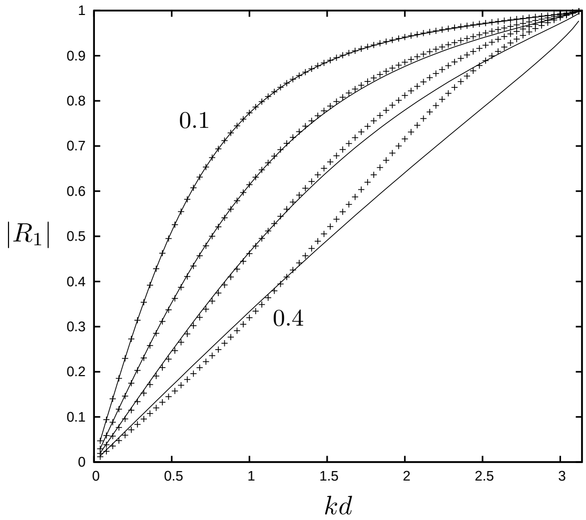

Fig. 2 Comparison of the small-gap approximation (crosses) against exact results (lines) for a single screen with $a/d = 0.1$, 0.2, 0.3, 0.4 (left to right).

for some constant C then, for small a/d,

$$\frac{1}{2d}\int_0^a F^{(n)}(y)dy = F_0^{(n)} \approx kC\cot\delta. \tag{3.10}$$

Before ending this section, we show in Fig. 2 a comparison between exact and approximate results for small gaps. The solid lines show variation of $|R_1|$ with kd for gap sizes $a/d = 0.1$, 0.2, 0.3, 0.4 employing the accurate numerical method described in Porter & Evans (1996). The crosses are computed using the small-gap result given in (3.8). It can be seen that the agreement up to a gap size of $a/d = 0.1$ is excellent over the whole range of $kd < \pi$ with more noticable divergence in agreement for larger a/d.

4. The solution for N screens with small gaps

Returning to the general case we have, from (2.12) to (2.14), as $y, t \to 0$,

$$K_1(y,t) \sim \frac{(-\operatorname{cosec} kb + E_1)}{2kd}, \tag{4.1}$$

$$K_2(y,t) - K(y,t) \sim \frac{(-\cot kb + E_2)}{2kd}, \tag{4.2}$$

and

$$K_3(y,t) - 2K(y,t) \sim \frac{(-\cot kb + E_2 + \mathrm{i})}{2kd}, \tag{4.3}$$

where $K(y,t)$ is the single screen kernel defined in (3.3) and

$$E_1 = 2kd\sum_{r=1}^{\infty}\frac{\operatorname{cosech}\alpha_r b}{\alpha_r d}, \qquad \text{and} \qquad E_2 = 2kd\sum_{r=1}^{\infty}\frac{\mathrm{e}^{-\alpha_r b}\operatorname{cosech}\alpha_r b}{\alpha_r d} \tag{4.4}$$

where the $E_i \to 0$ as $b/d \to \infty$, $i = 1,2$. Substituting the expressions (4.1)–(4.3) into (2.9)–(2.11) and using (3.10) gives the set of coupled algebraic equations

$$F_0^{(n+1)} - 2\cos\alpha F_0^{(n)} + F_0^{(n-1)} = 0, \quad n = 1,2,\ldots,N-2 \tag{4.5}$$

and

$$(p+\mathrm{i})F_0^{(0)} + 2 = qF_0^{(1)}, \tag{4.6}$$

$$(p+\mathrm{i})F_0^{(N-1)} = qF_0^{(N-2)}. \tag{4.7}$$

In the above

$$p = 2\tan\delta - (\cot kb - E_2), \qquad q = (-\operatorname{cosec} kb + E_1), \tag{4.8}$$

and we have defined

$$\cos\alpha = \frac{(\cos\alpha_0 - E_2\sin kb)}{(1 - E_1\sin kb)} \tag{4.9}$$

where

$$\cos\alpha_0 = \frac{\cos(\delta + kb)}{\cos\delta} \equiv \frac{\cos(\delta + kb)}{|T_1|}. \tag{4.10}$$

Clearly $\alpha \to \alpha_0$ as $E_i \to 0$ which corresponds to $b/d \to \infty$ and this is equivalent to adopting the classical wide-spacing approximation (WSA) from the outset. Parameter values such that $|\cos\alpha| < 1$ are said to lie in a pass band which indicates wave transmission through an infinite periodic array is possible. Other values lie in a stop band and transmission through the array is not possible. See, for example, Linton & McIver (2001, eqn. (6.52)), or consider an extended version of (4.5) which is allowed to hold for all integers n and satisfied by solutions of the form $F_0^{(n)} = C_\pm \mathrm{e}^{\pm\mathrm{i}n\alpha}$ for arbitrary $C_\pm$.

