Calculation of the hydrodynamic coefficients for cylinders of rectangular cross-section oscillating in the free surface

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Abstract

In this technical report we describe the method of solution to a variety of two-dimensional problems involving semi-immersed cylinders of rectangular cross-section piercing the free surface of a fluid of constant finite depth h. The aim is to be able to calculate the various components of added mass and radiation damping coefficients which can be used to characterise the two-dimensional time-harmonic oscillations of a cylinder in waves. The problems of a single cylinder making forced oscillations in sway, heave and roll in fluid of unbounded horizontal extent are considered first. Similar calculations are then made for a cylinder in forced motion placed next to a vertical wall.

1 Introduction

In this technical report we consider the forced motion of a cylinder of rectangular cross-section which extends through the free surface of a heavy fluid. The fluid is of finite depth and the motion of the cylinder and surrounding fluid is two-dimensional.

In the first part we consider a cylinder in isolation, meaning that there is a free surface either side of the cylinder extending towards infinity. In the second part, we place the same cylinder to the right of a vertical wall exteneding through the depth of the fluid. In both situations, we are interested in calculating the hydrodynamic coefficients that can be used to characterise the small-amplitude time-harmonic motion of the cylinder. These are the non-dimensional added-mass and radiation damping coefficients, being the real and imaginary components of the complex hydrodynamic force (or moment) on the cylinder. They apply to each of the three independent modes of motion of the cylinder in a two-dimensional plane are (in the terminology of ship hydrodynamics) sway, heave and roll. These refer, respectively, to translational oscillations in the horizontal and vertical directions with respect to a fixed frame of reference and the rolling oscillations about a fixed point.

The report has been written to accompany the paper of Evans & Porter (2008) in which approximations are developed for the hydrodynamic coefficients for a cylinder next to a wall in terms of those in the absence of a wall. The theoretical expressions derived in Evans & Porter (2008) are tested numerically in the cases of circular semi-immersed cylinders in addition to cylinders of rectangular cross-section.

This report contains the solution to six different problems. They are the forced sway, heave and roll of a cylinder in isolation and in the presence of a wall. The method of solution in each of the six problem is similar to the other five. The details do, however, vary to the extent that the solution to each problem but must presented separately. The solution to the first problem is outlined in some detail, and thereafter it is assumed that the reader should be able to follow the general approach that has been adopted and apply that approach to later problems. Though the mathemtical details become increasingly sparse towards the end of the report, there should be enough detail given to be able to reproduce results. It should be noted that the problem of rectangular cylinders in heave only has previously been the subject of a paper by Porter & Evans (2008), although we report a slightly different formulation of the solution here. Though we present an outline of the numerical method used to produce results, we do not present any results within this report. Some typical results are available in the paper of Evans & Porter (2008).

1.1 Definition of the problem

Cartesian coordinates are used with the origin in the undisturbed free surface, and y vertically downwards. When at rest, the submerged cross-section of the cylinder occupies $b - a \le x \le b + a$, $0 \le y \le d$. The fluid is of density ρ and of finite depth, h. The mass of the cylinder is $M = 2\rho ad$ (by Archimedes' principle). The roll axis of the cylinder is given by $(x, y) = (b, c)$. In section 2 when we consider a cylinder in isolation, we take $b = 0$. In section 3, in which the cylinder is situated next to a wall, the wall is located along $x = 0$, for $0 < y < h$, with $b > a$, so that the cylinder is separated from the wall.

The motion is two-dimensional and time-harmonic with angular frequency ω so that the velocity potential is given by $\Re\{\phi(x, y)e^{-i\omega t}\}$. The linearised free surface condition is $\phi_y + K\phi = 0$ on $y = 0$ where $K = \omega^2/g$ and g is gravity. At the bottom of the fluid, $y = h$, we impose $\phi_y = 0$. In the fluid, ϕ is harmonic. Other boundary conditions apply on the cylinder and are stated at the appropriate point in the text.

2 A swaying, heaving and rolling cylinder in isolation

As a reminder, we take $b = 0$ throughout this section so that the cylinder occupies the region $-a < x < a$, $0 < y < d$ of the fluid when at rest. When in motion, the linearised kinematic boundary conditions apply on $x = \pm a$, $0 < y < d$ and $y = d$, $-a < x < a$. In the construction of each of the solutions below, advantage is taken of the geometric symmetry of the rectangular cylinder to divide the solution to the hydrodynamic problem into components which are symmetric and antisymmetric about the centreline $x = 0$. For swaying and rolling cylinders, only the antisymmetric problem features and for the heaving cylinder, it is only the symmetric problem which features. Thus boundary conditions need only be placed on half of the cylinder surface in $x > 0$ with the appropriate condition placed on the line $x = 0$, $d < y < h$ extending vertically throughout the fluid. That is, for symmetric (antisymmetric) solutions about $x = 0$, we impose Neumann (Dirichlet) conditions on $x = 0$ for $d < y < h$. Accordingly, solutions are sought only in $x > 0$, and can be extended into $x < 0$ appropriately.

The superscripts s, h and r are used to identify problems associated with sway, heave and roll. Functions X, Y and Z with F, G and H and coefficients P_{ij}, S_{ij} and T_{ij} (in section 3, matrices $\mathbf{P}_{ij}$, $\mathbf{S}_{ij}$ and $\mathbf{T}_{ij}$) are used respectively for the sway, heave and roll problems.

2.1 Boundary conditions

For a swaying cylinder, we require

$$\phi^s_x = 1, \qquad \text{on } x = a,\ 0 < y < d, \tag{2.1}$$
$$\phi^s_y = 0, \qquad \text{on } y = d,\ 0 < x < a, \tag{2.2}$$
$$\phi^s = 0, \qquad \text{on } x = 0,\ d < y < h; \tag{2.3}$$

for a heaving cylinder we require

$$\phi^h_x = 0, \qquad \text{on } x = a,\ 0 < y < d, \tag{2.4}$$
$$\phi^h_y = 1, \qquad \text{on } y = d,\ 0 < x < a, \tag{2.5}$$
$$\phi^h_x = 0, \qquad \text{on } x = 0,\ d < y < h; \tag{2.6}$$

for a cylinder rolling about $y = c$ we have

$$\phi^r_x = (y - c), \qquad \text{on } x = a,\ 0 < y < d, \tag{2.7}$$
$$\phi^r_y = -x, \qquad \text{on } y = d,\ 0 < x < a, \tag{2.8}$$
$$\phi^r = 0, \qquad \text{on } x = 0,\ d < y < h, \tag{2.9}$$

2.2 Notation and definitions

Before embarking on the solutions we introduce the reader to some standard notation associated with water wave problems in finite depth. See Linton & McIver (2001) for more details.

The depth eigenfunctions associated with water of depth h with a free surface are defined by

$$\psi_n(y) = N_n^{-1/2} \cos k_n(h - y), \qquad N_n = \frac{1}{2}\left(1 + \frac{\sin 2k_nh}{2k_nh}\right) \tag{2.10}$$

with $K = \omega^2/g = -k_n \tan k_nh$ defining real $k_n \in ((n - \frac{1}{2})\pi, n\pi)$ for $n \geq 1$ and $k_0 \equiv -\mathrm{i}k$, k positive and real defined by $K = k \tanh kh$.

In the region under the rectangle, the eigenfunctions associated with a homogeneous Neumann condition on $y = d$ and $y = h$ are given by $\hat{\psi}_n(y) = \epsilon_n^{1/2} \cos \mu_n(y - h)$ and $\mu_n = n\pi/(h - d)$, $\epsilon_0 = 1$, $\epsilon_n = 2$, $n \geq 1$.

Both sets of eigenfunctions are orthogonal, satisfying the relations

$$\langle \hat{\psi}_n, \hat{\psi}_m \rangle \equiv \int_d^h \hat{\psi}_n(y)\hat{\psi}_m(y)\mathrm{d}y = (h-d)\delta_{mn}, \qquad \text{and} \qquad \int_0^h \psi_n(y)\psi_m(y)\mathrm{d}y = h\delta_{mn}. \tag{2.11}$$

Of continued use in this report are the quantities

$$L_0 \equiv \int_0^d \psi_0(y)\mathrm{d}y = N_0^{-1/2}\left[\frac{\sinh kh - \sinh k(h - d)}{k}\right] \tag{2.12}$$

and

$$L_n \equiv \int_0^d \psi_n(y)\mathrm{d}y = N_n^{-1/2}\left[\frac{\sin k_nh - \sin k_n(h - d)}{k_n}\right] \tag{2.13}$$

in addition to

$$M_0 \equiv \int_0^d (y-c)\psi_0(y)\mathrm{d}y$$
$$= \frac{-1}{N_0^{1/2}k}\left[(d-c)\sinh k(h-d) + c\sinh kh + \frac{\cosh k(h-d)}{k} - \frac{\cosh kh}{k}\right] \quad (2.14)$$

and

$$M_n \equiv \int_0^d (y-c)\psi_n(y)\mathrm{d}y$$
$$= \frac{-1}{N_n^{1/2}k_n}\left[(d-c)\sin k_n(h-d) + c\sin k_n h - \frac{\cos k_n(h-d)}{k_n} + \frac{\cos k_n h}{k_n}\right]. \quad (2.15)$$

2.3 A single cylinder in sway

Central to the method of solution are the particular solutions that we employ to deal with the inhomogeneous boundary conditions in each of the problems above (see equations (2.1)–(2.9)). These are designed in a piecewise fashion to fit in with the decomposition of the potential into rectangular subdomains of the fluid. We label region I, $0 < x < a$, $d < y < h$, and region II, $x > a$, $0 < y < h$ and use superscripts I and II to denote that the potentials are being defined in these regions.

For the swaying problem, we define a potential

$$X^{II} = -\frac{\mathrm{i}L_0}{kh}\psi_0(y)\mathrm{e}^{\mathrm{i}k(x-a)} - \sum_{n=1}^{\infty}\frac{L_n}{k_n h}\mathrm{e}^{-k_n(x-a)}\psi_n(y) \quad (2.16)$$

which satisfies $X_x^{II} = 1$ on $x = a$, $0 < y < d$ (as required by (2.1)) and also $X_x^{II} = 0$ on $x = a$, $d < y < h$ (this is not required by the problem, but chosen because it assists later algebraic manipulations) in addition to the required free surface and bottom conditions. In addition, note that on $x = a$,

$$X^{II}(a,y) = -\frac{\mathrm{i}L_0}{kh}\psi_0(y) + F^{II}(y), \qquad \text{where} \qquad F^{II}(y) = -\sum_{n=1}^{\infty}\frac{L_n}{k_n h}\psi_n(y). \quad (2.17)$$

With this we define the potentials in I and II for the swaying cylinder as, in II

$$\phi^s(x,y) = X^{II}(x,y) + (a_0^s + \mathrm{i}L_0/kh)\mathrm{e}^{\mathrm{i}k(x-a)}\psi_0(y) + \sum_{n=1}^{\infty} a_n^s \mathrm{e}^{-k_n(x-a)}\psi_n(y) \quad (2.18)$$

and in I

$$\phi^s(x,y) = b_0^s x + \sum_{n=1}^{\infty} b_n^s \sinh(\mu_n x)\hat{\psi}_n(y) \quad (2.19)$$

where $\mu_n = n\pi(h-d)$. These expansions are 'general solutions' to the problems, in which the inhomogeneity has been taken care of by the particular solution. It remains to determine the coefficients a_n^s and b_n^s. This is done by matching pressure and velocity across the common interface $x = a$, $d < y < h$ and by application of the boundary condition along $x = a$, $0 < y < d$ of the cylinder.

