The solution to water wave scattering and radiation problems involving semi-immersed circular cylinders

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Abstract

In this technical report we describe the method of solution to a variety of two-dimensional problems involving semi-immersed cylinders in the free surface. These include: (i) determining the reflection and transmission coefficients for the scattering of incident waves by a single fixed cylinder; (ii) determining the added mass and radiation damping for a cylinder forced to move with unit velocity in both heave and sway motions; (iii) determining the heave and sway added mass and damping coefficients for a cylinder in forced motion next to a vertical boundary on which either a Neumann or Dirichlet condition is posed, equivalent to a pair of cylinders forced to move in-phase or out of phase with each other.

In each case, the fluid is assumed to be of infinite depth and the fluid motion is two-dimensional.

The description of the method of solution for parts (i) and (ii) follows that of Martin & Dixon (1983). The formulation of the solution for the final third problem is similar to that performed by Wang & Wahab (1971) for catamarans in heave oscillation only.

1 Scattering by a fixed cylinder

Cartesian coordinates (x, y) are chosen with y directed vertically downwards and $y = 0$ coinciding with the mean free surface of the fluid. A time harmonic wave of angular frequency ω is incident from $x = -\infty$ upon a fixed semi-immersed cylinder of radius a. Polar coordinates based on the centre of the cylinder (r, θ) are also used with $y = r\cos\theta$ and $x = r\sin\theta$ so that $\theta = 0$ coincides with the positive y axis.

In the fluid the time-independent complex-valued velocity potential $\Phi_S(x, y)$ satisfies

$$\nabla^2\Phi_S = 0 \qquad (1.1)$$

with $|\nabla\Phi_S| \to 0$ as $y \to \infty$ and

$$\frac{\partial\Phi_S}{\partial y} + K\Phi_S = 0, \qquad y = 0, \qquad \text{with } K = \omega^2/g \qquad (1.2)$$

On the cylinder, we have

$$\frac{\partial\Phi_S}{\partial r} = 0, \qquad \text{on } r = a,\ -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \qquad (1.3)$$

is the potential for the incident wave, propagating from left to right. The radiation conditions are

$$\Phi_S(x,y) \sim \begin{cases} (\mathrm{e}^{\mathrm{i}Kx} + R\mathrm{e}^{-\mathrm{i}Kx})\mathrm{e}^{-Ky}, & x \to -\infty \\ T\mathrm{e}^{\mathrm{i}Kx}\mathrm{e}^{-Ky}, & x \to \infty \end{cases} \tag{1.4}$$

It is the purpose of this section to determine R and T.

We decompose Φ_S by writing $\Phi_S = \Phi_I + \Phi_D$ where

$$\Phi_I = \mathrm{e}^{\mathrm{i}Kx-Ky} = \sum_{m=0}^{\infty} \frac{(-Kr)^m}{m!}\mathrm{e}^{-\mathrm{i}m\theta} \tag{1.5}$$

is the potential for the incident wave, propagating from right to left, whilst the diffraction potential, responsible for the waves outgoing from the cylinder is written as

$$\Phi_D = \sum_{n=0}^{\infty}\{a_n^s\Phi_n^s + a_n^a\Phi_n^a\}. \tag{1.6}$$

In the above, $a_n^{s,a}$ are as yet unknown coefficients associated with the sets of expansion functions for the cylinder, described by $\Phi_n^{s,a}$ for $n = 0, 1, \ldots$. The superscripts s and a identify those functions which are symmetric and antisymmetric (respectively) about the vertical line $\theta = 0$. This set of functions involve a fundamental source at the origin (symmetric), horizontal dipole at the origin, and an infinite set of symmetric and antisymmetric wave-free potentials, all singular at $r = 0$. Ursell (1949) first introduced the symmetric set of potentials when looking at a the heave motion for a single cylinder semi-immersed in the free surface.

The symmetric source in the free surface given by

$$\Phi_0^s(r,\theta) = \int_0^{\infty} \frac{\mathrm{e}^{-ky}\cos kx}{k-K}\mathrm{d}k + \pi\mathrm{i}\mathrm{e}^{-Ky}\cos Kx \tag{1.7}$$

with symmetric wave-free potentials

$$\Phi_n^s = \left(\frac{a}{r}\right)^{2n}\cos 2n\theta + \frac{Ka}{(2n-1)}\left(\frac{a}{r}\right)^{2n-1}\cos(2n-1)\theta, \qquad (n \geq 1). \tag{1.8}$$

Antisymmetric functions are given by a horizontal dipole, defined by

$$\Phi_0^a(r,\theta) = -\frac{1}{K}\frac{\partial\Phi_0^s}{\partial x} = \frac{\sin\theta}{Kr} + \int_0^{\infty} \frac{\mathrm{e}^{-ky}\sin kx}{k-K}\mathrm{d}k + \pi\mathrm{i}\mathrm{e}^{-Ky}\sin Kx \tag{1.9}$$

and antisymmetric wave-free potentials

$$\Phi_n^a = \left(\frac{a}{r}\right)^{2n+1}\sin(2n+1)\theta + \frac{Ka}{2n}\left(\frac{a}{r}\right)^{2n}\sin 2n\theta, \qquad (n \geq 1). \tag{1.10}$$

As $|x| \to \infty$, $\Phi_0^s \sim \pi\mathrm{i}\mathrm{e}^{-Ky}\mathrm{e}^{\mathrm{i}K|x|}$, and $\Phi_0^a \sim \mathrm{sgn}(x)\pi\mathrm{e}^{-Ky}\mathrm{e}^{\mathrm{i}K|x|}$. All wave free potentials decay to zero at infinity. It follows from Yu & Ursell (1961), Appendix A, we can write

$$\Phi_0^s(r,\theta) = \Re\{f_1(Kz)\} + \pi\mathrm{i}\Re\{\mathrm{e}^{-Kz}\} \tag{1.11}$$

and

$$\Phi_0^a(r,\theta) = \frac{\sin\theta}{Kr} - \Im\{f_1(Kz)\} - \pi\mathrm{i}\Im\{\mathrm{e}^{-Kz}\} \tag{1.12}$$

where $z = re^{\mathrm{i}\theta}$ and where

$$f_1(\zeta) = -(\gamma + \ln(\zeta))\mathrm{e}^{-\zeta} + \sum_{m=1}^{\infty} \frac{(-\zeta)^m}{m!}\left(1 + \frac{1}{2} + \ldots + \frac{1}{m}\right). \tag{1.13}$$

and $\gamma = 0.5772\ldots$ is Euler's constant. Note the presentation here is slightly different to that used in Yu & Ursell (1961). It follows that

$$\Phi_0^s = \sum_{m=0}^{\infty} \frac{(-Kr)^m G_m(Kr)}{m!} \cos m\theta + \theta \sum_{m=0}^{\infty} \frac{(-Kr)^m}{m!} \sin m\theta \tag{1.14}$$

whilst

$$\Phi_0^a = \frac{\sin\theta}{Kr} - \sum_{m=1}^{\infty} \frac{(-Kr)^m G_m(Kr)}{m!} \sin m\theta + \theta \sum_{m=0}^{\infty} \frac{(-Kr)^m}{m!} \cos m\theta \tag{1.15}$$

where we have defined

$$G_0(z) = -(\ln(z) + \gamma - \pi\mathrm{i}), \qquad G_p(z) = G_{p-1}(z) + 1/p, \quad (p \geq 1)$$