The solution of (4.5) satisfying (4.7) is, for $n = 0,1,2,\ldots,N-1$,

$$F_0^{(n)} = F_0^{(N-1)}\big\{(p+\mathrm{i})U_{N-n-2} - qU_{N-n-3}\big\}/q \tag{4.11}$$

where $U_n(\alpha) = \sin(n+1)\alpha/\sin\alpha$, so that $U_{-2} = -1$, $U_{-1} = 0$, $U_0 = 1$. Thus from (2.7)

$$R_N = 1 + \mathrm{i}F_0^{(0)} = \frac{qF_0^{(1)} - (p-\mathrm{i})F_0^{(0)}}{qF_0^{(1)} - (p+\mathrm{i})F_0^{(0)}}$$

after using (4.7). Now (4.11) can be used to show

$$R_N = \frac{(p^2+1)U_{N-2} - 2pqU_{N-3} + q^2U_{N-4}}{(p+\mathrm{i})^2U_{N-2} - 2q(p+\mathrm{i})U_{N-3} + q^2U_{N-4}}. \tag{4.12}$$

Similarly

$$T_N = \frac{2\mathrm{i}q}{(p+\mathrm{i})^2U_{N-2} - 2q(p+\mathrm{i})U_{N-3} + q^2U_{N-4}}. \tag{4.13}$$

The above expressions turn out to hold for $N = 2$ also where only (4.6) and (4.7) are required. The energy condition $|R_N|^2 + |T_N|^2 = 1$ can be shown to be satisfied exactly but requires considerable algebra to do so.

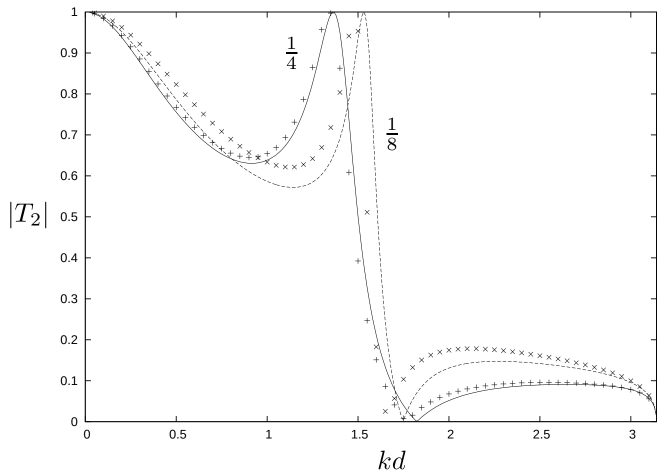

Fig. 3 Comparision of exact results for $|T_2|$ against kd (curves) against small-gap approximation (symbols) for two screens with $a/d = 0.1$ and for $b/d = \frac{1}{4}$ and $\frac{1}{8}$.

5. Results

5.1 *Two screens*

We start by assessing the accuracy of the method in a case where it is simple to check against exact results, namely when $N = 2$ where the methods of Porter & Evans (1996) can be used. Based on the agreement shown for a single screen in Fig. 2, we adopt a gap size of $a/d = 0.1$. Fig. 3 shows the variation of $|T_2|$ against kd up to the first channel cut off for two screen separations, $b/d = \frac{1}{4}$ and $b/d = \frac{1}{8}$. The curves show the exact computations based on Porter & Evans (1996) and the symbols are the values computed using the explicit expression (4.13). For values of b/d greater than $\frac{1}{4}$ the agreement between the exact results and those computed from (4.13) rapidly improves. Fig. 3 does show that the agreement is lost when b/d reduces in size and is comparable to a/d. The loss of agreement is likely to be because the small-gap assumption ($a/d \ll 1$) is forced to compete with a comparible dimensionless lengthscale (i.e. b/d) which has not been subjected to a similar or consistent approximation. Thus the conclusion is that the small-gap assumption is reliable for $b \gg a$.

Fig. 3 also illustrates two features of the results that we are interested in: total reflection and total transmission.