Equating velocities gives

$$U^s(y) \equiv \phi_x^s(a,y) = \begin{cases} X_x^{II}(a,y) + \mathrm{i}k(a_0^s + \mathrm{i}L_0/kh)\psi_0(y) - \sum_{n=1}^{\infty} k_n a_n^s \psi_n(y) \\ b_0^s \hat{\psi}_0(y) + \sum_{n=1}^{\infty} \mu_n b_n^s \cosh(\mu_n a)\hat{\psi}_n(y) \end{cases} \tag{2.20}$$

where $U^s(y)$ is an unknown function representing the horizontal component of velocity through the depth. Application of the orthogonality conditions gives

$$\mathrm{i}kha_0^s - L_0 = \langle U^s, \psi_0 \rangle, \qquad -k_n h a_n^s = \langle U^s, \psi_n \rangle \tag{2.21}$$

and

$$b_0^s(h-d) = \langle U^s, \hat{\psi}_0 \rangle, \qquad \mu_n(h-d) b_n^s \cosh(\mu_n a) = \langle U^s, \hat{\psi}_n \rangle. \tag{2.22}$$

Next, matching pressures gives

$$b_0^s a\hat{\psi}_0(y) + \sum_{n=1}^{\infty} b_n^s \sinh(\mu_n a)\hat{\psi}_n(y) = F^{II}(y) + a_0^s \psi_0(y) + \sum_{n=1}^{\infty} a_n^s \psi_n(y) \tag{2.23}$$

and then substitution for the coefficients in the above from (2.21), (2.22) gives

$$\int_d^h U^s(t) K_{asym}(y,t)\mathrm{d}t = a_0^s \psi_0(y) - b_0^s a\hat{\psi}_0(y) + df_3^s(y) \tag{2.24}$$

where

$$K_{asym}(y,t) = \sum_{n=1}^{\infty} \left[\frac{\psi_n(t)\psi_n(y)}{k_n h} + \frac{\hat{\psi}_n(t)\hat{\psi}_n(y)}{n\pi \coth \mu_n a} \right] \tag{2.25}$$

and $f_3^s(y) = F^{II}(y)/d$, with $f_1^s = \psi_0(y)$, $f_2^s = \hat{\psi}_0(y)$. Thus we have an integral equation for $U^s(y)$; one forced by a linear combination of three linearly-independent functions and two unknown coefficients.

We proceed by letting

$$U^s(t) = d\{\hat{a}_0^s u_1^s + \hat{b}_0^s u_2^s + u_3^s\} \tag{2.26}$$

where $\hat{a}_0^s = a_0/d$ and $\hat{b}_0^s = -b_0 a/d$ in which the functions $u_i^s(y)$ satisfy

$$\int_d^h u_i^s(t) K_{asym}(y,t)\mathrm{d}t = f_i^s(y), \qquad i = 1,2,3. \tag{2.27}$$

We define $P_{ij} = \langle f_i^s, u_j^s \rangle$. Then we insert (2.26) into the first equations in (2.21) to give

$$\mathrm{i}kh\hat{a}_0^s = (L_0/d) + \hat{a}_0^s P_{11} + \hat{b}_0^s P_{12} + P_{13} \tag{2.28}$$

and, similarly, into the first equation in (2.22) to give

$$-\hat{b}_0^s(h-d)/a = \hat{a}_0^s P_{12} + \hat{b}_0^s P_{22} + P_{23}. \tag{2.29}$$

These two equations (2.28), (2.29) can be solved for the unknown coefficients $\hat{a}_0^s$ and $\hat{b}_0^s$.

The sway component of the sway induced added-mass (A_{11}) and radiation damping (B_{11}) is defined as

$$\mathrm{i}\omega(A_{11} + \mathrm{i}B_{11}/\omega) = -2\mathrm{i}\omega\rho \int_0^d \phi^s(a,y)\mathrm{d}y \tag{2.30}$$

(the factor of 2 on the right-hand side is because we need to include the contribution from $x < 0$).

We do not evaluate the right-hand side integral above directly Instead, we apply Green's Identity over region II to ϕ^s and X^{II}, employing all the appropriate boundary conditions on the four sides of the semi-infinite rectangle. The contribution all but one side, $x = a$ and hence we are left with the identity

$$\int_0^h (\phi^s X_x^{II} - X^{II}\phi_x^s)\Big|_{x=a} \mathrm{d}y = 0. \tag{2.31}$$

Using the appropriate definitions of ϕ^s, X^{II} and their derivatives on each section of the interval from $0 < y < d$ and $d < y < h$ gives

$$\int_0^d \phi^s(a,y)\mathrm{d}y = \mathcal{X}^{II} + \int_d^h U^s(y)X^{II}(a,y)\mathrm{d}y \tag{2.32}$$

where, from using (2.17) with (2.12), (2.13),

$$\mathcal{X}^{II} = -\frac{\mathrm{i}L_0^2}{kh} - \sum_{n=1}^{\infty} \frac{L_n^2}{k_n h}. \tag{2.33}$$

The final integral in (2.32) above evaluates, using (2.17), to

$$\int_d^h U^s(y)X^{II}(a,y)\mathrm{d}y = -\frac{\mathrm{i}L_0}{kh}\langle U^s, \psi_0\rangle + d\langle U^s, f_3^s\rangle. \tag{2.34}$$

Bringing (2.33), (2.34) together in (2.32) and substituting in to (2.30) now gives

$$(A_{11} + \mathrm{i}B_{11}/\omega) = -2\rho d^2 \left[\hat{a}_0^s(L_0/d) - \sum_{n=1}^{\infty} \frac{(L_n/d)^2}{k_n h} + \hat{a}_0^s P_{31} + \hat{b}_0^s P_{32} + P_{33}\right]. \tag{2.35}$$

The non-dimensional added-mass and damping are defined as $\mu_{11} = A_{11}/M$ and $\nu_{11} = B_{11}/(\omega M)$ where M has dimensions of mass (we use the mass of the cylinder, assumed to float when at rest so that $M = 2\rho a d$ by Archimedes' principle).

2.4 A single cylinder in heave

We define a particular solution for the heaving problem, in region I, by

$$Y^I(x,y) = -\frac{1}{2}\frac{(h-y)^2}{(h-d)} + \frac{1}{2}\frac{(x^2-a^2)}{(h-d)} \tag{2.36}$$

(this is not a unique definition, but one which serves to simplify the algebra later on) and is such that

$$Y_x^I(a,y) = \frac{a}{(h-d)}; \qquad Y^I(a,y) = -G^I(y) \qquad \text{where} \qquad G^I(y) = \frac{1}{2}\frac{(h-y)^2}{(h-d)}. \tag{2.37}$$

Then the expansion for the potential in region I is written

$$\phi^h(x,y) = Y^I(x,y) + b_0^h + \sum_{n=1}^{\infty} b_n^h \cosh(\mu_n x)\hat{\psi}_n(y) \tag{2.38}$$

where Y^I accounts for the inhomogeneous boundary condition on $y = d$ and the remaining terms satisfy homogeneous boundary conditions there and elsewhere. In region II, the potential is expanded as

$$\phi^h(x,y) = a_0^h \mathrm{e}^{\mathrm{i}k(x-a)}\psi_0(y) + \sum_{n=1}^{\infty} a_n^h \mathrm{e}^{-k_n(x-a)}\psi_n(y). \tag{2.39}$$

The remainder of the solution follows in a manner very similar to in the previous section for sway. Thus, after defining $U^h(y) = \phi_x^h(a,y)$, for $d < y < h$, we have

$$\mathrm{i}kha_0^h = \langle U^h, \psi_0\rangle, \qquad -k_n h a_n^h = \langle U^h, \psi_n\rangle, \quad n \geq 1 \tag{2.40}$$

with

$$\mu_n(h-d)b_n^h \sinh(\mu_n a) = \langle U^h, \hat{\psi}_n\rangle, \qquad \text{and} \qquad \langle U^h, \hat{\psi}_0\rangle = a \tag{2.41}$$

(the latter equation incidentally an expression of conservation of flux). Then, matching ϕ^h across $x = a$, $d < y < h$ gives

$$\int_d^h U^h(t) K_{sym}(y,t)\mathrm{d}t = a_0^h\psi_0(y) - b_0^h\hat{\psi}_0(y) + G^I(y) \tag{2.42}$$

where $K_{sym}(y,t)$ is

$$K_{sym}(y,t) = \sum_{n=1}^{\infty}\left[\frac{\psi_n(t)\psi_n(y)}{k_n h} + \frac{\hat{\psi}_n(t)\hat{\psi}_n(y)}{n\pi\tanh\mu_n a}\right]. \tag{2.43}$$

Now we write

$$U^h(t) = a\{\hat{a}_0^h u_1^h(t) + \hat{b}_0^h u_2^h(t) + ((h-d)/a)u_3^h(t)\} \tag{2.44}$$

where $\hat{a}_0^h = a_0^h/a$ and $\hat{b}_0^h = b_0^h/a$ and solve

$$\int_d^h u_i^h(t) K_{sym}(y,t)\mathrm{d}t = f_i^h(y), \qquad i = 1,2,3 \tag{2.45}$$

with $f_1^h(y) = \psi_0(y)$, $f_2^h(y) = \hat{\psi}_0(y) \equiv 1$ as before and $f_3^h = G^I(y)/(h-d)$. Then, defining

$$S_{ij} = \langle f_i^h, u_j^h\rangle, \qquad i,j = 1,2,3 \tag{2.46}$$

we have

$$\mathrm{i}kh\hat{a}_0^h = \hat{a}_0^h S_{11} + \hat{b}_0^h S_{12} + ((h-d)/a)S_{13} \tag{2.47}$$

with

$$1 = \hat{a}_0^h S_{21} + \hat{b}_0^h S_{22} + ((h-d)/a)S_{23} \tag{2.48}$$

these two equations are used to determine the non-dimensional coefficients $\hat{a}_0^h$, $\hat{b}_0^h$.

All that remains is to calculate the heave component of the heave induced added mass and radiation damping, A_{22} and B_{22}, defined as

$$\mathrm{i}\omega(A_{22} + \mathrm{i}B_{22}/\omega) = -2\mathrm{i}\omega\rho \int_0^a \phi^h(x,d)\mathrm{d}x. \tag{2.49}$$

Application of Green's Identity to the harmonic functions ϕ^h and Y^I in region I (i.e. $d < y < h$, $0 < x < a$) gives, after using the known definitions of those functions and their derivatives on the boundaries of the rectangular domain,

$$\int_0^a \phi^h(x,d)\mathrm{d}x = \int_0^a Y^I(x,d)\mathrm{d}x + \frac{a}{(h-d)}\langle \phi^h, \hat{\psi}_0\rangle + (h-d)\langle U^h, f_3^h\rangle. \tag{2.50}$$

In the above the first term evaluates to

$$\mathcal{Y}^I \equiv \int_0^a Y^I(x,d)\mathrm{d}x = -a^2\left[\frac{(h-d)}{2a} + \frac{a}{3(h-d)}\right] \tag{2.51}$$

whilst the middle term is

$$\frac{a}{(h-d)}\langle \phi^h, \hat{\psi}_0\rangle = -a^2\hat{b}_0^h - \frac{a}{(h-d)}\langle G^I, \hat{\psi}_0\rangle = -a^2\hat{b}_0^h - \tfrac{1}{6}a(h-d). \tag{2.52}$$

Then we find that

$$(A_{22} + \mathrm{i}B_{22}/\omega) = 2\rho a^2\left[\frac{1}{3}\left(\frac{a}{h-d}\right) + \hat{b}_0^h - \frac{(h-d)}{a}\left[a_0^h S_{31} + b_0^h S_{32} + ((h-d)/a)S_{33} + \tfrac{2}{3}\right]\right] \tag{2.53}$$

and the non-dimensional coefficients are $\mu_{22} = A_{22}/M$ and $\nu_{22} = B_{22}/(\omega M)$.

2.5 A single cylinder in roll

This is more complicated than the previous two problems, because there are inhomogeneous boundary conditions on all both vertical and horizontal sides of the rectangular cylinder.

In the region I, $\{0 < x < a,\ d < y < h\}$ we define a particular solution accounting for the inhomogeneous boundary condition on $y = d$, $Z^I(x,y)$, satisfying $\nabla^2 Z^I = 0$, in I with the four boundary conditions $Z_y^I(x,h) = 0$, $Z_y^I(x,d) = -x$, $Z^I(0,y) = 0$ and $Z_x^I(a,y) = 0$. The solution is readily determined by separation of variables and is given by

$$Z^I(x,y) = 2\sum_{n=0}^{\infty} \frac{(-1)^n \sin(\lambda_n x)\cosh\lambda_n(h-y)}{\lambda_n^3 a \sinh\lambda_n(h-d)} \tag{2.54}$$

where $\lambda_n = (n+\frac{1}{2})\pi/a$. Quantities of specific interest are

$$H^I(y) = Z^I(a,y) = 2\sum_{n=0}^{\infty} \frac{\cosh\lambda_n(h-y)}{\lambda_n^3 a \sinh\lambda_n(h-d)} \tag{2.55}$$

in addition to

$$Z^I(x,d) = 2\sum_{n=0}^{\infty} \frac{(-1)^n \coth\lambda_n(h-d)}{\lambda_n^3 a}\sin(\lambda_n x) \tag{2.56}$$

In region II, being $\{x > a,\ 0 < y < h\}$, the function $Z^{II}(x,y)$ is designed to absorb the inhomogeneous boundary condition on $x = a$ is harmonic and satisfies the conditions $Z_x^{II}(a,y) = (y-c)$ on $x = a$ for $0 < y < d$ whilst $Z_x^{II}(a,y) = 0$ for $d < y < h$. As $x \to \infty$, only outgoing waves are possible. Again, the solution is straightforward to determine via separation of variables and we find

$$Z^{II}(x,y) = \frac{-\mathrm{i}M_0}{kh}\mathrm{e}^{\mathrm{i}k(x-a)}\psi_0(y) - \sum_{n=1}^{\infty}\frac{M_n}{k_n h}\mathrm{e}^{-k_n(x-a)}\psi_n(y) \tag{2.57}$$

where M_n are defined by (2.14), (2.15).