Applying of the cylinder boundary condition (1.3) – representing no-flow on the cylinder – and making the resulting expression, dependent on $\theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, orthogonal to $\cos 2m\theta$ and $\sin(2m+1)\theta$, results in the decoupled systems of equations,

$$\sum_{n=0}^{\infty} A_{mn}^{s,a} a_n^{s,a} = F_m^{s,a}, \qquad m = 0, 1, 2, \ldots \tag{1.16}$$

$$A_{mn}^s = -\tfrac{1}{4}\pi(2m)\delta_{mn} - (-1)^{m+n} \frac{(2n-1)Ka}{4m^2 - (2n-1)^2} \tag{1.17}$$

and

$$A_{mn}^a = -\tfrac{1}{4}\pi(2m+1)\delta_{mn} + (-1)^{m+n} \frac{(2n)Ka}{4n^2 - (2m+1)^2} \tag{1.18}$$

for $m = 0, 1, \ldots$ and $n = 1, 2, \ldots$. For $n = 0$, things are more complicated. We have,

$$A_{00}^s = -\tfrac{1}{2}\pi \cos Ka - \sum_{j=0}^{\infty} \frac{(-1)^j (Ka)^{2j+1}}{(2j+1)!} G_{2j+1}(Ka) \tag{1.19}$$

with, for $m \geq 1$,

$$\begin{aligned} A_{m0}^s = \frac{\pi}{4} \frac{(Ka)^{2m}}{(2m-1)!} G_{2m-1}(Ka) - \sum_{j=0}^{\infty} \frac{(-1)^{m+j}(Ka)^{2j+1} G_{2j}(Ka)}{(2j+1)!(1 - [2m/(2j+1)]^2)} \\ + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j (Ka)^j}{(j-1)!} (S_{j+2m} + S_{j-2m}) \end{aligned} \tag{1.20}$$

followed by, for all m,

$$\begin{aligned} A_{m0}^a = -\frac{\pi}{4(Ka)}\delta_{m0} + \frac{\pi}{4} \frac{(Ka)^{2m+1}}{(2m)!} G_{2m}(Ka) + \sum_{j=1}^{\infty} \frac{(-1)^{m+j}(Ka)^{2j} G_{2j-1}(Ka)}{(2j)!(1 - [(2m+1)/(2j)]^2)} \\ + \frac{1}{2} \sum_{j=1}^{\infty} \frac{(-1)^j (Ka)^j}{(j-1)!} (S_{j+2m+1} - S_{j-2m-1}). \end{aligned} \tag{1.21}$$

where we have defined

$$S_0 = 0, \qquad S_p = \frac{\sin(\frac{1}{2}p\pi) - (\frac{1}{2}p\pi)\cos(\frac{1}{2}p\pi)}{p^2}, \quad (p \geq 1).$$

The right-hand sides of (1.16) are given by $F_0^s = \sin(Ka)$, with

$$F_m^s = -\frac{\pi}{4}\frac{(Ka)^{2m}}{(2m-1)!} + \sum_{j=0}^{\infty} \frac{(-1)^{m+j}(Ka)^{2j+1}}{(2j+1)!(1-[2m/(2j+1)]^2)}, \qquad (m \geq 1) \tag{1.22}$$

and

$$F_m^a = -\mathrm{i}\frac{\pi}{4}\frac{(Ka)^{2m+1}}{(2m)!} - \mathrm{i}\sum_{j=1}^{\infty} \frac{(-1)^{m+j}(Ka)^{2j}}{(2j)!(1-[(2m+1)/(2j)]^2)}, \qquad (m \geq 0). \tag{1.23}$$

The reflection and transmission coefficients are given by

$$R = \pi(\mathrm{i}a_0^s - a_0^a), \qquad T = 1 + \pi(\mathrm{i}a_0^s + a_0^a). \tag{1.24}$$

These expressions precisely those to be found in Martin & Dixon (1983), correcting a single typographical error in their work (the power of Ka in their first sum defining A_{n1}^s should be $2j$ and not $2j+1$). Note also that our use of notation is slightly different to that of Martin & Dixon's.

2 The forced heave and sway motion of the cylinder

Next we consider a single cylinder, semi-immersed in the free surface undergoing forced small-amplitude heaving or swaying oscillations of unit velocity. We denote the associated velocity potentials Φ_1 (sway) and Φ_2 (heave) which will be symmetric and antisymmetric, respectively.

The aim is to determine the heave and sway added-mass and damping coefficients, which will be defined shortly.

The equations satisfied by Φ_j, $j = 1, 2$ are the same as for Φ_S in the previous section, apart that from on the cylinder boundary, the forced motion condition are

$$\left.\frac{\partial \Phi_1}{\partial r}\right|_{r=a} = \sin\theta, \qquad \text{and} \qquad \left.\frac{\partial \Phi_2}{\partial r}\right|_{r=a} = \cos\theta, \qquad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \tag{2.1}$$

whilst in the far field,

$$\Phi_j \sim [\mathrm{sgn}(x)]^j A_j \mathrm{e}^{\mathrm{i}K|x| + Ky}, \qquad \text{as } |x| \to \infty \tag{2.2}$$

and A_1 and A_2 are the far-field radiated wave amplitudes at $x = +\infty$ due to sway and heave motions respectively.

We write

$$\Phi_1 = a\sum_{n=0}^{\infty} b_n^a \Phi_n^a, \qquad \Phi_2 = a\sum_{n=0}^{\infty} b_n^s \Phi_n^s \tag{2.3}$$

(the multiplicative factor of a is introduced for algebraic convenience) where $b_n^{s,a}$ are coefficients to be determined and $\Phi_n^{s,a}$ are those functions defined previously in section 1.

By imposing the boundary conditions (2.1) and making the result of these operations orthogonal to $\sin(2m+1)\theta$ and $\cos 2m\theta$ (respectively) over $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, we obtain the uncoupled systems of equations

$$\sum_{n=0}^{\infty} A_{mn}^{s} b_n^s = \frac{(-1)^m}{1-4m^2}, \qquad \text{and} \qquad \sum_{n=0}^{\infty} A_{mn}^{a} b_n^a = \frac{\pi}{4}\delta_{m0}, \qquad m = 0, 1, \dots \tag{2.4}$$

where $A_{mn}^{s,a}$ are defined by (1.17)–(1.21). The added-mass and damping, a_{jj} and b_{jj}, are defined by

$$-(b_{jj} - \mathrm{i}\omega a_{jj}) = -\mathrm{i}\omega\rho \int_{-\pi/2}^{\pi/2} \Phi_j(a,\theta)\cos(\theta - \tfrac{1}{2}(2-j)\pi) a \mathrm{d}\theta, \qquad (j = 1, 2)$$