5.2 *Total reflection*

We turn our attention first to the more obvious result to interpret though not the main focus of the paper, that of total reflection. The vanishing of the transmission coefficient is a relatively unusual phenomenon. The first example in the water wave problem was shown by Evans & Morris (1972) in considering the scattering of waves by a pair of partially-immersed vertical screens.

Thus it is clear from (4.13) that $T_N = 0$ if $q = 0$ for all $N > 1$ which is obvious on physical grounds. Less obvious is the fact that the condition is independent of the smallness of the

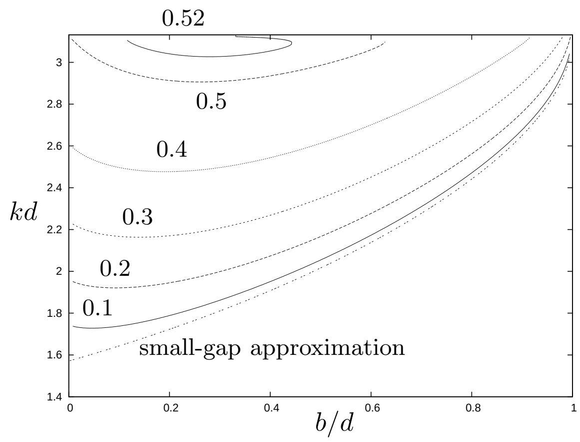

Fig. 4 Computations for a pair of screens showing curves relating kd to b/d upon which $T_2 = 0$. Each curve is sketched for a different value of a/d (shown on figure). The lowest chained curve represents solutions to (38) based on the small-gap assumption.

gap. Now from (4.8) and (4.4), $q = 0$ implies

$$\sin(\kappa\lambda) = \left[\sum_{r=1}^{\infty} \frac{2\kappa}{(r^2\pi^2 - \kappa^2)^{1/2} \sinh \lambda (r^2\pi^2 - \kappa^2)^{1/2}}\right]^{-1}, \tag{5.1}$$

where we have written $\kappa = kd$ and $\lambda = b/d$.

It is clear that provided the right-hand side of (5.1) is less than unity, solutions of the form $\lambda(\kappa)$ exist, but that there will be a cut-off at, say, $\lambda = \lambda_c(\kappa)$ above which no solution is possible. The existence of a cut-off is consistent with the wide-spacing approximation which is valid for large λ and therefore predicts no solution. Solutions of (5.1) are included in Fig. 4 as the lowest (chained) curve illustrating that this critical value is $\lambda_c = 1$. It is not difficult to confirm this analytically by showing that if $\kappa = \pi - \epsilon$ where $\epsilon \ll 1$ then $\lambda = 1 - \epsilon^2\sigma$ where $\sigma = \frac{3}{2}\pi^{-2} - 2\sum_{n=1}^{\infty}(n\pi \sinh(n\pi))^{-1} = 0.0956\ldots$

On the other hand when $\lambda \to 0$ then (5.1) can be approximated by

$$\kappa\lambda \approx \left(\frac{1}{\lambda}\sum_{r=1}^{\infty}\frac{2\kappa}{r^2\pi^2 - \kappa^2}\right)^{-1} = \left(\frac{1}{\lambda}\left(\frac{1}{\kappa} - \cot\kappa\right)\right)^{-1}$$

which implies $\kappa \to \frac{1}{2}\pi$, again confirmed in Fig. 4.

Total reflection by multiple closely-spaced screens is not a new result. In Porter & Evans (1996) it was shown, using exact computations with no assumption of a small gap, that a pair of sufficiently closely-spaced screens were able to reflect all incident wave energy at certain frequencies. This phenomenon is evident in Fig. 3. What was not discussed in that paper was the entire range of values of spacings and gap sizes for which total reflection could occur. One reason for coming back to this question here is to consider the possibility of what one might call 'extraordinary reflection'. This term could be used to describe a situation in which the introduction of a small geometric perturbation results in total reflection when otherwise there would be total transmission.

Results in Fig. 4 show that as $a/d \to 0$ the exact results tend to those predicted by the small-gap approximation (5.1) which, as already mentioned, is independent of a/d. As a/d increases in size, the values of kd at which $T_2 = 0$ increase away from the approximation of (5.1) towards the channel cut-off at $kd = \pi$. The results shown in Fig. 4 in the range $b/d < 1$, are consistent with the previous observation that the small-gap approximation requires b to be significantly greater than a for good agreement with exact results. Zeros of transmission are lost as a/d increases just beyond the value of 0.52 as shown in Fig. 4. In fact, numerically, it is found that the zero exists up to $a/d \approx 0.5255$ where $b/d \approx 0.28$, $kd \approx 3.092$. In conclusion, it has been shown that it is not possible to decrease the size of the screens indefinitely towards zero and retain the property of total reflection.