We shall be interested in

$$Z^{II}(a,y) = \frac{-\mathrm{i}M_0}{kh}\psi_0(y) + H^{II}(y), \qquad \text{where} \qquad H^{II}(y) = -\sum_{n=1}^{\infty}\frac{M_n}{k_n h}\psi_n(y) \tag{2.58}$$

Armed with these two functions, we now expand the potential in each of the regions I and II, using Z^I and Z^{II} to take account of the inhomogeneous boundary conditions (2.7), (2.8) that ϕ^r must satisfy and expanding the remaining part of the potential in terms of functions satisfying the homogeneous boundary conditions. In the region II ($x \geq a$) we write

$$\phi^r(x,y) = Z^{II}(x,y) + (a_0^r + \mathrm{i}M_0/kh)\mathrm{e}^{\mathrm{i}k(x-a)}\psi_0(y) + \sum_{n=1}^{\infty} a_n^r \mathrm{e}^{-k_n(x-a)}\psi_n(y) \tag{2.59}$$

and in region I ($0 \leq x \leq a$) we write

$$\phi^r(x,y) = Z^I(x,y) + b_0^r x + \sum_{n=0}^{\infty} b_n^r \sinh(\mu_n x)\hat{\psi}_n(y). \tag{2.60}$$

It follows that

$$U^r(y) \equiv \phi_x^r(a,y) = Z_x^{II}(a,y) + \mathrm{i}k(a_0^r + \mathrm{i}M_0/kh)\psi_0(y) - \sum_{n=1}^{\infty} k_n a_n^r \psi_n(y), \qquad 0 < y < d \tag{2.61}$$

$$= Z_x^I(a,y) + b_0^r\hat{\psi}_0(y) + \sum_{n=1}^{\infty}\mu_n b_n^r \cosh(\mu_n a)\hat{\psi}_n(y) \qquad d < y < h. \tag{2.62}$$

Using orthogonality of $\psi_n(y)$ and $\hat{\psi}_n(y)$ gives

$$\mathrm{i}kha_0^r - M_0 = \langle U^r, \psi_0\rangle, \qquad \text{and} \qquad -k_n h a_n^r = \langle U^r, \psi_n\rangle, \tag{2.63}$$

and

$$b_0^r(h-d) = \langle U^r, \hat{\psi}_0\rangle, \qquad \text{and} \qquad \mu_n(h-d)b_n^r\cosh(\mu_n a) = \langle U^r, \hat{\psi}_n\rangle, \tag{2.64}$$

in which the conditions on the vertical face of the rectangular cylinder have been used.

Next, matching ϕ^r from each region across $x = a$ gives

$$H^I(y) + b_0^r a\hat{\psi}_0(y) + \sum_{n=1}^{\infty} b_n^r \sinh(\mu_n a)\hat{\psi}_n(y) = H^{II}(y) + a_0^r\psi_0(y) + \sum_{n=1}^{\infty} a_n^r\psi_n(y). \tag{2.65}$$

Inserting the definitions of a_n^r and b_n^r from above, gives

$$\int_d^h U^r(t) K_{asym}(y,t)\mathrm{d}t = a_0^r f_1^r(y) - b_0^r a f_2^r(y) + d^2 f_3^r(y) \tag{2.66}$$

where $f_1^r(y) = \psi_0(y)$, $f_2^r(y) = \hat{\psi}_0(y) = 1$, $f_3^r(y) = (H^{II}(y) - H^I(y))/d^2$

Now we let $u_i^r(t)$ be the solution of

$$\int_d^h u_i^r(t) K_{asym}(y,t)\mathrm{d}t = f_i^r(y), \qquad d < y < h, \quad i = 1,2,3. \tag{2.67}$$

such that

$$U^r(t) = d^2\{\hat{a}_0^r u_1^r(t) + \hat{b}_0^r u_2^r(t) + u_3^r(t)\} \tag{2.68}$$

and $\hat{a}_0^r = a_0^r/d^2$, $\hat{b}_0^r = -b_0^r a/d^2$. We also define

$$T_{ij} = \langle f_i^r, u_j^r \rangle, \qquad i,j = 1,2,3 \tag{2.69}$$

and substituting (2.68) into (2.63) and (2.64) we find

$$\mathrm{i}kh\hat{a}_0^r = \hat{a}_0^r T_{11} + \hat{b}_0^r T_{12} + T_{13} + (M_0/d^2) \tag{2.70}$$

and

$$-\hat{b}_0^r \frac{(h-d)}{a} = \hat{a}_0^r T_{21} + \hat{b}_0^r T_{22} + T_{23}. \tag{2.71}$$

These two equations are used to determine the non-dimensional coefficients $\hat{a}_0^r$ and $\hat{b}_0^r$.

Again, we calculate the roll component of the roll-induced added inertia and radiation damping, A_{33} and B_{33}, defined as the real and imaginary parts of the complex torque on the rectangular cylinder about $(0,c)$. Thus we have

$$\mathrm{i}\omega(A_{33} + \mathrm{i}B_{33}/\omega) = -2\mathrm{i}\omega\rho\int_0^a (-x)\phi^r(x,d)\mathrm{d}x - 2\mathrm{i}\omega\rho\int_0^d (y-c)\phi^r(a,y)\mathrm{d}y \tag{2.72}$$

We deal with each of the integrals on the right-hand side in turn, but in much the same way as one another. First, we apply Green's Identity in region I to the functions $\phi^h(x,y)$ and $Z^I(x,y)$ and find

$$\int_0^a (-x)\phi^r(x,d)\mathrm{d}x = \int_0^a (-x)Z^I(x,d)\mathrm{d}x - \langle U^r, H^I \rangle \tag{2.73}$$

on account of the way that the function $Z^I(x,y)$ has been defined upon its boundaries.

The integral above can be evaluated directly from the definition of Z^I to give

$$\mathcal{Z}^I \equiv \int_0^a (-x)Z^I(x,d)\mathrm{d}x = -2\sum_{n=0}^{\infty} \frac{\coth\lambda_n(h-d)}{\lambda_n^5 a}. \tag{2.74}$$

Moving onto the second integral in (2.72. We again use Green's Identity, this time in region II with ϕ^r and Z^{II} and find

$$\int_0^d (y-c)\phi^r(a,d)\mathrm{d}y = \int_0^d (y-c)Z^{II}(a,y)\mathrm{d}y + \langle U^r, H^{II} \rangle - (\mathrm{i}M_0/kh)\langle U^r, \psi_0 \rangle. \tag{2.75}$$

The first integral in (2.75) can be evaluated directly from the definition of Z^{II} to give

$$\mathcal{Z}^{II} = \int_0^d (y-c)Z^{II}(a,y)\mathrm{d}y = \frac{-\mathrm{i}M_0^2}{kh} - \sum_{n=1}^{\infty} \frac{M_n^2}{k_n h}. \qquad (2.76)$$

where (2.14), (2.15) have been used. Combining (2.73)–(2.76) and inserting in (2.72) we obtain

$$(A_{33} + \mathrm{i}B_{33}/\omega) = -2\rho\left[\mathcal{Z}^{II} + \mathcal{Z}^{I} + \mathrm{i}M_0^2/kh + d^2\langle U^r, f_3^r\rangle + d^2\hat{a}_0^r M_0\right] \qquad (2.77)$$

after using $\mathrm{i}kha_0^r - M_0 = \langle U^r, \psi_0\rangle$. Then finally,

$$(A_{33} + \mathrm{i}B_{33}/\omega) = -2\rho d^4\left[\hat{a}_0^r(M_0/d^2) - \sum_{n=1}^{\infty}\frac{(M_n/d^2)^2}{k_n h} + \mathcal{Z}^{I} + \{\hat{a}_0^r T_{31} + \hat{b}_0^r T_{32} + T_{33}\}\right]. \qquad (2.78)$$

These are to be non-dimensionalised by a moment of inertia, I, having dimensions equal to the fourth power of length. Thus, non-dimensional coefficients $\mu_{33} = A_{33}/I$, $\nu_{33} = B_{33}/(\omega I)$.

2.6 The remaining hydrodynamic coefficients: the cross terms

Clearly, we have $A_{21} = A_{23} = A_{12} = A_{32} = 0$ and $B_{21} = B_{23} = B_{12} = B_{32} = 0$, as heave motion is symmetric and sway and roll are both antisymmetric about the geometric line of symmetry, $x = 0$. That is, symmetric motions cannot induce antisymmetric forces and moments and vice versa.

The first of the non-zero terms to consider is the roll-component of added mass and radiation damping due to forced sway motion. Thus we have

$$\mathrm{i}\omega(A_{13} + \mathrm{i}B_{13}/\omega) = -2\mathrm{i}\omega\rho\int_0^d \phi^s(a,y)(y-c)\mathrm{d}y - 2\mathrm{i}\omega\rho\int_0^a \phi^s(x,d)(-x)\mathrm{d}x. \qquad (2.79)$$

We use Green's Identity to for the first integral, in region II with ϕ_s and Z^{II} to give

$$\int_0^d \phi^s(a,y)(y-c)\mathrm{d}y = \int_0^d Z^{II}(a,y)\mathrm{d}y + \int_d^h U^s(y)Z^{II}(a,y)\mathrm{d}y. \qquad (2.80)$$

The first integral is

$$\int_0^d Z^{II}(a,y)\mathrm{d}y = \frac{-\mathrm{i}M_0L_0}{kh} - \sum_{n=1}^{\infty}\frac{M_nL_n}{k_nh}. \qquad (2.81)$$

The second integral is

$$\int_d^h U^s(y)Z^{II}(a,y)\mathrm{d}y = \frac{-\mathrm{i}M_0}{kh}(\mathrm{i}kha_0^s - L_0) + \langle U^s, H^{II}\rangle. \qquad (2.82)$$

Thus

$$\int_0^d \phi^s(a,y)(y-c)\mathrm{d}y = d\hat{a}_0^s M_0 - \sum_{n=1}^{\infty}\frac{M_nL_n}{k_nh} + \langle U^s, H^{II}\rangle. \qquad (2.83)$$

In a similar manner, we use Green's Identity in region I with ϕ^s and Z^I to give

$$\int_0^a (-x)\phi^s(x,d)\mathrm{d}x = -\int_d^h U^s(y)Z^I(a,y)\mathrm{d}y = -\langle U^s, H^I\rangle. \qquad (2.84)$$

The net outcome is that

$$A_{13}+\mathrm{i}B_{13}/\omega = -2\rho d^3 \left[\hat{a}_0^s(M_0/d^2) - \sum_{n=1}^{\infty} \frac{(M_n/d^2)(L_n/d)}{k_n h} + \{\hat{a}_0^s\langle u_1^s, f_3^r\rangle + \hat{b}_0^s\langle u_2^s, f_3^r\rangle + \langle u_3^s, f_3^r\rangle\} \right]. \tag{2.85}$$

Non-dimensional coefficients are defined by $\mu_{13} = A_{13}/\sqrt{MI}$ and $\nu_{13} = B_{13}/(\omega\sqrt{IM})$. Similarly for the 31 components below.