The non-dimensional added-mass and damping defined by $\mu_j = a_{jj}/M$ and $\nu_j = b_{jj}\omega/M$ where $M = \frac{1}{2}\rho\pi a^2$ is the mass of the cylinder are then

$$\mu_j + \mathrm{i}\nu_j = -\frac{4}{\pi}\sum_{n=0}^{\infty} b_n^{a,s} V_n^{a,s} \tag{2.5}$$

where $j = 1(2)$ correspond to the use of superscripts $a(s)$. In the above,

$$V_0^s = G_0(Ka) - \frac{\pi}{4}(Ka)G_1(Ka) + \sum_{j=1}^{\infty} \frac{(-1)^j (Ka)^{2j} G_{2j}(Ka)}{(2j)!(1-4j^2)} + \frac{1}{2}\sum_{j=1}^{\infty} \frac{(-Ka)^j}{j!}(S_{j+1} + S_{j-1}) \tag{2.6}$$

and

$$V_n^s = \frac{\pi}{4}(Ka)\delta_{n1} + \frac{(-1)^n}{1-4n^2}, \qquad (n \geq 1) \tag{2.7}$$

whilst

$$V_0^a = \frac{\pi}{4Ka} + \frac{\pi}{4}(Ka)G_1(Ka) + S_1 - \sum_{j=1}^{\infty} \frac{(-1)^j (Ka)^{2j} G_{2j}(Ka)}{(2j-1)!(1-4j^2)} + \frac{1}{2}\sum_{j=1}^{\infty} \frac{(-Ka)^j}{j!}(S_{j+1} - S_{j-1}) \tag{2.8}$$

and

$$V_n^a = \frac{(-1)^n (Ka)}{1-4n^2}, \qquad (n \geq 1). \tag{2.9}$$

The far field wave amplitude is determined from (2.2) and the and the far field behaviour of the functions $\Phi_0^{s,a}$ expressed in the previous section, which gives $A_1 = a\pi b_0^a$ and $A_2 = a\pi \mathrm{i} b_0^s$.

Various relations exist between the various problems considered in this and the previous section. Thus, the Newman relations state

$$R + (-1)^j T = -A_j/\bar{A}_j \tag{2.10}$$

which reduce to $R + T = b_0^s/\overline{b_0}^s$ and $R - T = -b_0^a/\overline{b_0}^a$. Additionally, the relations

$$b_{jj} = \rho\omega|A_j|^2 \tag{2.11}$$

translate to the non-dimensional versions $\nu_1 = 2\pi|b_0^a|^2$, and $\nu_2 = 2\pi|b_0^s|^2$. These conditions can be used to check the accuracy of the numerical procedure.

3 Calculation of the hydrodynamic coefficients for a cylinder next to a wall

In this section we detail the analysis needed to calculate the added mass and damping coefficients in the jth direction due to forced motion in the ith direction for a cylinder placed a distance b from a vertical boundary on which either a Neumann or Dirichlet condition is to be placed. We continue this section as though it is the former of these two conditions which is to be imposed, which is equivalent to having a wall along $x = -b$. The changes for the case of a Dirichlet condition are simple and outlined at the end of this section.

The effect of the wall is modelled by an image cylinder centred on $(-2b, 0)$ which moves in an equal and opposite manner. The potential for the radiation of waves due to unit forced motion in the sway ($j = 1$) and heave ($j = 2$) directions is given by

$$\Phi_j^w = a \sum_{n=0}^{\infty} \{ c_n^{(j)} \Psi_n^a + d_n^{(j)} \Psi_n^s \}, \qquad (j = 1, 2) \tag{3.1}$$

where

$$\Psi_n^{s,a}(r, \theta) = \Phi_n^{s,a}(r, \theta) + \Phi_n^{s,a}(r', -\theta') \tag{3.2}$$

and $z' = r' \mathrm{e}^{\mathrm{i}\theta'} = z + 2\mathrm{i}b$ define polar coordinates based on the point $(-2b, 0)$. The superscript w makes Φ_i^w here distinct from Φ_i used for the radiation potential for a cylinder in the absence of a wall.

The functions $\Psi_n^{s,a}$ have been constructed so as to satisfy the condition $\partial \Psi_n^{s,a}/\partial x = 0$ on $x = -b$, Unlike the radiation problems in section 2, there is no longer symmetry or antisymmetry in the fluid motion about a vertical line through the origin, hence the need for a combination of both sets of singular solutions in the expansion above. There is no particular significance in the use of superscripts s and a here; they merely serve to remind us of the functions from which they were derived and act as a convenient labelling system.

Using the translation of variables expressed in complex form, we can expand the second term above involving singularities at $(-2b, 0)$ in terms of (r, θ) variables and after some lengthy and detailed algebra we find that, for $n \geq 1$

$$\Psi_n^s(r, \theta) = \Phi_n^s(r, \theta) + \sum_{j=0}^{\infty} \left(\frac{r}{a} \right)^j (C_{j,n}^{ss} \cos j\theta + C_{j,n}^{sa} \sin j\theta) \tag{3.3}$$

provided $r < 2b$ (this restriction holds for all subsequent definitions), where

$$C_{2j,n}^{ss} = (-1)^{j+n} k_{2j,2n}, \qquad C_{2j+1,n}^{ss} = -(-1)^{j+n} \frac{Ka}{(2n-1)} k_{2j+1,2n-1}, \tag{3.4}$$

$$C_{2j,n}^{sa} = -(-1)^{j+n} \frac{Ka}{(2n-1)} k_{2j,2n-1}, \qquad C_{2j+1,n}^{sa} = -(-1)^{j+n} k_{2j+1,2n}, \tag{3.5}$$

and

$$k_{j,n} = \left(\frac{a}{2b} \right)^{j+n} \frac{(j+n-1)!}{j!(n-1)!}. \tag{3.6}$$

Similarly we have

$$\Psi_n^a(r, \theta) = \Phi_n^a(r, \theta) + \sum_{j=0}^{\infty} \left(\frac{r}{a} \right)^j (C_{j,n}^{as} \cos j\theta + C_{j,n}^{aa} \sin j\theta) \tag{3.7}$$

for $n \geq 1$ where

$$C^{as}_{2j,n} = -(-1)^{j+n} k_{2j,2n+1}, \qquad C^{as}_{2j+1,n} = +(-1)^{j+n} \frac{Ka}{(2n)} k_{2j+1,2n}, \tag{3.8}$$

$$C^{aa}_{2j,n} = (-1)^{j+n} \frac{Ka}{(2n)} k_{2j,2n}, \qquad C^{aa}_{2j+1,n} = (-1)^{j+n} k_{2j+1,2n+1}. \tag{3.9}$$