5.3 *Total transmission*

It is the phenomenon of total transmission which is the main interest in this paper. This is a far more common occurrence than total reflection and one which frequently occurs when there are two or more sources of scattering. Nevertheless, our interest here stems from the fact that transmission through a single screen is limited by the small-gap assumption.

Thus the numerator in (4.12) is real so it can be expected that $R_N = 0$ for certain values of p, q and α. It is informative to consider, as a special case, wide screen spacings when E_i in (4.4) tend to zero. After considerable algebra it can be shown that (4.12) and (4.13) become

$$R_N = \frac{U_{N-1}R_1}{U_{N-1} - T_1 \mathrm{e}^{\mathrm{i}kb} U_{N-2}} \tag{5.2}$$

$$T_N = \frac{T_1}{U_{N-1} - T_1 \mathrm{e}^{\mathrm{i}kb} U_{N-2}} \tag{5.3}$$

in agreement with Martin (2014, eqn. (21)). These are the wide-spacing approximations (WSAs) to R_N and T_N.

The WSA condition $R_N = 0$ for total transmission is now simply $U_{N-1}(\alpha_0) = 0$ or $\alpha_0 = m\pi/N$, $m = 1, 2, \ldots, N-1$ so that from (4.10)

$$\cos\frac{m\pi}{N} = \frac{\cos(\delta + kb)}{\cos\delta}, \qquad m = 1, 2, \ldots, N-1. \tag{5.4}$$

Thus for example, for $N = 2$ we have $\cos(\delta + kb) = 0$ or $\delta + kb = (2p-1)\pi/2$, p an integer.

The result (5.4) shows that a WSA approximation provides $N-1$ equations to determine when $R_N = 0$ and $|T_N| = 1$ for each region in which $\cos\alpha_0 < 1$ and we are in a pass-band, and we might expect that to be the case generally when the numerator of (4.12) vanishes. This is confirmed in Fig. 5 where $N = 4$ and $a/d = 0.1$ throughout, on the basis of the good agreement demonstrated in Fig. 2. The solid lines show, in $(kd, b/d)-$space, where $R_4 = 0$ and the crosses are based on the WSA. Thus for example for $kd = 1$ as b/d increases there are three different spacings at which $R_4 = 0$ followed by a gap before a further three cut in and so on. Alternatively, Fig. 5 shows that at a given spacing there are three distinct wavenumbers for which total transmission occurs with further groups of three at higher frequencies occurring for larger spacings. It is also clear that for most purposes the WSA is entirely adequate in predicting the results. The solution for $b/d \lesssim a/d$ is less clear and should be disregarded as it conflicts with the small-gap approximation. Also shown in Fig.

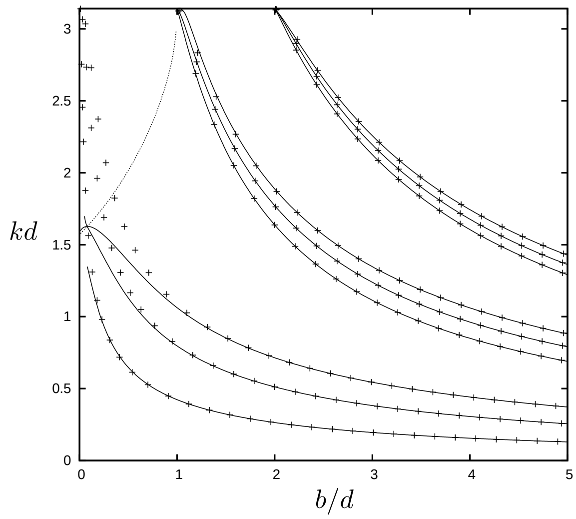

Fig. 5 Solid curves/points show $R_4 = 0$ for $N = 4$ screens with $a/d = 0.1$ using exact/WSA theory. The dotted line is $T_4 = 0$.

5 is a dotted line on which $T_4 = 0$ derived from (5.1), is identical to the lowest curve in Fig. 4 and which has no counterpart in a WSA.