Next, consider the reciprocal terms, which represent the sway component of added mass and radiation damping due to forced roll motion. In other words,

$$\mathrm{i}\omega(A_{31} + \mathrm{i}B_{31}/\omega) = -2\mathrm{i}\omega\rho \int_0^d \phi^r(a, y)\mathrm{d}y. \tag{2.86}$$

We use Green's Identity with ϕ^r and X^{II} to give

$$\int_0^a \phi^r(a, y)\mathrm{d}y = \int_0^a (y - c)X^{II}(a, y)\mathrm{d}y - \frac{\mathrm{i}L_0}{kh}\langle U^r, \psi_0\rangle + \langle U^r, F^{II}\rangle. \tag{2.87}$$

Doing the same things as before we get

$$A_{31}+\mathrm{i}B_{31}/\omega = -2\rho d^3 \left[\hat{a}_0^r(L_0/d) - \sum_{n=1}^{\infty} \frac{(L_n/d)(M_n/d^2)}{k_n h} + \{\hat{a}_0^s\langle u_1^r, f_3^s\rangle + \hat{b}_0^s\langle u_2^r, f_3^s\rangle + \langle u_3^r, f_3^s\rangle\} \right] \tag{2.88}$$

2.7 Numerical solution of the integral equations

The numerical procedure for solving the integral equations is as follows and is based on the Galerkin method in which we write

$$u_i^{s,h,r}(y) \simeq \sum_{n=0}^{N} a_{n,i}^{s,h,r} v_n(y) \tag{2.89}$$

(the notation here is a little awkward) for given functions $v_n(y)$, $n = 0, \dots, N$ and, by making the residual of the integral equation orthogonal to the space spanned by the functions $v_m(y)$, $m = 0, \dots, N$ we derive the system of equations

$$\sum_{n=0}^{N} a_{n,i}^{s,r} K_{m,n}^{(asym)} = F_m^{(s,r;i)}, \qquad m = 0, \dots, N \tag{2.90}$$

and

$$\sum_{n=0}^{N} a_{n,i}^{h} K_{m,n}^{(sym)} = F_m^{(h;i)}, \qquad m = 0, \dots, N \tag{2.91}$$

for $i = 1, 2, 3$ whose solutions determine $a_{n,i}^{s,h,r}$. In the above

$$K_{m,n}^{(asym)} = \sum_{r=1}^{\infty} \left[\frac{F_{r,n}^{(1)} F_{r,m}^{(1)}}{k_r h} + \frac{F_{r,n}^{(2)} F_{r,m}^{(2)}}{r\pi \coth \mu_r a} \right], \qquad \text{and} \qquad K_{m,n}^{(sym)} = \sum_{r=1}^{\infty} \left[\frac{F_{r,n}^{(1)} F_{r,m}^{(1)}}{k_r h} + \frac{F_{r,n}^{(2)} F_{r,m}^{(2)}}{r\pi \tanh \mu_r a} \right] \tag{2.92}$$

and

$$F^{(1)}_{r,m} = \int_d^h v_m(y)\psi_r(y)\mathrm{d}y, \qquad F^{(2)}_{r,m} = \int_d^h v_m(y)\hat{\psi}_r(y)\mathrm{d}y, \tag{2.93}$$

with

$$F^{(s,h,r;3)}_m = \int_d^h v_m(y) f_3^{s,h,r}(y)\mathrm{d}y. \tag{2.94}$$

Then finally

$$P_{ij} \approx \sum_{n=0}^{N} a^s_{n,j} F^{(s;i)}_{0,n}, \qquad S_{ij} \approx \sum_{n=0}^{N} a^h_{n,j} F^{(h;i)}_{0,n}, \qquad T_{ij} \approx \sum_{n=0}^{N} a^r_{n,j} F^{(r;i)}_{0,n}. \tag{2.95}$$

We choose the standard sets of function for $v_m(y)$, defined by

$$v_m(y) = \frac{(-1)^m 2^{1/6}(2m)!\Gamma(\frac{1}{6})}{\pi\Gamma(2m+\frac{1}{3})(h-d)^{1/3}} \frac{C^{(1/6)}_{2m}((h-y)/(h-d))}{[(h-d)^2-(h-y)^2]^{1/3}} \tag{2.96}$$

where $C^{(\nu)}_m(\cdot)$ is the ultraspherical Gegenbauer polynomial. The normalisation is chosen to give the simplest form for $F^{(i)}_{r,m}$. Thus we find that

$$F^{(1)}_{r,m} = N_r^{-1/2} \frac{J_{2m+1/6}(k_r(h-d))}{[k_r(h-d)]^{1/6}}, \qquad \text{and} \qquad F^{(2)}_{r,m} = \sqrt{2}\frac{J_{2m+1/6}(r\pi)}{(r\pi)^{1/6}}, \qquad \text{for } r \geq 1. \tag{2.97}$$

Next,

$$F^{(s,h,r;1)}_m = (-1)^m N_0^{-1/2} \frac{I_{2m+1/6}(k(h-d))}{[k(h-d)]^{1/6}}, \qquad \text{and} \qquad F^{(s,h,r;2)}_m = \frac{2^{-1/6}}{\Gamma(\frac{7}{6})}\delta_{0m}. \tag{2.98}$$

Finally

$$F^{(s;3)}_m = -\sum_{r=1}^{\infty} \frac{(L_r/d)}{k_r h} F^{(1)}_{r,m}. \tag{2.99}$$

and

$$F^{(h;3)}_0 = \frac{9}{7}\frac{2^{-1/6}}{\Gamma(\frac{1}{6})}, \qquad F^{(h;3)}_1 = -\frac{54}{91}\frac{2^{-1/6}}{\Gamma(\frac{1}{6})}, \qquad \text{and} \qquad F^{(h;3)}_m = 0, \quad \text{for } m \geq 2. \tag{2.100}$$

and

$$F^{(r;3)}_m = -\sum_{r=1}^{\infty} \frac{(M_r/d^2)}{k_r h} F^{(1)}_{r,m} - 2\sum_{r=0}^{\infty} \frac{(-1)^m I_{2m+1/6}(\lambda_r(h-d))}{\lambda_r^3 a d^2 [\lambda_r(h-d)]^{1/6}\sinh\lambda_r(h-d)}. \tag{2.101}$$

2.8 Low-frequency asymptotics

We consider the result on ν_{ij}, the non-dimensional radiation damping, of letting the frequency ω tend to zero. Accordingly, the dimensionless wavenumber $kh \to 0$, whilst all other length scales are held fixed with respect to h.

As will be shown, we can ascertain the leading order behaviour of ν_{ij} as $kh \to 0$ without having to solve the integral equations, although the arguments given below are not particularly rigorous. We start by looking at ν_{22}.

As a result of letting $kh \to 0$, we note from (2.10) that

$$N_0 \to 1, \qquad \psi_0(y) \to 1 + O((kh)^2) \tag{2.102}$$

Hence $f_1^h(y) = f_2^h(y) + O((kh)^2)$. Accordingly, we have from (2.43)

$$u_1^h(y) = u_2^h(y) + (kh)^2 \hat{u}(y) + \dots \tag{2.103}$$

to where $\hat{u}(y)$ is some function we are not concerned with. It follows that

$$S_{11} = S_{12} + O((kh)^2), \qquad S_{11} = S_{21} + O((kh)^2), \qquad S_{11} = S_{22} + O((kh)^2) \tag{2.104}$$

and also that $S_{13} = S_{23} + O((kh)^2)$. Using these estimates in (2.47) and (2.48) we can easily obtain leading order estimates on the coefficients $\hat{a}_0^h$, $\hat{b}_0^h$ so that

$$\hat{a}_0^h \sim -\frac{\mathrm{i}}{kh}, \qquad \text{and} \qquad \hat{b}_0^h \sim \frac{\mathrm{i}}{kh} \tag{2.105}$$

It is important to note that, inspite of any approximations being made, S_{ij} are real and symmetric. Using (2.105) in (2.53) and equating imaginary parts gives the estimate

$$B_{22} \sim \frac{2\rho\omega a^2}{kh}, \qquad \text{as } kh \to 0 \tag{2.106}$$

In non-dimensional terms, with $M = 2\rho a d$, we have $\nu_{22} \sim a/(khd)$ as $kh \to 0$. Following similar arguments for the other non-zero components of the radiation damping, it is straightforward to see that $\nu_{ij} \to 0$ as $kh \to 0$ for $i, j \neq 2$.

3 A swaying, heaving and rolling cylinder next to a wall

In this section we repeat the calculations of section 2, but now the cylinder is centred on $x = b \neq 0$ with a wall at $x = 0$. The problems are now more complicated, since there is no line of geometric symmetry. Accordingly, the fluid domain is now divided into three rectangular subdomains, being the region between the wall and the cylinder, under the cylinder and from the right of the cylinder. This inevitably makes the solution process more complicated. Moreover, the effect of the wall means that none of the nine pairs of hydrodynamic coefficients are trivially zero.

In this section, we re-use the notation of the last section.

3.1 Boundary conditions

For a swaying cylinder, we require

$$\phi_x^s = 1, \qquad \text{on } x = b \pm a,\ 0 < y < d, \tag{3.1}$$
$$\phi_y^s = 0, \qquad \text{on } y = d,\ b - a < x < b + a; \tag{3.2}$$

for a heaving cylinder we require

$$\phi_x^h = 0, \qquad \text{on } x = b \pm a,\ 0 < y < d, \tag{3.3}$$
$$\phi_y^h = 1, \qquad \text{on } y = d,\ b - a < x < b + a; \tag{3.4}$$

for a rolling cylinder we have

$$\phi_x^r = (y - c), \qquad \text{on } x = b \pm a,\ 0 < y < d, \tag{3.5}$$
$$\phi_y^r = (b - x), \qquad \text{on } y = d,\ b - a < x < b + a. \tag{3.6}$$

3.2 A swaying cylinder next to a wall

We start by defining the particular solutions in each case of the regions.

In region I, we have $X_x^I(b-a, y) = 1$ for $0 < y < d$ and $X_x^I(b-a, y) = 0$ for $d < y < h$ in addition to the wall condition on $x = 0$, so that we write

$$X^I(x, y) = -\frac{L_0}{kh \sin k(b-a)} \cos(kx)\psi_0(y) + \sum_{n=1}^{\infty} \frac{L_n}{k_n h \sinh k_n(b-a)} \cosh(k_n x)\psi_n(y) \tag{3.7}$$

so that

$$X^I(b-a, y) = -\frac{L_0}{kh} \cot k(b-a)\psi_0(y) + F^I(y), \qquad F^I(y) = \sum_{n=1}^{\infty} \frac{L_n}{k_n h} \coth k_n(b-a)\psi_n(y). \tag{3.8}$$

Then in region III, we need $X_x^{III}(b+a, y) = 1$ for $0 < y < d$ and $X_x^{III}(b+a, y) = 0$ for $d < y < h$ and outgoing waves at infinity.

$$X^{III}(x, y) = -\frac{\mathrm{i}L_0}{kh}\psi_0(y)\mathrm{e}^{\mathrm{i}k(x-b-a)} + \sum_{n=1}^{\infty} \frac{L_n}{k_n h} \mathrm{e}^{-k_n(x-b-a)}\psi_n(y). \tag{3.9}$$

Then,

$$X^{III}(b+a,y) = -\frac{\mathrm{i}L_0}{kh}\psi_0(y) + F^{III}(y), \qquad \text{where} \qquad F^{III}(y) = -\sum_{n=1}^{\infty} \frac{L_n}{k_n h}\psi_n(y). \quad (3.10)$$

With this we define the potentials in I, II and III for the swaying cylinder as, in I

$$\phi^s(x,y) = X^I(x,y) + \left(a_0^s + \frac{L_0}{kh\sin(k(b-a))}\right)\cos(kx)\psi_0(y) + \sum_{n=1}^{\infty} a_n^s \cosh(k_n x)\psi_n(y) \quad (3.11)$$

in II

$$\phi^s(x,y) = (b_0^s(x-b) + c_0^s)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}\Big(b_n^s \sinh \mu_n(x-b) + c_n^s \cosh \mu_n(x-b)\Big)\hat{\psi}_n(y) \quad (3.12)$$

and in III

$$\phi^s(x,y) = X^{III}(x,y) + \left(d_0^s + \frac{\mathrm{i}L_0}{kh}\right)\mathrm{e}^{\mathrm{i}k(x-b-a)}\psi_0(y) + \sum_{n=1}^{\infty} d_n^s \mathrm{e}^{-k_n(x-a)}\psi_n(y). \quad (3.13)$$

Then we let $U_1^s(y) = \phi_x^s(b-a,y)$ so that we obtain

$$-kha_0^s \sin k(b-a) - L_0 = \langle U_1^s, \psi_0 \rangle, \qquad k_n h \sinh k_n(b-a) a_n^s = \langle U_1^s, \psi_n \rangle \quad (3.14)$$

and

$$b_0^s(h-d) = \langle U_1^s, \hat{\psi}_0 \rangle, \qquad n\pi(b_n^s \cosh(\mu_n a) - c_n^s \sinh(\mu_n a)) = \langle U_1^s, \hat{\psi}_n \rangle. \quad (3.15)$$

Now letting $U_2^s(y) = \phi_x^s(b+a,y)$ we have

$$\mathrm{i}khd_0^s - L_0 = \langle U_2^s, \psi_0 \rangle, \qquad -k_n h d_n^s = \langle U_2^s, \psi_n \rangle \quad (3.16)$$

and

$$b_0^s(h-d) = \langle U_2^s, \hat{\psi}_0 \rangle, \qquad n\pi(b_n^s \cosh(\mu_n a) + c_n^s \sinh(\mu_n a)) = \langle U_2^s, \hat{\psi}_n \rangle. \quad (3.17)$$