We move onto the $n = 0$ terms: these require more effort. As above, we are only concerned at how the image singularities at $x = -2b$, $y = 0$ can be expanded about the origin. From the integral expression for a source in the free surface we can write

$$\Psi^s_0(r, \theta) = \Phi^s_0(r, \theta) - \Re\{f_2(Kz)\} + \pi \mathrm{i} \mathrm{e}^{-Ky} \mathrm{e}^{\mathrm{i}K(x+2b)} \tag{3.10}$$

and

$$\Psi^a_0(r, \theta) = \Phi^a_0(r, \theta) - \frac{\sin\theta'}{Kr'} - \Im\{f_2(Kz)\} - \pi \mathrm{e}^{-Ky} \mathrm{e}^{\mathrm{i}K(x+2b)} \tag{3.11}$$

where

$$\frac{\sin\theta'}{Kr'} = \frac{1}{2Kb} \sum_{j=0}^{\infty} (-1)^j \left\{ \left(\frac{r}{2b}\right)^{2j} \cos 2j\theta - \left(\frac{r}{2b}\right)^{2j+1} \sin(2j+1)\theta \right\} \tag{3.12}$$

provides the expansion of the horizontal dipole at $(-2b, 0)$ about the origin whilst

$$f_2(\zeta) = ⨍_0^\infty \frac{\mathrm{e}^{-v(\zeta+2\mathrm{i}Kb)}}{1-v} \mathrm{d}v \tag{3.13}$$

with $z = y + \mathrm{i}x = r\mathrm{e}^{\mathrm{i}\theta}$, the contour being deformed below the pole. Then, assuming $x > -2b$, we deform the contour onto the negative imaginary axis to give

$$f_2(\zeta) = -\int_0^\infty \frac{\mathrm{e}^{\mathrm{i}v\zeta} \mathrm{e}^{-2vKb}}{v - \mathrm{i}} \mathrm{d}v = -\sum_{n=0}^{\infty} \frac{(\mathrm{i}\zeta)^n}{n!} I_n \tag{3.14}$$

with

$$I_n = \int_0^\infty \frac{v^n \mathrm{e}^{-\lambda v}}{v - \mathrm{i}} \mathrm{d}v = \frac{(n-1)!}{(2Kb)^n} + \mathrm{i} I_{n-1} \tag{3.15}$$

for $n \geq 1$ whilst

$$I_0 = \int_0^\infty \frac{\mathrm{e}^{-2vKb}}{v - \mathrm{i}} \mathrm{d}v = \mathrm{e}^{-2\mathrm{i}Kb} E_1(-2\mathrm{i}Kb) \tag{3.16}$$

and E_1 is an exponential integral (see Abramowitz & Stegun (1964)) defined by

$$E_1(z) = -\gamma - \ln(z) - \sum_{n=1}^{\infty} \frac{(-z)^n}{n n!}, \qquad |\arg(z)| < \pi. \tag{3.17}$$

It follows from (3.15) that

$$I_n = \mathrm{i}^n \sum_{j=1}^{n} \frac{\mathrm{e}^{-\mathrm{i}\pi j/2} (j-1)!}{(2Kb)^j} + \mathrm{i}^n I_0, \qquad n \geq 1 \tag{3.18}$$

and so

$$f_2(\zeta) = -I_0 \sum_{n=0}^{\infty} \frac{(-\zeta)^n}{n!} - \sum_{n=0}^{\infty} \left(-\frac{\zeta}{2Kb}\right)^n W_n, \qquad W_n = \sum_{j=1}^{n} \frac{\mathrm{e}^{-\mathrm{i}j\pi/2}(j-1)!(2Kb)^{n-j}}{n!}, \quad n \geq 1. \tag{3.19}$$

and $W_0 = 0$. We note the convenient recurrence relation $W_n = (2KbW_{n-1} + \mathrm{i}^{-n})/n$. Note also that the series above converges provided $r < 2b$. We decompose quantities above into their real and imaginary parts via $W_n = W_n^{(r)} + \mathrm{i}W_n^{(i)}$ and $I_0 = I_0^{(r)} + \mathrm{i}I_0^{(i)}$.

Thus we have

$$\begin{aligned}\Psi_0^s(r,\theta) &= \Phi_0^s(r,\theta) \\ &+ (I_0^{(r)} + \pi\mathrm{i}\mathrm{e}^{2\mathrm{i}Kb}) \sum_{j=0}^{\infty} \frac{(-Kr)^j}{j!} \cos j\theta + (-I_0^{(i)} + \pi\mathrm{e}^{2\mathrm{i}Kb}) \sum_{j=0}^{\infty} \frac{(-Kr)^j}{j!} \sin j\theta \\ &+ \sum_{j=0}^{\infty} \left(\frac{r}{a}\right)^j \left(-\frac{a}{2b}\right)^j (W_j^{(r)} \cos j\theta - W_j^{(i)} \sin j\theta)\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\Psi_0^a(r,\theta) &= \Phi_0^a(r,\theta) - \frac{1}{2Kb} \sum_{j=0}^{\infty} (-1)^j \left\{ \left(\frac{r}{2b}\right)^{2j} \cos 2j\theta - \left(\frac{r}{2b}\right)^{2j+1} \sin(2j+1)\theta \right\} \\ &+ (I_0^{(i)} - \pi\mathrm{e}^{2\mathrm{i}Kb}) \sum_{j=0}^{\infty} \frac{(-Kr)^j}{j!} \cos j\theta + (I_0^{(r)} + \pi\mathrm{i}\mathrm{e}^{2\mathrm{i}Kb}) \sum_{j=0}^{\infty} \frac{(-Kr)^j}{j!} \sin j\theta \\ &+ \sum_{j=0}^{\infty} \left(\frac{r}{a}\right)^j \left(-\frac{a}{2b}\right)^j (W_j^{(i)} \cos j\theta + W_j^{(r)} \sin j\theta)\end{aligned} \tag{3.21}$$

We impose the conditions

$$\frac{\partial \Phi_1^w}{\partial r} = \sin\theta, \qquad \frac{\partial \Phi_2^w}{\partial r} = \cos\theta, \qquad \text{on } r = a \text{ for } -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \tag{3.22}$$

and make the result of each equation orthogonal to $\sin(2m+1)\theta$ and $\cos 2m\theta$, $m = 0, 1, \ldots$ which gives a pair of coupled infinite systems equations for the coefficients $c_n^{(j)}$, $d_n^{(j)}$ for $j = 1, 2$.

The systems of equations we end up with can be written

$$\sum_{n=0}^{\infty} (c_n^{(1)}(A_{m,n}^a + B_{m,n}^{aa}) + d_n^{(1)} B_{m,n}^{sa}) = \frac{1}{4}\pi\delta_{m0} \tag{3.23}$$

and

$$\sum_{n=0}^{\infty} (c_n^{(1)} B_{m,n}^{as} + d_n^{(1)}(A_{m,n}^s + B_{m,n}^{ss})) = 0 \tag{3.24}$$

whilst the system for the coefficients $c_n^{(2)}$, $d_n^{(2)}$ differ from the above only in the right hand side terms which are replaced by 0 and $(-1)^m/(1-(2m)^2)$.