Fig. 6 shows a plot of $|R_4|$ against b/d based on the exact small-gap theory with the WSA results overlaid. It shows clearly the triplets of zeros of R_4 separated by stop bands.

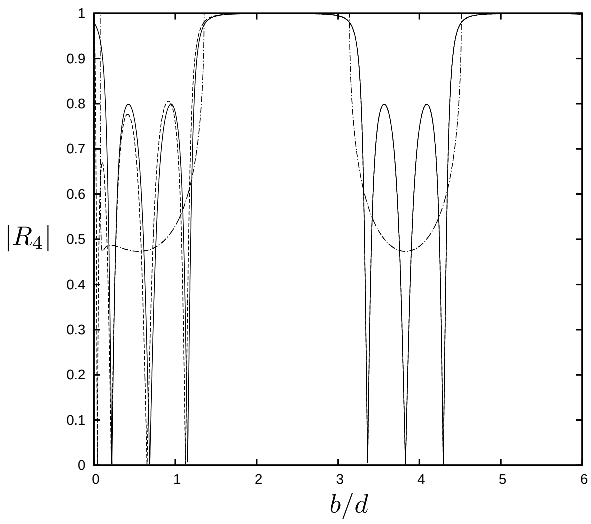

Fig. 6 Solid/dashed lines are exact/WSA results for $N = 4$ screens with $a/d = 0.1$ and $kd = 1$. The chained line is $|R_\infty|$.

5.4 *Reflection by a semi-infinite periodic array*

It is possible to consider scattering by a semi-infinite array of screens by overlooking the end condition (4.7) and assuming $F_0^{(n)} = C_{\pm}\mathrm{e}^{\pm in\alpha}$ as a solution of (4.5) whence (4.6) gives

$$R_\infty = \frac{q\mathrm{e}^{\pm \mathrm{i}\alpha} - (p - \mathrm{i})}{q\mathrm{e}^{\pm \mathrm{i}\alpha} - (p + \mathrm{i})} \tag{5.5}$$

the sign chosen so that $|R_\infty|$ does not exceed unity. This subtlety is associated with the satisfaction of the radiation condition far into the array which must ensure that the direction of energy propagation is in the positive x-direction. In this case, α assumes the role of a Bloch wavenumber and a phase relation $\mathrm{e}^{\mathrm{i}\alpha}$ from the nth to the $(n+1)$th screen far into the semi-infinite periodic array does not imply that the the energy is propagating in the positive x-direction. Instead, one has to consider the slope of the curve $d\alpha/dk$ which indicates the sign of the group velocity and hence the direction of energy propagation.

We remark the that WSA counterpart of (5.5) requires replacing α by α_0.

Fig. 6 includes a chained curve showing $|R_\infty|$ computed from (5.5). It is clear that $|R_\infty| = 1$ when there is a stop band and α becomes imaginary. Within a pass band, where α is real the reflection coefficient, in some loose sense, represents the average of the the oscillatory behaviour experienced for large N.

6. Conclusions

This paper has considered the phenomenon of total transmission of wave energy through small gaps in infinitesimally-thin screens spanning a waveguide. It is shown that there are frequencies at which total transmission exist when there are two or more parallel screens with small gaps. The solution is computed using a small-gap approximation to simplify the kernel of coupled systems of integral equations, a vital step in the reductio to linear difference equations. Consequently reflection and transmission coefficients are expressed in closed form and the analysis of the solutions reveals the structure of zeros of transmission and their connection to stop and pass bands in the corresponding infinite periodic system. When two screens are placed close enough together there also exist frequencies at which total reflection occurs even when the gaps occupy nearly half the width of the channel. Total transmission cannot occur when waves are incident on a semi-infinite periodic array of screens with small gaps.

Prof. John Chaplin (private communication) has performed experiments on water wave propagation in his narrow wave tank at the University of Southampton with a single screen having a gap occupying 10% of the tank width. However, the sharp edges of the thin screens give rise to significant viscous losses due to vortex shedding. Thus the results presented here have only theoretical interest in the setting of water waves and should be applied to electromagnetics or possibly acoustics.

It should be possible to make a similar small-gap approximation to the problem considered by Porto *et al* (1999) and Lu *et al* (2007) by setting up the solution using the integral equation approach of Fernhyough & Evans (1995). Similarly, for rows of circular cylinders considered by Linton & Evans (1993) it would be interesting to examine solutions for this arrangement in the limit of small gaps.

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