Matching pressures across $x = b-a$ for $d < y < h$ gives

$$F^I(y) + a_0^s \cos k(b-a)\psi_0(y) + \sum_{n=1}^{\infty} a_n^s \cosh k_n(b-a)\psi_n(y) =$$
$$(-b_0^s a + c_0^s)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}(-b_n^s \sinh(\mu_n a) + c_n^s \cosh(\mu_n a))\hat{\psi}_n(y). \quad (3.18)$$

And matching pressures across $x = b+a$ for $d < y < h$ gives

$$F^{III}(y) + d_0^s\psi_0(y) + \sum_{n=1}^{\infty} d_n^s \psi_n(y) = (b_0^s a + c_0^s)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}(b_n^s \sinh(\mu_n a) + c_n^s \cosh(\mu_n a))\hat{\psi}_n(y). \quad (3.19)$$

Using the relations (3.16), (3.17) in (3.18), (3.19) gives

$$\int_d^h U_1^s(t)K_{11}(y,t)\mathrm{d}t + \int_d^h U_2^s(t)K_{12}(y,t)\mathrm{d}t = -a_0^s \cos k(b-a)\psi_0(y) - (b_0^s a - c_0^s)\hat{\psi}_0(y) - F^I(y) \tag{3.20}$$

and

$$\int_d^h U_1^s(t)K_{21}(y,t)\mathrm{d}t + \int_d^h U_2^s(t)K_{22}(y,t)\mathrm{d}t = d_0^s\psi_0(y) - (b_0^s a + c_0^s)\hat{\psi}_0(y) + F^{III}(y) \tag{3.21}$$

where

$$K_{11}(y,t) = \sum_{n=1}^{\infty}\left[\frac{\coth k_n(b-a)}{k_n h}\psi_n(t)\psi_n(y) + \frac{\coth(2\mu_n a)}{n\pi}\hat{\psi}_n(t)\hat{\psi}_n(y)\right] \tag{3.22}$$

and

$$K_{12}(y,t) = K_{21}(y,t) = -\sum_{n=1}^{\infty}\frac{\mathrm{cosech}(2\mu_n a)}{n\pi}\hat{\psi}_n(t)\hat{\psi}_n(y) \tag{3.23}$$

and finally

$$K_{22}(y,t) = \sum_{n=1}^{\infty}\left[\frac{\psi_n(t)\psi_n(y)}{k_n h} + \frac{\coth(2\mu_n a)}{n\pi}\hat{\psi}_n(t)\hat{\psi}_n(y)\right]. \tag{3.24}$$

We can write the coupled integral equations in matrix/vector form by introducing some new notation. Hence we define

$$\mathbf{K}(y,t) = \begin{pmatrix} K_{11}(y,t) & K_{12}(y,t) \\ K_{21}(y,t) & K_{22}(y,t) \end{pmatrix}, \qquad \mathbf{U}^s(y) = \begin{pmatrix} U_1^s(y) \\ U_2^s(y) \end{pmatrix}. \tag{3.25}$$

We write

$$\mathbf{U}^s(y) = d\left(\mathbf{U}_1^s(y)\mathbf{A}_1^s + \mathbf{U}_2^s(y)\boldsymbol{A}_2^s + \mathbf{U}_3^s(y)\right), \tag{3.26}$$

where

$$\mathbf{U}_1^s(y) = \begin{pmatrix} u_1^s(y) & u_3^s(y) \\ u_2^s(y) & u_4^s(y) \end{pmatrix}, \quad \mathbf{U}_2^s(y) = \begin{pmatrix} u_5^s(y) & u_7^s(y) \\ u_6^s(y) & u_8^s(y) \end{pmatrix}, \quad \mathbf{U}_3^s(y) = \begin{pmatrix} u_9^s(y) \\ u_{10}^s(y) \end{pmatrix}, \tag{3.27}$$

and

$$\mathbf{A}_1^s = \begin{pmatrix} \hat{a}_0^s \\ \hat{d}_0^s \end{pmatrix}, \qquad \mathbf{A}_2^s = \begin{pmatrix} \hat{b}_0^s + \hat{c}_0^s \\ \hat{b}_0^s - \hat{c}_0^s \end{pmatrix}, \tag{3.28}$$

where $\hat{a}_0^s = -a_0^s \cos k(b-a)/d$, $\hat{b}_0^s = -b_0^s a/d$, $\hat{c}_0^s = c_0^s/d$ and $\hat{d}_0^s = d_0^s/d$.

Finally, we define

$$\mathbf{F}_1^s(y) = \begin{pmatrix} \psi_0(y) & 0 \\ 0 & \psi_0(y) \end{pmatrix}, \quad \mathbf{F}_2^s(y) = \begin{pmatrix} \hat{\psi}_0(y) & 0 \\ 0 & \hat{\psi}_0(y) \end{pmatrix}, \quad \mathbf{F}_3^s(y) = \begin{pmatrix} -F^I(y)/d \\ F^{III}(y)/d \end{pmatrix}. \tag{3.29}$$

Then if $\mathbf{U}_i^s(t)$ are defined to be the solutions of the coupled integral equation system

$$\int_d^h \mathbf{K}(y,t)\mathbf{U}_i^s(t)\mathrm{d}t = \mathbf{F}_i^s(y), \qquad d < y < h, \ i = 1,2,3. \tag{3.30}$$

it follows from (3.26) that $U_1^s(y)$, $U_2^s(y)$ satisfy (3.19), (3.20). Following on from this, we define the matrices

$$\mathbf{P}_{ij} = \langle \mathbf{F}_i^{s\mathrm{T}}, \mathbf{U}_j^s \rangle, \qquad i,j = 1,2,3. \tag{3.31}$$

whose dimensions are inherited from the definitions of the components of the inner product. For example, $\mathbf{P}_{11}$ is a 2×2 matrix, whereas $\mathbf{P}_{33}$ is a 1×1 matrix (a coefficient). From a vectorisation of earlier relations (3.13), (3.15) we have

$$\begin{pmatrix} kh\hat{a}_0^s \tan k(b-a) \\ \mathrm{i}kh\hat{d}_0^s \end{pmatrix} - \begin{pmatrix} L_0/d \\ L_0/d \end{pmatrix} = \frac{1}{d}\int_d^h \mathbf{U}^s(y)\psi_0(y)\mathrm{d}y = \mathbf{P}_{11}\mathbf{A}_1 + \mathbf{P}_{12}\mathbf{A}_2 + \mathbf{P}_{13} \quad (3.32)$$

and of (3.14), (3.16) we have

$$-\begin{pmatrix} \hat{b}_0^s(h-d)/a \\ \hat{b}_0^s(h-d)/a \end{pmatrix} = \frac{1}{d}\int_d^h \mathbf{U}^s(y)\hat{\psi}_0(y)\mathrm{d}y = \mathbf{P}_{21}\mathbf{A}_1 + \mathbf{P}_{22}\mathbf{A}_2 + \mathbf{P}_{23}. \quad (3.33)$$

These four linear equations contained in (3.32), (3.33) determine the coefficients $\hat{a}_0^s$, $\hat{b}_0^s$, $\hat{c}_0^s$ and $\hat{d}_0^s$.

The sway component of the sway induced added-mass and radiation damping, A_{11}^w and B_{11}^w (the superscript w is used here to indicate the presence of the wall and distinguish from the results of §2), are defined by

$$\mathrm{i}\omega(A_{11}^w + \mathrm{i}B_{11}^w/\omega) = -\mathrm{i}\omega\rho\int_0^d \phi^s(b+a,y)\mathrm{d}y + \mathrm{i}\omega\rho\int_0^d \phi^s(b-a,y)\mathrm{d}y. \quad (3.34)$$

We apply Green's Identity over region I to ϕ^s and X^I and this ends up giving us the relation

$$\begin{aligned}\int_0^d \phi^s(b-a,y)\mathrm{d}y &= \int_0^d X^I(b-a,y)\mathrm{d}y + \int_d^h X^I(b-a,y)U_1^s(y)\mathrm{d}y \\ &= -d\hat{a}_0^s L_0 + \sum_{n=1}^{\infty}\frac{L_n^2\coth k_n(b-a)}{k_n h} + \langle U_1^s, F^I\rangle. \end{aligned} \quad (3.35)$$

Next, in region III, use of ϕ^s and X^{III} gives, in much the same way

$$\int_0^d \phi^s(b+a,y)\mathrm{d}y = d\hat{d}_0^s L_0 - \sum_{n=1}^{\infty}\frac{L_n^2}{k_n h} + \langle U_2^s, F^{III}\rangle. \quad (3.36)$$

Thus, finally,

$$\begin{aligned}(A_{11}^w + \mathrm{i}B_{11}^w/\omega) &= \\ -\rho d^2\left[(\hat{d}_0^s + \hat{a}_0^s)(L_0/d) - \sum_{n=1}^{\infty}\frac{(L_n/d)^2}{k_n h}(1+\coth k_n(b-a)) + \mathbf{A}_1^{\mathrm{T}}\mathbf{P}_{31} + \mathbf{A}_2^{\mathrm{T}}\mathbf{P}_{32} + \mathbf{P}_{33}\right]&. \end{aligned} \quad (3.37)$$

The non-dimensional added-mass and damping are $\mu_{11}^w = A_{11}^w/M$ and $\nu_{11}^w = B_{11}^w/(M\omega)$.

3.3 A heaving cylinder next to a wall

The particular solution is defined by $Y_y^{II}(x,d) = 1$, $b-a < x < b+a$, and we define

$$Y^{II}(x,y) = -\frac{1}{2}\frac{(h-y)^2}{(h-d)} + \frac{1}{2}\frac{(x-b)^2 - a^2}{(h-d)}. \quad (3.38)$$

This function is such that

$$Y^{II}(b \pm a, y) = -G^{II}(y) = -\frac{1}{2}\frac{(h-y)^2}{(h-d)} \tag{3.39}$$

$$Y_x^{II}(b \pm a, y) = \pm\frac{a}{h-d}. \tag{3.40}$$

Using this, we expand the potential in II as

$$\phi^h(x,y) = Y^{II}(x,y) + (b_0^h(x-b) + c_0^h)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}(b_n^h \cosh \mu_n(x-b) + c_n^h \sinh \mu_n(x-b))\hat{\psi}_n(y). \tag{3.41}$$

In region I we have

$$\phi^h(x,y) = a_0^h \cos(kx)\psi_0(y) + \sum_{n=1}^{\infty} a_n^h \cosh(k_n x)\psi_n(y) \tag{3.42}$$

and in region III we have

$$\phi^h = d_0^h \mathrm{e}^{\mathrm{i}k(x-b-a)}\psi_0(y) + \sum_{n=1}^{\infty} d_n^h \mathrm{e}^{-k_n(x-a)}\psi_n(y). \tag{3.43}$$

Then we let $U_1^h(y) = \phi_x^h(b-a,y)$ so that we obtain

$$-kha_0^h \sin k(b-a) = \langle U_1^h, \psi_0\rangle, \qquad k_n h \sinh k_n(b-a) a_n^h = \langle U_1^h, \psi_n\rangle \tag{3.44}$$

and

$$-a + b_0^h(h-d) = \langle U_1^h, \hat{\psi}_0\rangle, \qquad n\pi(b_n^h \cosh(\mu_n a) - c_n^h \sinh(\mu_n a)) = \langle U_1^h, \hat{\psi}_n\rangle. \tag{3.45}$$

Now letting $U_2^h(y) = \phi_x^h(b+a,y)$ we have

$$\mathrm{i}khd_0^h = \langle U_2^h, \psi_0\rangle, \qquad -k_n h d_n^h = \langle U_2^h, \psi_n\rangle \tag{3.46}$$

and

$$a + b_0^h(h-d) = \langle U_2^h, \hat{\psi}_0\rangle, \qquad n\pi(b_n^h \cosh(\mu_n a) + c_n^h \sinh(\mu_n a)) = \langle U_2^h, \hat{\psi}_n\rangle. \tag{3.47}$$

Matching pressures across $x = b-a$ for $d < y < h$ gives

$$a_0^h \cos k(b-a)\psi_0(y) + \sum_{n=1}^{\infty} a_n^h \cosh k_n(b-a)\psi_n(y) =$$

$$-G^{II}(y) + (-b_0^h a + c_0^h)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}(-b_n^h \sinh(\mu_n a) + c_n^h \cosh(\mu_n a))\hat{\psi}_n(y) \tag{3.48}$$

And matching pressures across $x = b+a$ for $d < y < h$ gives

$$d_0^h\psi_0(y) + \sum_{n=1}^{\infty} d_n^h \psi_n(y) = -G^{II}(y) + (b_0^h a + c_0^h)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}(b_n^h \sinh(\mu_n a) + c_n^h \cosh(\mu_n a))\hat{\psi}_n(y). \tag{3.49}$$