The quantities $A^{s,a}_{m,n}$ are defined earlier in (1.17)–(1.21) for all m and n. For $n \geq 1$ and $m \geq 0$ we have

$$B^{ss}_{m,n} = (-1)^{m+n}\left[\frac{\pi}{4}(2m)k_{2m,2n} - \frac{Ka}{2n-1}\sum_{j=0}^{\infty}\frac{k_{2j+1,2n-1}}{1-[2m/(2j+1)]^2}\right] \tag{3.25}$$

$$B^{as}_{m,n} = (-1)^{m+n}\left[-\frac{\pi}{4}(2m)k_{2m,2n+1} + \frac{Ka}{2n}\sum_{j=0}^{\infty}\frac{k_{2j+1,2n}}{1-[2m/(2j+1)]^2}\right] \tag{3.26}$$

and

$$B^{sa}_{m,n} = (-1)^{m+n}\left[-\frac{\pi}{4}(2m+1)k_{2m+1,2n} + \frac{Ka}{2n-1}\sum_{j=1}^{\infty}\frac{k_{2j,2n-1}}{1-[(2m+1)/(2j)]^2}\right] \tag{3.27}$$

$$B^{aa}_{m,n} = (-1)^{m+n}\left[\frac{\pi}{4}(2m+1)k_{2m+1,2n+1} - \frac{Ka}{2n}\sum_{j=1}^{\infty}\frac{k_{2j,2n}}{1-[(2m+1)/(2j)]^2}\right] \tag{3.28}$$

The tricky terms are $n = 0$ where we have

$$B^{ss}_{m,0} = -(I^{(r)}_0 + \pi \mathrm{i}\mathrm{e}^{2\mathrm{i}Kb})F^s_m + \frac{\pi}{4}(2m)(a/2b)^{2m}W^{(r)}_{2m} - \sum_{j=0}^{\infty}\frac{(-1)^{m+j}(a/2b)^{2j+1}W^{(r)}_{2j+1}}{1-[2m/(2j+1)]^2} \tag{3.29}$$

and

$$\begin{aligned} B^{as}_{m,0} = & -\frac{\pi m(-1)^m}{4Kb}\left(\frac{a}{2b}\right)^{2m} \\ & -(I^{(i)}_0 - \pi \mathrm{e}^{2\mathrm{i}Kb})F^s_m + \frac{\pi}{4}(2m)(a/2b)^{2m}W^{(i)}_{2m} - \sum_{j=0}^{\infty}\frac{(-1)^{m+j}(a/2b)^{2j+1}W^{(i)}_{2j+1}}{1-[2m/(2j+1)]^2} \end{aligned} \tag{3.30}$$

with

$$\begin{aligned} B^{sa}_{m,0} = & \ \mathrm{i}(I^{(i)}_0 - \pi \mathrm{e}^{2\mathrm{i}Kb})F^a_m \\ & + \frac{\pi}{4}(2m+1)(a/2b)^{2m+1}W^{(i)}_{2m+1} + \sum_{j=1}^{\infty}\frac{(-1)^{m+j}(a/2b)^{2j}W^{(i)}_{2j}}{1-[(2m+1)/(2j)]^2} \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} B^{aa}_{m,0} = & \ \frac{\pi(m+\frac{1}{2})(-1)^m}{4Kb}\left(\frac{a}{2b}\right)^{2m+1} \\ & -\mathrm{i}(I^{(r)}_0 + \pi \mathrm{i}\mathrm{e}^{2\mathrm{i}Kb})F^a_m - \frac{\pi}{4}(2m+1)(a/2b)^{2m+1}W^{(r)}_{2m+1} - \sum_{j=1}^{\infty}\frac{(-1)^{m+j}(a/2b)^{2j}W^{(r)}_{2j}}{1-[(2m+1)/(2j)]^2} \end{aligned} \tag{3.32}$$

where F^s_m, F^a_m are defined earlier in (1.23) and (1.24).

The quantities of interest are the non-dimensional added mass and damping coefficients which are defined as

$$\mu^w_{jk} + \mathrm{i}\nu^w_{jk} = -\frac{2}{\pi a}\int_{-\pi/2}^{\pi/2}\Phi^w_j(a,\theta)\cos(\theta - \tfrac{1}{2}(2-k)\pi)\mathrm{d}\theta, \qquad (j,k=1,2) \tag{3.33}$$

(having being non-dimensionalised by M and $M\omega$ as in Section 2). This gives

$$\mu_{j1}^w + \mathrm{i}\nu_{j1}^w = -\frac{4}{\pi}\left[\sum_{n=0}^{\infty} c_n^{(j)}(V_n^a + Y_n^{aa}) + d_n^{(j)} Y_n^{sa}\right] \tag{3.34}$$

and

$$\mu_{j2}^w + \mathrm{i}\nu_{j2}^w = -\frac{4}{\pi}\left[\sum_{n=0}^{\infty} c_n^{(j)} Y_n^{as} + d_n^{(j)}(V_n^s + Y_n^{ss})\right] \tag{3.35}$$

for $i = j, 2$, where for $n \geq 1$ we have

$$Y_n^{ss} = (-1)^n\left[-\frac{\pi}{4}\frac{Ka}{2n-1}k_{1,2n-1} + \sum_{j=0}^{\infty}\frac{k_{2j,2n}}{1-(2j)^2}\right] \tag{3.36}$$

$$Y_n^{as} = (-1)^n\left[\frac{\pi}{4}\frac{Ka}{2n}k_{1,2n} - \sum_{j=0}^{\infty}\frac{k_{2j,2n+1}}{1-(2j)^2}\right] \tag{3.37}$$

$$Y_n^{sa} = (-1)^n\left[-\frac{\pi}{4}k_{1,2n} - \frac{Ka}{2n-1}\sum_{j=0}^{\infty}\frac{(2j)k_{2j,2n-1}}{1-(2j)^2}\right] \tag{3.38}$$

$$Y_n^{sa} = (-1)^n\left[\frac{\pi}{4}k_{1,2n+1} + \frac{Ka}{2n}\sum_{j=0}^{\infty}\frac{(2j)k_{2j,2n}}{1-(2j)^2}\right] \tag{3.39}$$

Again, the calculations are more complicated for $n = 0$ where we find

$$Y_0^{ss} = -(I_0^{(r)} + \pi\mathrm{i}\mathrm{e}^{2\mathrm{i}Kb})\left[\frac{\pi}{4}Ka + H_1^+\right] - \frac{\pi}{4}(a/2b)W_1^{(r)} + \sum_{j=0}^{\infty}\frac{(-1)^j(a/2b)^{2j}W_{2j}^{(r)}}{1-(2j)^2} \tag{3.40}$$

and

$$Y_0^{as} = -(I_0^{(i)} - \pi\mathrm{e}^{2\mathrm{i}Kb})\left[\frac{\pi}{4}Ka + H_1^+\right] - H_2 - \frac{\pi}{4}(a/2b)W_1^{(i)} + \sum_{j=0}^{\infty}\frac{(-1)^j(a/2b)^{2j}W_{2j}^{(i)}}{1-(2j)^2} \tag{3.41}$$

where

$$H_1^+ = \sum_{j=0}^{\infty}\frac{(-1)^j(Ka)^{2j}}{(2j)!((2j)^2-1)} = -\tfrac{1}{2}(\cos(Ka) + \sin(Ka)/Ka + Ka\,\mathrm{Si}(Ka)) \tag{3.42}$$

and $\mathrm{Si}(\cdot)$ is the sine integral (Abramowitz & Stegun (1964)),

$$H_2 = \frac{1}{2Kb}\sum_{j=0}^{\infty}\frac{(a/2b)^{2j}}{1-(2j)^2} = \frac{1}{4Kb}\left(1 + \left(\frac{2b}{a} - \frac{a}{2b}\right)\tanh^{-1}(a/2b)\right) \tag{3.43}$$