Upon substituting in for a_n^h, b_n^h, c_n^h, d_n^h for $n \geq 1$ from (3.43)–(3.46) into (3.48), (3.49 above, we derive the coupled integral equations for the functions $U_1^h(y)$ and $U_2^h(y)$,

$$\int_d^h U_1^s(t)K_{11}(y,t)\mathrm{d}t + \int_d^h U_2^s(t)K_{12}(y,t)\mathrm{d}t = -a_0^h \cos k(b-a)\psi_0(y) - (b_0^h a - c_0^h)\hat{\psi}_0(y) - G^{II}(y) \tag{3.50}$$

and

$$\int_d^h U_1^h(t)K_{21}(y,t)\mathrm{d}t + \int_d^h U_2^h(t)K_{22}(y,t)\mathrm{d}t = d_0^h\psi_0(y) - (b_0^h a + c_0^h)\hat{\psi}_0(y) + G^{II}(y) \tag{3.51}$$

where $K_{11}(y,t)$ $K_{12}(y,t)$, $K_{21}(y,t)$ and $K_{22}(y,t)$ are as before in the swaying cylinder problem. We can write the coupled integral equations in matrix/vector form as

$$\mathbf{U}^h(y) = \begin{pmatrix} U_1^h(y) \\ U_2^h(y) \end{pmatrix}, \qquad \mathbf{U}^h(y) = (h-d)\left(\mathbf{U}_1^h(y)\mathbf{A}_1^h + \mathbf{U}_2^h(y)\mathbf{A}_2^h + \mathbf{U}_3^h(y)\right) \tag{3.52}$$

where

$$\mathbf{U}_1^h(y) = \begin{pmatrix} u_1^h(y) & u_3^h(y) \\ u_2^h(y) & u_4^h(y) \end{pmatrix}, \qquad \mathbf{U}_2^h(y) = \begin{pmatrix} u_5^h(y) & u_7^h(y) \\ u_6^h(y) & u_8^h(y) \end{pmatrix}, \qquad \mathbf{U}_3^h(y) = \begin{pmatrix} u_9^h(y) \\ u_{10}^h(y) \end{pmatrix} \tag{3.53}$$

and

$$\mathbf{A}_1^h = \begin{pmatrix} \hat{a}_0^h \\ \hat{d}_0^h \end{pmatrix}, \qquad \mathbf{A}_2^h = \begin{pmatrix} \hat{b}_0^h + \hat{c}_0^h \\ \hat{b}_0^h - \hat{c}_0^h \end{pmatrix} \tag{3.54}$$

where $\hat{a}_0^h = -a_0^h \cos k(b-a)/(h-d)$, $\hat{b}_0^h = -b_0^h a/(h-d)$, $\hat{c}_0^h = c_0^h/(h-d)$ and $\hat{d}_0^h = d_0^h/(h-d)$.

Also, we let

$$\mathbf{F}_1^h(y) = \begin{pmatrix} \psi_0(y) & 0 \\ 0 & \psi_0(y) \end{pmatrix}, \qquad \mathbf{F}_2^h(y) = \begin{pmatrix} \hat{\psi}_0(y) & 0 \\ 0 & \hat{\psi}_0(y) \end{pmatrix},$$
$$\mathbf{F}_3^h(y) = \begin{pmatrix} -G^{II}(y)/(h-d) \\ G^{II}(y)/(h-d) \end{pmatrix}. \tag{3.55}$$

Then $\mathbf{U}_i^h(t)$ are the solutions of the integral equations

$$\int_d^h \mathbf{K}(y,t)\mathbf{U}_i^h(t)\mathrm{d}t = \mathbf{F}_i^h(y), \qquad d < y < h,\ i = 1,2,3 \tag{3.56}$$

Following from this, we define the matrices

$$\mathbf{S}_{ij} = \langle \mathbf{F}_i^{h\,\mathrm{T}}, \mathbf{U}_j^h \rangle, \qquad i,j = 1,2,3. \tag{3.57}$$

From the earlier relations we have

$$\begin{pmatrix} kh\hat{a}_0^h \tan k(b-a) \\ \mathrm{i}kh\hat{d}_0^h \end{pmatrix} = \frac{1}{(h-d)}\int_d^h \mathbf{U}^h(y)\psi_0(y)\mathrm{d}y = \mathbf{S}_{11}\mathbf{A}_1 + \mathbf{S}_{12}\mathbf{A}_2 + \mathbf{S}_{13} \tag{3.58}$$

and

$$\begin{pmatrix} -a/(h-d) - \hat{b}_0^h(h-d)/a \\ a/(h-d) - \hat{b}_0^h(h-d)/a \end{pmatrix} = \frac{1}{(h-d)}\int_d^h \mathbf{U}^h(y)\hat{\psi}_0(y)\mathrm{d}y = \mathbf{S}_{21}\mathbf{A}_1 + \mathbf{S}_{22}\mathbf{A}_2 + \mathbf{S}_{23}. \tag{3.59}$$

These four equations are sufficient to determine the coefficients $\hat{a}_0^h$, $\hat{b}_0^h$, $\hat{c}_0^h$ and $\hat{d}_0^h$.

The heave component of the heave induced added-mass and radiation damping is defined as

$$\mathrm{i}\omega(A_{22}^w + \mathrm{i}B_{22}^w/\omega) = -\mathrm{i}\omega\rho \int_{b-a}^{b+a} \phi^h(x,d)\mathrm{d}x. \tag{3.60}$$

We apply Green's Identity over region I to ϕ^h and Y^{II} and this ends up giving us the relation

$$\begin{aligned}\int_{b-a}^{b+a} \phi^h(x,d)\mathrm{d}x &= \int_{b-a}^{b+a} Y^{II}(x,d)\mathrm{d}x - \langle U_1^s, G^{II}\rangle + \langle U_2^s, G^{II}\rangle \\ &\quad + \left(\frac{a}{h-d}\right)\int_d^h \phi(b+a,y)\mathrm{d}y + \left(\frac{a}{h-d}\right)\int_d^h \phi(b-a,y)\mathrm{d}y.\end{aligned} \tag{3.61}$$

Then we have

$$\int_{b-a}^{b+a} Y^{II}(x,d)\mathrm{d}x = -a(h-d) - \frac{2}{3}\frac{a^3}{(h-d)}. \tag{3.62}$$

whilst

$$\left(\frac{a}{h-d}\right)\int_d^h \phi(b\pm a,y)\mathrm{d}y = a(h-d)(\hat{c}_0^h \mp \hat{b}_0^h) - \tfrac{1}{6}a(h-d). \tag{3.63}$$

So altogether we get

$$\begin{aligned}\int_{b-a}^{b+a} \phi^h(x,d)\mathrm{d}x = -a(h-d) - \frac{2}{3}\frac{a^3}{(h-d)} + 2a(h-d)\hat{c}_0^h - \frac{1}{3}a(h-d) \\ +(h-d)^2\{\mathbf{S}_{31}\mathbf{A}_1^h + \mathbf{S}_{32}\mathbf{A}_2^h + \mathbf{S}_{33}\}.\end{aligned} \tag{3.64}$$

Thus, finally we get

$$(A_{22}^w + \mathrm{i}B_{22}^w/\omega) = -a(h-d)\rho\left[(2\hat{c}_0^h - \tfrac{4}{3}) - \frac{2}{3}\frac{a^2}{(h-d)^2} + ((h-d)/a)\{\mathbf{S}_{31}\mathbf{A}_1^h + \mathbf{S}_{32}\mathbf{A}_2^h + \mathbf{S}_{33}\}\right]. \tag{3.65}$$

The non-dimensional coefficients are $\mu_{22}^w = A_{22}^w/M$ and $\nu_{22}^w = B_{22}^w/(\omega M)$, where M is the cylinder mass.

3.4 A rolling cylinder next to a wall

We start by defining the particular solutions in each case.

In region I, we have $Z_x^I(b-a,y) = (y-c)$ for $0 < y < d$ and $Z_x^I(b-a,y) = 0$ for $d < y < h$ in addition to the wall condition on $x = 0$, so that we write

$$Z^I(x,y) = -\frac{M_0}{kh\sin k(b-a)}\cos(kx)\psi_0(y) + \sum_{n=1}^{\infty}\frac{M_n}{k_n h\sinh k_n(b-a)}\cosh(k_n x)\psi_n(y) \tag{3.66}$$

where M_n are defined in (2.14), (2.15) so that

$$Z^I(b-a,y) = -\frac{M_0}{kh}\cot k(b-a)\psi_0(y) + H^I(y), \qquad H^I(y) = \sum_{n=1}^{\infty}\frac{M_n}{k_n h}\coth k_n(b-a)\psi_n(y). \tag{3.67}$$

Next, in region III, we need $Z_x^{III}(b+a,y) = (y-c)$ for $0 < y < d$ and $Z_x^{III}(b+a,y) = 0$ for $d < y < h$ and outgoing waves at infinity.

$$Z^{III}(x,y) = -\frac{\mathrm{i}M_0}{kh}\psi_0(y)\mathrm{e}^{\mathrm{i}k(x-b-a)} + \sum_{n=1}^{\infty}\frac{M_n}{k_n h}\mathrm{e}^{-k_n(x-b-a)}\psi_n(y). \tag{3.68}$$

Then,

$$Z^{III}(a,y) = -\frac{\mathrm{i}M_0}{kh}\psi_0(y) + H^{III}(y), \qquad \text{where} \qquad H^{III}(y) = -\sum_{n=1}^{\infty}\frac{M_n}{k_n h}\psi_n(y). \tag{3.69}$$

We also need a particular solution operating the region II. The harmonic potential $Z^{II}(x,y)$ needs to satisfy $Z_y^{II}(x,d) = b-x$ for $b-a < x < b+a$ and $Z_x^{II}(b\pm a,y) = 0$. Then, we have

$$Z^{II}(x,y) = 2\sum_{n=0}^{\infty}\frac{(-1)^n \sin\lambda_n(x-b)\cosh\lambda_n(h-y)}{\lambda_n^3 a\sinh\lambda_n(h-d)} \tag{3.70}$$

which is a translated version of (2.54) and is such that

$$Z^{II}(b\pm a,y) = \pm H^{II}(y), \qquad H^{II}(y) = 2\sum_{n=0}^{\infty}\frac{\cosh\lambda_n(h-y)}{\lambda_n^3 a\sinh\lambda_n(h-d)}. \tag{3.71}$$

With this we define the potentials in I, II and III for the rolling cylinder as, in I

$$\phi^r(x,y) = Z^I(x,y) + \left(a_0^r + \frac{M_0}{kh\sin(k(b-a))}\right)\cos(kx)\psi_0(y) + \sum_{n=1}^{\infty}a_n^r\cosh(k_n x)\psi_n(y) \tag{3.72}$$

in II

$$\phi^r(x,y) = Z^{II}(x,y) + (b_0^r(x-b)+c_0^r)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}\Big(b_n^r\sinh\mu_n(x-b) + c_n^r\cosh\mu_n(x-b)\Big)\hat{\psi}_n(y) \tag{3.73}$$

and in III

$$\phi^r(x,y) = Z^{III}(x,y) + \left(d_0^r + \frac{\mathrm{i}M_0}{kh}\right)\mathrm{e}^{\mathrm{i}k(x-b-a)}\psi_0(y) + \sum_{n=1}^{\infty}d_n^r\mathrm{e}^{-k_n(x-a)}\psi_n(y). \tag{3.74}$$

Then we let $U_1^r(y) = \phi_x^r(b-a,y)$ so that we obtain

$$-kha_0^r\sin k(b-a) - M_0 = \langle U_1^r,\psi_0\rangle, \qquad k_n h\sinh k_n(b-a)a_n^r = \langle U_1^r,\psi_n\rangle \tag{3.75}$$

and

$$b_0^r(h-d) = \langle U_1^s,\hat{\psi}_0\rangle, \qquad n\pi(b_n^r\cosh(\mu_n a) - c_n^r\sinh(\mu_n a)) = \langle U_1^r,\hat{\psi}_n\rangle. \tag{3.76}$$

Now letting $U_2^r(y) = \phi_x^r(b+a,y)$ we have

$$\mathrm{i}khd_0^r - M_0 = \langle U_2^r,\psi_0\rangle, \qquad -k_n hd_n^r = \langle U_2^r,\psi_n\rangle \tag{3.77}$$

and

$$b_0^r(h-d) = \langle U_2^r, \hat{\psi}_0 \rangle, \qquad n\pi(b_n^r \cosh(\mu_n a) + c_n^r \sinh(\mu_n a)) = \langle U_2^r, \hat{\psi}_n \rangle. \tag{3.78}$$

Matching pressures across $x = b - a$ for $d < y < h$ gives

$$H^I(y) + a_0^r \cos k(b-a)\psi_0(y) + \sum_{n=1}^{\infty} a_n^r \cosh k_n(b-a)\psi_n(y) =$$

$$-H^{II}(y) + (-b_0^r a + c_0^r)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}(-b_n^r \sinh(\mu_n a) + c_n^r \cosh(\mu_n a))\hat{\psi}_n(y) \tag{3.79}$$

And matching pressures across $x = b + a$ for $d < y < h$ gives

$$H^{III}(y) + d_0^r \psi_0(y) + \sum_{n=1}^{\infty} d_n^r \psi_n(y) =$$

$$H^{II}(y) + (b_0^r a + c_0^r)\hat{\psi}_0(y) + \sum_{n=1}^{\infty}(b_n^r \sinh(\mu_n a) + c_n^r \cosh(\mu_n a))\hat{\psi}_n(y) \tag{3.80}$$

Using the relations above in these two equations gives

$$\int_d^h U_1^r(t)K_{11}(y,t)\mathrm{d}t + \int_d^h U_2^r(t)K_{12}(y,t)\mathrm{d}t =$$

$$-a_0^r \cos k(b-a)\psi_0(y) - (b_0^r a - c_0^r)\hat{\psi}_0(y) - (H^I(y) + H^{II}(y)) \tag{3.81}$$

and

$$\int_d^h U_1^r(t)K_{21}(y,t)\mathrm{d}t + \int_d^h U_2^r(t)K_{22}(y,t)\mathrm{d}t = d_0^r\psi_0(y) - (b_0^r a + c_0^r)\hat{\psi}_0(y) + (H^{III}(y) - H^{II}(y)) \tag{3.82}$$

where $K_{11}(y,t)$ $K_{12}(y,t)$, $K_{21}(y,t)$ and $K_{22}(y,t)$ are as before.

We can write the coupled integral equations in matrix/vector form as

$$\mathbf{U}^r(y) = \begin{pmatrix} U_1^r(y) \\ U_2^r(y) \end{pmatrix}, \qquad \mathbf{U}^r(y) = d^2\left(\mathbf{U}_1^r(y)\mathbf{A}_1^r + \mathbf{U}_2^r(y)\mathbf{A}_2^r + \mathbf{U}_3^r(y)\right) \tag{3.83}$$

where

$$\mathbf{U}_1^r(y) = \begin{pmatrix} u_1^r(y) & u_3^r(y) \\ u_2^r(y) & u_4^r(y) \end{pmatrix}, \quad \mathbf{U}_2^r(y) = \begin{pmatrix} u_5^r(y) & u_7^r(y) \\ u_6^r(y) & u_8^r(y) \end{pmatrix}, \quad \mathbf{U}_3^r(y) = \begin{pmatrix} u_9^r(y) \\ u_{10}^r(y) \end{pmatrix} \tag{3.84}$$

and

$$\mathbf{A}_1^r = \begin{pmatrix} \hat{a}_0^r \\ \hat{d}_0^r \end{pmatrix}, \qquad \mathbf{A}_2^r = \begin{pmatrix} \hat{b}_0^r + \hat{c}_0^r \\ \hat{b}_0^r - \hat{c}_0^r \end{pmatrix} \tag{3.85}$$

where $\hat{a}_0^r = -a_0^r \cos k(b-a)/d^2$, $\hat{b}_0^r = -b_0^r a/d^2$, $\hat{c}_0^r = c_0^r/d^2$ and $\hat{d}_0^r = d_0^r/d^2$.

Also, we let

$$\mathbf{F}_1^r(y) = \begin{pmatrix} \psi_0(y) & 0 \\ 0 & \psi_0(y) \end{pmatrix}, \qquad \mathbf{F}_2^r(y) = \begin{pmatrix} \hat{\psi}_0(y) & 0 \\ 0 & \hat{\psi}_0(y) \end{pmatrix},$$

$$\mathbf{F}_3^r(y) = \begin{pmatrix} (-H^I(y) - H^{II}(y))/d^2 \\ (H^{III}(y) - H^{II}(y))/d^2 \end{pmatrix}. \tag{3.86}$$

Then $\mathbf{U}_i^r(t)$ are the solutions of the integral equations

$$\int_d^h \mathbf{K}(y,t)\mathbf{U}_i^r(t)\mathrm{d}t = \mathbf{F}_i^r(y), \qquad d < y < h,\ i = 1,2,3. \tag{3.87}$$

Following from this, we define the matrices

$$\mathbf{T}_{ij} = \langle \mathbf{F}_i^{r\mathrm{T}}, \mathbf{U}_j^r \rangle, \qquad i,j = 1,2,3. \tag{3.88}$$

From the earlier relations, (3.77), (3.78) we have

$$\begin{pmatrix} kh\hat{a}_0^r \tan k(b-a) \\ \mathrm{i}kh\hat{d}_0^r \end{pmatrix} - \begin{pmatrix} M_0/d^2 \\ M_0/d^2 \end{pmatrix} = \frac{1}{d^2}\int_d^h \mathbf{U}^r(y)\psi_0(y)\mathrm{d}y = \mathbf{T}_{11}\mathbf{A}_1 + \mathbf{T}_{12}\mathbf{A}_2 + \mathbf{T}_{13} \tag{3.89}$$

and

$$-\begin{pmatrix} \hat{b}_0^r(h-d)/a \\ \hat{b}_0^r(h-d)/a \end{pmatrix} = \frac{1}{d}\int_d^h \mathbf{U}^r(y)\hat{\psi}_0(y)\mathrm{d}y = \mathbf{T}_{21}\mathbf{A}_1 + \mathbf{T}_{22}\mathbf{A}_2 + \mathbf{T}_{23}. \tag{3.90}$$

These four equations are sufficient to determine the coefficients $\hat{a}_0^r$, $\hat{b}_0^r$, $\hat{c}_0^r$ and $\hat{d}_0^r$.

The roll component of the roll induced added-mass and radiation damping is defined as

$$\begin{aligned} \mathrm{i}\omega(A_{33}^w + \mathrm{i}B_{33}^w/\omega) = -\mathrm{i}\omega\rho\int_0^d \phi^r(b+a,y)(y-c)\mathrm{d}y + \mathrm{i}\omega\rho\int_0^d \phi^r(b-a,y)(y-c)\mathrm{d}y \\ -\mathrm{i}\omega\rho\int_{b-a}^{b+a} \phi^r(x,d)(b-x)\mathrm{d}x. \end{aligned} \tag{3.91}$$

We apply Green's Identity over region I to ϕ^r and Z^I and this ends up giving us the relation

$$\begin{aligned} \int_0^d \phi^r(b-a,y)(y-c)\mathrm{d}y &= \int_0^d Z^I(b-a,y)(y-c)\mathrm{d}y + \int_d^h Z^I(b-a,y)U_1^r(y)\mathrm{d}y \\ &= -d^2\hat{a}_0^r M_0 + \sum_{n=1}^{\infty}\frac{M_n^2\coth k_n(b-a)}{k_n h} + \langle U_1^r, H^I\rangle. \end{aligned} \tag{3.92}$$

Next, in region III, use of ϕ^r and Z^{III} gives, in much the same way

$$\int_0^d \phi^r(b+a,y)(y-c)\mathrm{d}y = d^2\hat{d}_0^r M_0 - \sum_{n=1}^{\infty}\frac{M_n^2}{k_n h} + \langle U_2^r, H^{III}\rangle. \tag{3.93}$$

In region II, we use ϕ^r and Z^{II} and get

$$\int_{b-a}^{b+a} \phi^r(x,d)(b-x)\mathrm{d}x = \int_{b-a}^{b+a} Z^{II}(x,d)(b-x)\mathrm{d}x - \langle U_1^r, H^{II}\rangle - \langle U_2^r, H^{II}\rangle. \tag{3.94}$$

Using (3.92)–(3.94) in (3.91) gives

$$\begin{aligned} (A_{33}^w + \mathrm{i}B_{33}^w/\omega) = -\rho d^4\Bigg[(\hat{d}_0^s + \hat{a}_0^s)(M_0/d^2) - \sum_{n=1}^{\infty}\frac{(M_n/d^2)^2}{k_n h}(1+\coth k_n(b-a)) \\ -4\sum_{n=0}^{\infty}\frac{\coth\lambda_n(h-d)}{\lambda_n^5 a d^4} + \mathbf{A}_1^{\mathrm{T}}\mathbf{T}_{31} + \mathbf{A}_2^{\mathrm{T}}\mathbf{T}_{32} + \mathbf{T}_{33}\Bigg]. \end{aligned} \tag{3.95}$$

The non-dimensional added-mass and damping are $\mu_{33}^w = A_{33}^w/I$ and $\nu_{33}^w = B_{33}^w/(I\omega)$ where I is the moment of inertia of the cylinder about its point of rotation.

3.5 The remaining hydrodynamic coefficients: the cross terms

Having dealt with A^w_{ii} and B^w_{ii} for $i = 1, 2, 3$ in each of the three preceding sections, we move onto calculating the off-diagonal coefficients. It is easy to establish reciprocity relations, namely that $A^w_{ij} = A^w_{ji}$ and $B^w_{ij} = B^w_{ji}$ for $i, j = 1, 2, 3$.

The components of added-mass and radiation damping in sway, induced by forced heave motion, are defined as

$$\mathrm{i}\omega(A^w_{21} + \mathrm{i}B^w_{21}/\omega) = -\mathrm{i}\omega\rho\int_0^d \phi^h(b+a,y)\mathrm{d}y + \mathrm{i}\omega\rho\int_0^d \phi^h(b-a,y)\mathrm{d}y. \qquad (3.96)$$

For the first integral, we use Green's Identity in region III with ϕ^h and X^{III} to give

$$\int_0^d \phi^h(b+a,y)\mathrm{d}y = \int_d^h U^h_2(y)\mathrm{d}y = -\frac{\mathrm{i}L_0}{kh}\langle U^h_2, \psi_0\rangle + \langle U^h_2, F^{III}\rangle = (h-d)L_0\hat{d}^h_0 + \langle U^h_2, F^{III}\rangle. \qquad (3.97)$$

Next, in region I apply Green's Identity to ϕ^h and X^I. Then

$$\int_0^d \phi^h(b-a,y)\mathrm{d}y = \int_d^h U^h_2(y)\mathrm{d}y = -\frac{L_0\cot k(b-a)}{kh}\langle U^h_1, \psi_0\rangle + \langle U^h_1, F^I\rangle$$
$$= -(h-d)L_0\hat{a}^h_0 + \langle U^h_1, F^I\rangle. \qquad (3.98)$$

Using (3.97), (3.98) in (3.96)

$$A^w_{21} + \mathrm{i}B^w_{21}/\omega = -d(h-d)\rho\left[(L_0/d)(\hat{d}^h_0 + \hat{a}^h_0) + (h-d)^{-1}\langle \mathbf{F}^{sT}_3, \mathbf{U}^h\rangle\right]. \qquad (3.99)$$

The final inner product can be expanded as

$$(h-d)^{-1}\langle \mathbf{F}^{sT}_3, \mathbf{U}^h\rangle = \langle \mathbf{F}^{sT}_3, \mathbf{U}^h_1\rangle\mathbf{A}^h_1 + \langle \mathbf{F}^{sT}_3, \mathbf{U}^h_2\rangle\mathbf{A}^h_2 + \langle \mathbf{F}^{sT}_3, \mathbf{U}^h_3\rangle. \qquad (3.100)$$

The reciprocal of this is the added-mass and radiation damping in heave, induced by forced sway motion

$$\mathrm{i}\omega(A^w_{12} + \mathrm{i}B^w_{12}/\omega) = -\mathrm{i}\omega\rho\int_{b-a}^{b+a} \phi^s(x,d)\mathrm{d}x. \qquad (3.101)$$

Use of Green's Identity in region II with ϕ^s and Y^{II} gives

$$\int_{b-a}^{b+a} \phi^s(x,d)\mathrm{d}x = \left(\frac{a}{h-d}\right)\int_d^h \phi^s(b+a,y)\mathrm{d}y + \left(\frac{a}{h-d}\right)\int_d^h \phi^s(b-a,y)\mathrm{d}y$$
$$-\langle U^s_1, G^{II}\rangle + \langle U^s_2, G^{II}\rangle. \qquad (3.102)$$

It follows that

$$A^w_{12} + \mathrm{i}B^w_{12}/\omega = -ad\rho\left[2\hat{c}^s_0 + \frac{(h-d)}{ad}\langle \mathbf{F}^{h\mathrm{T}}_3, \mathbf{U}^s\rangle\right]. \qquad (3.103)$$