Also,

$$Y_0^{sa} = (I_0^{(i)} - \pi\mathrm{e}^{2\mathrm{i}Kb})\left[\frac{\pi}{4}Ka + H_1^-\right] + \frac{\pi}{4}(a/2b)W_1^{(i)} - \sum_{j=1}^{\infty}\frac{(-1)^j(a/2b)^{2j}(2j)W_{2j}^{(i)}}{1-(2j)^2} \tag{3.44}$$

and

$$Y_0^{aa} = \frac{\pi(a/2b)}{8Kb} - (I_0^{(r)} + \pi \mathrm{i} e^{2\mathrm{i}Kb}) \left[\frac{\pi}{4}Ka + H_1^-\right] - \frac{\pi}{4}(a/2b)W_1^{(r)} + \sum_{j=1}^{\infty} \frac{(-1)^j (a/2b)^{2j} (2j) W_{2j}^{(r)}}{1-(2j)^2}. \tag{3.45}$$

where

$$H_1^- = \sum_{j=1}^{\infty} \frac{(-1)^j (Ka)^{2j}}{(2j-1)!((2j)^2-1)} = -\tfrac{1}{2}(\cos(Ka) - \sin(Ka)/Ka + Ka\,\mathrm{Si}(Ka)) \tag{3.46}$$

The changes required to consider a Dirichlet condition on the wall $x = -b$ are simply that the sign between the two sets of functions defining $\Psi_n^{s,a}$ are to be reversed. This has the effect of reversing the sign in front of the matrix coefficients $B_{mn}^{sa,sa}$ and $Y_n^{sa,sa}$.

4 Low frequency asymptotics

In this section, we consider the effect of letting $Ka \to 0$. For a single cylinder in isolation, this was considered in infinite depth by Ursell (1949) and in finite depth again by Ursell (1976). We can infer Ursell's asymptotic results from our systems of equations by taking the limit as $Ka \to 0$ and retaining leading order terms only. Thus,

$$A_{00}^a \sim -\frac{\pi}{4Ka}, \quad A_{00}^s \sim -\frac{\pi}{2},$$

and

$$A_{mn}^a \sim -\frac{\pi}{4}(2m+1)\delta_{mn}, \quad A_{mn}^s \sim -\frac{\pi}{4}(2m)\delta_{mn}, \qquad n \neq 0$$

with

$$F_0^a \sim -\mathrm{i}\frac{\pi}{4}Ka, \quad F_n^a \sim o(Ka), \qquad \text{and} \qquad F_0^s \sim Ka, \quad F_n^s \sim Ka\frac{(-1)^n}{(1-(2n)^2)}, \qquad n \neq 0.$$

Thus, we find that asymptotically, the coefficients defined by the solution of the heaving and swaying problems are, as $Ka \to 0$,

$$b_0^a \sim -Ka, \quad b_n^a \sim 0 \qquad \text{and} \qquad b_0^s \sim -\frac{2}{\pi}, \quad b_n^s \sim -\frac{4}{\pi}\frac{1}{(2n)}\frac{(-1)^n}{(1-(2n)^2)}$$

In order to work out the added mass and radiation we need the following estimates

$$V_0^a \sim \pi/(4Ka), \quad V_n^a \sim 0, \qquad n \neq 0$$

and

$$V_0^s \sim -\log(Ka) - \gamma + \pi\mathrm{i}, \quad V_n^s \sim \frac{(-1)^n}{(1-(2n)^2)},$$

This now means that

$$\mu_1 + \mathrm{i}\nu_1 \sim -\frac{4}{\pi}(-Ka)\left(\frac{\pi}{4Ka}\right) = 1$$

so that $\mu_1 \sim 1$, $\nu_1 \sim 0$. For the heave added mass and damping we have

$$\mu_2 + \mathrm{i}\nu_2 \sim -\frac{4}{\pi}\left[\frac{2}{\pi}(\log(Ka) + \gamma - \pi\mathrm{i}) - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n)(1-(2n)^2)^2}\right]$$

This latter sum is readily evaluated to $\frac{3}{4} - \log 2$ so that

$$\mu_2 \sim -\frac{8}{\pi^2}\left(\log(Ka) + \gamma - \tfrac{3}{2} + 2\log 2\right), \qquad \text{and} \qquad \nu_2 \sim \frac{8}{\pi}.$$

These are the results derived by Ursell (1949, 1976).

We move onto the consideration of the low frequency asymptotics for a cylinder in motion next to a wall. First, we equip ourselves with estimates of certain terms, needed for later calculations. For example

$$W_0 = 0, \quad W_1 = -\mathrm{i}, \quad W_n \sim \frac{(-1)^n \mathrm{i}^n}{n}$$

so $W_{2n}^{(r)} \sim (-1)^n/(2n)$, $W_{2n+1}^{(i)} \sim -(-1)^n/(2n+1)$ with $W_{2n+1}^{(r)} \sim 0$ and $W_{2n}^{(i)} \sim 0$.

Also,

$$I_0^{(r)} \sim -\log(2Kb) - \gamma, \quad I_0^{(i)} \sim \pi/2$$

Using this, we find that, for $n \geq 1$,

$$B_{mn}^{ss} \sim (-1)^{m+n}\frac{\pi}{4}(2m)\frac{(2m+2n-1)!}{(2m)!(2n-1)!}\left(\frac{a}{2b}\right)^{2m+2n}$$

with

$$B_{0n}^{ss} \sim -Ka(-1)^n \sum_{j=0}^{\infty} \frac{(2j+2n-1)!}{(2j+1)!(2n-1)!}\left(\frac{a}{2b}\right)^{2j+2n}$$

and, when $n = 0$, but $m \neq 0$,

$$B_{m0}^{ss} \sim (-1)^m \frac{\pi}{4}\left(\frac{a}{2b}\right)^{2m},$$

else $B_{00}^{ss} = O(Ka)$. Next, for $n \geq 1$,

$$B_{mn}^{as} \sim -(-1)^{m+n}\frac{\pi}{4}(2m)\frac{(2m+2n)!}{(2m)!(2n)!}\left(\frac{a}{2b}\right)^{2m+2n+1}$$

with

$$B_{0n}^{as} \sim Ka(-1)^n \sum_{j=0}^{\infty} \frac{(2j+2n)!}{(2j+1)!(2n)!}\left(\frac{a}{2b}\right)^{2j+2n+1}$$

and, when $n = 0$,

$$B_{m0}^{as} \sim -(-1)^m \frac{\pi}{4}\frac{(2m)}{Ka}\left(\frac{a}{2b}\right)^{2m+1} + f_1(m, a/2b)$$

where

$$f_1(m, a/2b) = \sum_{j=0}^{\infty} \frac{(-1)^m (a/2b)^{2j+1}}{(2j+1)(1 - [(2m)/(2j+1)]^2)}$$