The final inner product can be expanded using

$$d^{-1}\langle \mathbf{F}^{hT}_3, \mathbf{U}^s\rangle = \langle \mathbf{F}^{hT}_3, \mathbf{U}^s_1\rangle\mathbf{A}^s_1 + \langle \mathbf{F}^{hT}_3, \mathbf{U}^s_2\rangle\mathbf{A}^s_2 + \langle \mathbf{F}^{hT}_3, \mathbf{U}^s_3\rangle. \qquad (3.104)$$

Moving onto the heave components of added mass and radiation damping induced by roll motion we have

$$\mathrm{i}\omega(A^w_{32} + \mathrm{i}B^w_{32}/\omega) = -\mathrm{i}\omega\rho\int_{b-a}^{b+a} \phi^r(x,d)\mathrm{d}x. \qquad (3.105)$$

Use of Green's Identity in region II with ϕ^s and Y^{II} gives

$$\int_{b-a}^{b+a} \phi^r(x,d)\mathrm{d}x = \left(\frac{a}{h-d}\right)\int_d^h \phi^r(b+a,y)\mathrm{d}y + \left(\frac{a}{h-d}\right)\int_d^h \phi^r(b-a,y)\mathrm{d}y$$
$$+\int_{b-a}^{b+a}(b-x)Y^{II}(x,d)\mathrm{d}x - \langle U_1^r, G^{II}\rangle + \langle U_2^r, G^{II}\rangle. \quad (3.106)$$

The 3rd integral evaluates to zero, the first two are evaluated easily from the expressions for ϕ^r on $x = b \pm a$ in (3.72)–(3.74) and so we get

$$A_{32}^w + \mathrm{i}B_{32}^w/\omega = -ad^2\rho\left[2\hat{c}_0^r + \frac{(h-d)}{ad^2}\langle \mathbf{F}_3^{h\mathrm{T}}, \mathbf{U}^r\rangle\right]. \quad (3.107)$$

The reciprocal of this is roll component of added-mass and radiation induced by heave motion is

$$\mathrm{i}\omega(A_{23}^w + \mathrm{i}B_{23}^w/\omega) = -\mathrm{i}\omega\rho\int_0^d \phi^h(b+a,y)(y-c)\mathrm{d}y + \mathrm{i}\omega\rho\int_0^d \phi^h(b-a,y)(y-c)\mathrm{d}y$$
$$-\mathrm{i}\omega\rho\int_{b-a}^{b+a}\phi^h(x,d)(b-x)\mathrm{d}x. \quad (3.108)$$

We apply Green's Identity over region I to ϕ^h and Z^I and this ends up giving us the relation

$$\begin{aligned}\int_0^d \phi^h(b-a,y)(y-c)\mathrm{d}y &= \int_d^h Z^I(b-a,y)U_1^h(y)\mathrm{d}y \\ &= -(h-d)\hat{a}_0^h M_0 + \langle U_1^h, H^I\rangle. \quad (3.109)\end{aligned}$$

Next, in region III, use of ϕ^h and Z^{III} gives, in much the same way

$$\int_0^d \phi^h(b+a,y)(y-c)\mathrm{d}y = (h-d)\hat{d}_0^h M_0 + \langle U_2^h, H^{III}\rangle. \quad (3.110)$$

In region II, we use ϕ^h and Z^{II} and get

$$\int_{b-a}^{b+a}\phi^h(x,d)(b-x)\mathrm{d}x = -\langle U_1^h, H^{II}\rangle - \langle U_2^h, H^{II}\rangle. \quad (3.111)$$

Thus, finally,

$$A_{23}^w + \mathrm{i}B_{23}^w/\omega = -\rho(h-d)d^2\left[(\hat{d}_0^h + \hat{a}_0^h)(M_0/d^2) + (h-d)^{-1}\langle \mathbf{F}_3^{r\mathrm{T}}, \mathbf{U}^h\rangle\right]. \quad (3.112)$$

The non-dimensional added-mass and damping are $\mu_{23}^w = A_{23}^w/\sqrt{MI}$ and $\nu_{23}^w = B_{23}^w/(\sqrt{MI}\omega)$ where M and I are the mass and moment of inertia of the cylinder. Similarly for μ_{32}^w and μ_{32}^w, as well as the 13 and 31 components below.

Finally, the last set of two coefficients turn out, using similar methods to above, to be

$$A_{13}^w + \mathrm{i}B_{13}^w/\omega = -\rho d^3\left[(\hat{d}_0^s + \hat{a}_0^s)(M_0/d^2) - \sum_{n=1}^{\infty}\frac{(M_nL_n/d^3)}{k_nh}(1+\coth k_n(b-a)) + d^{-1}\langle \mathbf{F}_3^{r\mathrm{T}}, \mathbf{U}^s\rangle\right] \quad (3.113)$$

with

$$A_{31}^w + \mathrm{i}B_{31}^w/\omega = -\rho d^3\left[(\hat{d}_0^r + \hat{a}_0^r)(L_0/d) - \sum_{n=1}^{\infty}\frac{(M_nL_n/d^3)}{k_nh}(1+\coth k_n(b-a)) + d^{-1}\langle \mathbf{F}_3^{s\mathrm{T}}, \mathbf{U}^r\rangle\right]. \quad (3.114)$$

3.6 Numerical implementation of the solutions of section 3

In each problem, the solution is decomposed into ten unknown functions which we approximate in terms of a series solution as

$$u_i^{s,h,r}(y) \approx \sum_{n=0}^{N} \alpha_{n,i}^{s,h,r} v_n(y), \qquad d < y < h \tag{3.115}$$

$(i = 1, \ldots, 10)$. Here, $v_n(y)$ are the same as defined earlier towards the end of section 2.

We apply the Galerkin scheme as before. In matrix form, this process leads to a system of linear equations for the ten sets of coefficients, which we write in the form

$$\begin{pmatrix} K_{m,n}^{(11)} & K_{m,n}^{(12)} \\ K_{m,n}^{(21)} & K_{m,n}^{(22)} \end{pmatrix} \boldsymbol{\alpha}_{s,h,r} = \boldsymbol{f}_{s,h,r} \tag{3.116}$$

where

$$\boldsymbol{\alpha}_{s,h,r} = \left(\begin{array}{cc|cc|c} \alpha_{n,1}^{s,h,r} & \alpha_{n,3}^{s,h,r} & \alpha_{n,5}^{s,h,r} & \alpha_{n,7}^{s,h,r} & \alpha_{n,9}^{s,h,r} \\ \alpha_{n,2}^{s,h,r} & \alpha_{n,4}^{s,h,r} & \alpha_{n.6}^{s,h,r} & \alpha_{n,8}^{s,h,r} & \alpha_{n,10}^{s,h,r} \end{array} \right) \tag{3.117}$$

and

$$\boldsymbol{f}_{s,h,r} = \left(\begin{array}{cc|cc|c} F_m^{(s,h,r;1)} & 0 & F_m^{(s,h,r;2)} & 0 & F_m^{(s,h,r;9)} \\ 0 & F_m^{(s,h,r;1)} & 0 & F_m^{(s,h,r;2)} & F_m^{(s,h,r;10)} \end{array} \right) \tag{3.118}$$

for $m = 0, \ldots, N$ (where summation is implied by repeated suffices). The partitioning indicates the separation between, and the structure of, the three separate integral equations. In the above,

$$K_{m,n}^{(11)} = \sum_{r=1}^{\infty} \left\{ \frac{\coth k_r(b-a)}{k_r h} F_{r,m}^{(1)} F_{r,n}^{(1)} + \frac{\coth(2\mu_r a)}{r\pi} F_{r,m}^{(2)} F_{r,n}^{(2)} \right\} \tag{3.119}$$

$$K_{m,n}^{(12)} = K_{m,n}^{(21)} = -\sum_{r=1}^{\infty} \frac{\operatorname{cosech}(2\mu_r a)}{r\pi} F_{r,m}^{(2)} F_{r,n}^{(2)} \tag{3.120}$$

and

$$K_{m,n}^{(22)} = \sum_{r=1}^{\infty} \left\{ \frac{1}{k_r h} F_{r,m}^{(1)} F_{r,n}^{(1)} + \frac{\coth(2\mu_r a)}{r\pi} F_{r,m}^{(2)} F_{r,n}^{(2)} \right\} \tag{3.121}$$

where $F_{r,m}^{(1)}$, $F_{r,m}^{(2)}$ and $F_m^{(s,h,r;1)}$, $F_m^{(s,h,r;2)}$ are all defined as in section 2. The new terms here are

$$F_m^{(s;9)} = -\sum_{r=1}^{\infty} \frac{L_r \coth k_r(b-a)}{k_r h} F_{r,m}^{(1)}, \qquad F_m^{(s;10)} = F_m^{(s;3)} \tag{3.122}$$

with

$$F_m^{(h;9)} = F_m^{(h;3)}, \qquad F_m^{(h;10)} = -F_m^{(h;3)} \tag{3.123}$$

and

$$F_m^{(r;9)} = -\sum_{r=1}^{\infty} \frac{M_r \coth k_r(b-a)}{k_r h} F_{r,m}^{(1)} - 2\sum_{r=0}^{\infty} \frac{(-1)^m I_{2m+1/6}(\lambda_r(h-d))}{\lambda_r^3 a d^2 [\lambda_r(h-d)]^{1/6} \sinh \lambda_r(h-d)}, \tag{3.124}$$

and $F_m^{(r;10)} = F_m^{(r;3)}$. These new terms are all expressed in terms of expressions defined in section 2. The simplest way of expressing the matrices $\mathbf{P}_{ij}$ is in the block partitioned form

$$\begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} \\ \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{P}_{33} \end{pmatrix} = \boldsymbol{f}_s^{\mathrm{T}} \boldsymbol{\alpha}_s \tag{3.125}$$

where the matrices are 5×5 and partitioned into a (2,2,1) row and column structure. For example, $\mathbf{P}_{13}$ is a vector of two rows and one column, $\mathbf{P}_{33}$ is a 1×1 matrix (i.e. a coefficient). Similarly,

$$\begin{pmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \mathbf{S}_{13} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \mathbf{S}_{23} \\ \mathbf{S}_{31} & \mathbf{S}_{32} & \mathbf{S}_{33} \end{pmatrix} = \boldsymbol{f}_h^{\mathrm{T}} \boldsymbol{\alpha}_h \tag{3.126}$$

and

$$\begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{pmatrix} = \boldsymbol{f}_r^{\mathrm{T}} \boldsymbol{\alpha}_r \tag{3.127}$$

There are also other inner products lurking in amongst the expressions for the off-diagonal hydrodynamic coefficients.

For example, the set of matrix inner products required in the calculation of A_{21}, B_{21} are $\langle \mathbf{F}_3^{s\,\mathrm{T}}, \mathbf{U}_i^h \rangle$ for $i = 1, 2, 3$. These can be extracted from the appropriately partitioned elements of the bottom row of the 5×5 matrix formed by the multiplication of $\boldsymbol{f}_s^{\mathrm{T}} \boldsymbol{\alpha}_h$.

3.7 Low frequency asymptotics

We consider the exact results for the heave-induced heave radiation damping, given by the imaginary part of (3.65), in the limit as $kd \to 0$. This result is analogous to the one considered for a single cylinder in heave in §2.8.

We follow the same initial reasoning as described in §2.8, which implies that $\mathbf{F}_1^h \sim \mathbf{F}_2^h$ (which means the terms ignored are order $(kh)^2$ and that the correspondance is element by element) which implies that $\mathbf{U}_1^h \sim \mathbf{U}_2^h$. This in turn means that $\mathbf{S}_{11} \sim \mathbf{S}_{12} \sim \mathbf{S}_{21} \sim \mathbf{S}_{22}$ whilst $\mathbf{S}_{13} \sim \mathbf{S}_{23}$.

These approximations are used in the four equations (3.57), (3.58) defining the four coefficients a_0^h, b_0^h, c_0^h and d_0^h. By structuring the equations appropriately, one can form a matrix equation for the vector of four unknowns in which the coefficient matrix is symmetric and involves the four elements of $\mathbf{S}_{11}$ only (under this approximation). Using a symbolic algebra package, this system is inverted and the imaginary part extracted. As a result, it is then easy to confirm that, as $kh \to 0$,

$$\Im\{a_0^h\} \sim \frac{2\mathrm{i}a}{kh(h-d)}, \qquad \Im\{b_0^h\} \sim 0, \qquad \Im\{c_0^h\} \sim \frac{-2\mathrm{i}a}{kh(h-d)}, \qquad \Im\{d_0^h\} \sim \frac{-2\mathrm{i}a}{kh(h-d)} \tag{3.128}$$

Using this in (3.65) shows that $\nu_{22} \sim 2a/(khd)$ as $kh \to 0$, exactly the same result as for an isolated cylinder in

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