Moving on, for $n \geq 1$, we have

$$B_{mn}^{sa} \sim -(-1)^{m+n}\frac{\pi}{4}\frac{(2m+2n)!}{(2m)!(2n-1)!}\left(\frac{a}{2b}\right)^{2m+2n+1}$$

and, when $n = 0$,

$$B^{sa}_{m0} \sim -(-1)^m \frac{\pi}{4}\left(\frac{a}{2b}\right)^{2m+1}.$$

Finally, for $n \geq 1$,

$$B^{aa}_{mn} \sim (-1)^{m+n}\frac{\pi}{4}\frac{(2m+2n+1)!}{(2m)!(2n)!}\left(\frac{a}{2b}\right)^{2m+2n+2}$$

with

$$B^{aa}_{m0} \sim (-1)^m \frac{\pi}{4}\frac{(2m+1)}{Ka}\left(\frac{a}{2b}\right)^{2m+2}$$

Inserting this into the system of equations for the coefficients for $c_n^{(1)}$ and $d_n^{(1)}$ and retaining leading order gives the following. For $m = 0$, we have from the first equation

$$\frac{c_0^{(1)}}{Ka}\left(\left(\frac{a}{2b}\right)^2 - 1\right) - d_0^{(1)}\left(\frac{a}{2b}\right) + \sum_{n=1}^{\infty} c_n^{(1)}(-1)^n(2n+1)(a/2b)^{2n+2} - \sum_{n=1}^{\infty} d_n^{(1)}(-1)^n(2n)(a/2b)^{2n+1} = 1 \qquad (4.1)$$

whilst for $m \geq 1$ we have

$$\frac{c_0^{(1)}}{Ka}\left(\frac{a}{2b}\right)^{2m+2} - d_0^{(1)}\left(\frac{1}{(2m+1)}\left(\frac{a}{2b}\right)^{2m+1}\right) - (-1)^m c_m^{(1)} + \sum_{n=1}^{\infty} c_n^{(1)}(-1)^n \frac{(2m+2n+1)!}{(2m+1)!(2n)!}\left(\frac{a}{2b}\right)^{2m+2n+2} - \sum_{n=1}^{\infty} d_n^{(1)}(-1)^n \frac{(2m+2n)!}{(2m+1)!(2n-1)!}\left(\frac{a}{2b}\right)^{2m+2n+1} = 0 \qquad (4.2)$$

Similarly, for the second system of equations, we have for $m = 0$,

$$c_0^{(1)}\tanh^{-1}(a/2b) - \frac{\pi}{2}d_0^{(1)} + Ka\sum_{n=1}^{\infty} c_n^{(1)}(-1)^n \sum_{j=0}^{\infty}\frac{(2j+2n)!}{(2j+1)!(2n)!}\left(\frac{a}{2b}\right)^{2n+2j+1} + Ka\sum_{n=1}^{\infty} d_n^{(1)}\frac{(-1)^n}{(2n-1)} - Ka\sum_{n=1}^{\infty} d_n^{(1)}(-1)^n\sum_{j=0}^{\infty}\frac{(2j+2n-1)!}{(2j+1)!(2n-1)!}\left(\frac{a}{2b}\right)^{2n+2j} = 0 \qquad (4.3)$$

and $f_1(0, a/2b) = \tanh^{-1}(a/2b)$ has been used. In the above, we have retained terms to $O(Ka)$ for reasons which will become clear.

For $m \geq 1$ we have

$$-\frac{c_0^{(1)}}{Ka}\left(\frac{a}{2b}\right)^{2m+1} + d_0^{(1)}\left(\frac{1}{(2m)}\left(\frac{a}{2b}\right)^{2m}\right) - (-1)^m d_m^{(1)} - \sum_{n=1}^{\infty} c_n^{(1)}(-1)^n\frac{(2m+2n)!}{(2m)!(2n)!}\left(\frac{a}{2b}\right)^{2m+2n+1} + \sum_{n=1}^{\infty} d_n^{(1)}(-1)^n\frac{(2m+2n-1)!}{(2m)!(2n-1)!}\left(\frac{a}{2b}\right)^{2m+2n} = 0 \qquad (4.4)$$

In the system of equations for $c_0^{(2)}$ and $d_0^{(2)}$, the right-hand sides of (4.1), (4.3) and (4.4) are replaced by 0, 1 and $(4/\pi)/(2m(1-(2m)^2))$ respectively

What is immediately apparent is that $c_0^{(j)} \sim O(Ka)$ for both $j = 1, 2$. It follows from the homogeneous equation () when $j = 1$ that $d_0^{(1)} \sim O(Ka)$ whilst the inhomogeneous version for $j = 1$ immediately yields $d_0^{(2)} \sim -2/\pi$ as $Ka \to 0$. Beyond this immediate observation, the systems of equations are still coupled and to complicated to produce explicit approximations to the coefficients as we were able to do in the case of a single cylinder in motion.

Instead, we assume use the fact that $a/2b < \frac{1}{2}$ and propose an expansion in terms of the parameter $\epsilon \equiv a/2b$. This has been done using the symbolic algebra package Maple and gives the following series

$$\frac{c_0^{(1)}}{Ka} \sim -(1 + \epsilon^2 + \epsilon^4 + 3\epsilon^6 + \ldots), \qquad \frac{d_0^{(1)}}{Ka} \sim -\frac{2}{\pi}(\epsilon - \tfrac{1}{3}\epsilon^3 - \tfrac{6}{5}\epsilon^5 + \ldots) \tag{4.5}$$

and

$$\frac{c_0^{(2)}}{Ka} \sim \frac{2}{\pi}(\epsilon + \tfrac{1}{3}\epsilon^3 + \tfrac{6}{5}\epsilon^5 + \ldots), \qquad d_0^{(2)} \sim -\frac{2}{\pi} \tag{4.6}$$

We can also make leading order approximation estimate to the remaining coefficients, by balancing terms in (4.2), (4.4) which gives

$$\left.\begin{aligned} c_n^{(1)} &\sim -(-1)^n\epsilon^{2n+2} \\ d_n^{(1)} &\sim (-1)^n\epsilon^{2n+1} \end{aligned}\right\}, \qquad \left.\begin{aligned} c_n^{(2)} &\sim \frac{(2/\pi)(-1)^n\epsilon^{2n+1}}{(2n+1)} \\ d_n^{(2)} &\sim -\frac{(4/\pi)(-1)^n}{((2n)(1-(2n)^2))} \end{aligned}\right\} \tag{4.7}$$

and we will also need the next order of approximation in $d_1^{(1)} \sim -\epsilon^3 - \epsilon^5$.

We turn to the expressions for the added mass and damping. Before this, we derive estimates to various terms. Hence

$$Y_0^{ss} \sim -\log(2Kb) - \gamma + \pi\mathrm{i} + f_2(a/2b), \qquad f_2 = \sum_{j=1}^{\infty} \frac{(a/2b)^{2j}}{(2j)(1-(2j)^2)}$$

$$Y_0^{as} \sim -\frac{\hat{H}_2}{Ka}, \qquad \hat{H}_2 = (a/2b)(\tfrac{1}{2} + \tfrac{1}{2}((2b/a) - (a/2b))\tanh^{-1}(a/2b))$$

$$Y_0^{sa} \sim -\frac{\pi}{4}(a/2b)$$

$$Y_0^{aa} \sim \frac{\pi}{4Ka}(a/2b)^2$$

In addition to this,

$$Y_n^{ss} \sim (-1)^n \sum_{j=0}^{\infty} \frac{(2j+2n-1)!}{(1-(2j)^2)(2j)!(2n-1)!} \left(\frac{a}{2b}\right)^{2n+2j}$$

$$Y_n^{as} \sim -(-1)^n \sum_{j=0}^{\infty} \frac{(2j+2n)!}{(1-(2j)^2)(2j)!(2n)!} \left(\frac{a}{2b}\right)^{2n+2j+1}$$

$$Y_n^{sa} \sim -(-1)^n\frac{\pi}{4}(2n)\left(\frac{a}{2b}\right)^{2n+1}$$

$$Y_n^{aa} \sim (-1)^n\frac{\pi}{4}(2n+1)\left(\frac{a}{2b}\right)^{2n+2}$$

Now, we can use this to approximate the added mass and damping coefficients. Starting with

$$\begin{aligned}\mu_{j1}^w + \mathrm{i}\nu_{j1}^w &\sim -\frac{4}{\pi}\bigg[c_0^{(j)}\left(\frac{\pi}{4Ka} + \frac{\pi}{4Ka}\left(\frac{a}{2b}\right)^2\right) - d_0^{(j)}\frac{\pi}{4}\left(\frac{a}{2b}\right) \\ &\quad + \frac{\pi}{4}\sum_{n=1}^{\infty} c_n^{(j)}(-1)^n(2n+1)(a/2b)^{2n+2} - \frac{\pi}{4}\sum_{n=1}^{\infty} d_n^{(j)}(-1)^n(2n)(a/2b)^{2n+1}\bigg] \\ &= -\delta_{j1} - 2\frac{c_0^{(j)}}{Ka} \end{aligned} \qquad (4.8)$$

after (4.1) has been used. We note that when considering the low frequency asymptotics of a single cylinder in the absence of the wall in heave and sway, the result for sway was very simple, but the result for heave required more work. The same occurs here in that the result above in (4.8) is simple, but the next result much less so. Thus, we have

$$\begin{aligned}\mu_{j2}^w + \mathrm{i}\nu_{j2}^w &\sim -\frac{4}{\pi}\bigg[c_0^{(j)}\left(-\frac{\hat{H}_2}{Ka}\right) + d_0^{(j)}\left(-\log(2K^2ab) - 2\gamma + 2\pi\mathrm{i} + f_2(a/2b)\right) \\ &\quad - \sum_{n=1}^{\infty} c_n^{(j)}(-1)^n \sum_{m=0}^{\infty}\frac{(2m+2n)!}{(1-(2m)^2)(2m)!(2n)!}\left(\frac{a}{2b}\right)^{2m+2n+1} \\ &\quad + \sum_{n=1}^{\infty} d_n^{(j)}(-1)^n \sum_{m=0}^{\infty}\frac{(2m+2n-1)!}{(1-(2m)^2)(2m)!(2n-1)!}\left(\frac{a}{2b}\right)^{2m+2n} \\ &\quad + \sum_{n=1}^{\infty} d_n^{(j)}\frac{(-1)^n}{(1-(2n)^2)}\bigg]. \end{aligned} \qquad (4.9)$$

Note that from taking (4.4), dividing by $(1-(2m)^2)$ and summing over $m = 1, 2, \dots$ we have

$$\begin{aligned} &-\frac{c_0^{(j)}}{Ka}(\hat{H}_2 - a/2b) + d_0^{(j)} f_2(a/2b) - \sum_{n=1}^{\infty} d_n^{(j)}\frac{(-1)^n}{(1-(2n)^2)} \\ &- \sum_{n=1}^{\infty} c_n^{(j)}(-1)^n \sum_{m=1}^{\infty}\frac{(2m+2n)!}{(1-(2m)^2)(2m)!(2n)!}\left(\frac{a}{2b}\right)^{2m+2n+1} \\ &+ \sum_{n=1}^{\infty} d_n^{(j)}(-1)^n \sum_{m=1}^{\infty}\frac{(2m+2n-1)!}{(1-(2m)^2)(2m)!(2n-1)!}\left(\frac{a}{2b}\right)^{2m+2n} = \delta_{j2}\sum_{m=1}^{\infty}\frac{(4/\pi)}{(2m)(1-(2m)^2)^2}. \end{aligned}$$

This can be substituted into (4.9) above to eliminate many of the terms to give

$$\begin{aligned}\mu_{j2}^w + \mathrm{i}\nu_{j2}^w &\sim -\frac{4}{\pi}\bigg[-\frac{c_0^{(j)}}{Ka}\left(\frac{a}{2b}\right) + d_0^{(j)}\left(-\log(2K^2ab) - 2\gamma + 2\pi\mathrm{i}\right) \\ &\quad - \sum_{n=1}^{\infty} c_n^{(j)}(-1)^n\left(\frac{a}{2b}\right)^{2n+1} + \sum_{n=1}^{\infty} d_n^{(j)}(-1)^n\left(\frac{a}{2b}\right)^{2n} \\ &\quad + 2\sum_{n=1}^{\infty} d_n^{(j)}\frac{(-1)^n}{(1-(2n)^2)} + \frac{4}{\pi}(\tfrac{3}{4} - \log 2)\delta_{j2}\bigg] \end{aligned} \qquad (4.10)$$

We can now put everything together and formulate approximate asymptotic expressions for the added mass and damping coefficients, first using (4.5), (4.6) in (4.8) to give

$$\mu_{11} \sim 1 + 2(\epsilon^2 + \epsilon^4 + 3\epsilon^6 + \dots) \qquad \nu_{11} \sim 0$$

and

$$\mu_{21}^w \sim -\frac{4}{\pi}(\epsilon + \tfrac{1}{3}\epsilon^3 + \tfrac{6}{5}\epsilon^5 \ldots), \qquad \nu_{21}^w \sim 0.$$

Now using both (4.5), (4.6) and (4.7) in (4.10) and retaining all terms to $O(\epsilon^5)$ gives

$$\mu_{12}^w \sim -\frac{4}{\pi}(\epsilon + \tfrac{1}{3}\epsilon^3 + \tfrac{6}{5}\epsilon^5 \ldots), \qquad \nu_{12}^w \sim 0.$$

confirming the reciprocity relations, and

$$\mu_{22}^w \sim -\frac{8}{\pi^2}\left(\log(2K^2ab) + 2\gamma - \tfrac{3}{2} + 2\log 2\right), \qquad \nu_{22}^w \sim \frac{16}{\pi}.$$

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