ON THE CONNECTION BETWEEN STEP APPROXIMATIONS AND DEPTH-AVERAGED MODELS FOR WAVE SCATTERING BY VARIABLE BATHYMETRY

by R. PORTER

(*School of Mathematics, University of Bristol, BS8 1TW, UK*)

Summary

Two popular and computationally-inexpensive class of methods for approximating the propagation of surface waves over two-dimensional variable bathymetry are "step approximations" and "depth-averaged models". In the former, the bathymetry is discretised into short sections of constant depth connected by vertical steps. Scattering across the bathymetry is calculated from the product of 2×2 transfer matrices whose entries encode scattering properties at each vertical step taken in isolation from all others. In the latter, a separable depth dependence is assumed in the underlying velocity field and a vertical averaging process is implemented leading to a 2nd order Ordinary Differential Equation (ODE).

In this paper the step approximation is revisited and shown to be equivalent to an ODE describing a depth-averaged model in the limit of zero-step length. The ODE depends on how the solution to the canonical vertical step problem is approximated. If a shallow-water approximation is used, then the well-known linear shallow water equation results. If a plane-wave variational approximation is used, then a new variant of the Mild-Slope Equations is recovered.

1. Introduction

A problem of longstanding interest in the setting of classical linearised water wave theory is how to determine the two-dimensional scattering of incident plane-crested surface waves by changes in the fluid depth. Only one exact solution is known, due to (**1**), for a particular class of bathymetry described by a smooth transition from one depth to another. In all other cases an approximate solution must be sought and a number of approaches can be employed.

Numerical methods that apply directly to the original boundary-value problem fall into two main classes. Finite element methods place the governing equation and boundary conditions on a discretised domain and connecting boundary. They are normally applied to problems posed with a non-linear free surface condition (e.g. (**2**)). Boundary integral methods use Green's functions to reformulate the boundary-value problem, without approximation, into integral equations over one-dimensional curves in space. Those integral equations can subsequently be approximated by discretisation, whence they are referred to as boundary element methods (hydrodynamic solvers including WAMIT and NEMOH are based on this – e.g. see (**3**)), or by employing spectral methods (e.g. (**4**), (**5**)).

The approaches described above have the property that approximations converge to the exact solution with increasing refinement of the numerical scheme. However, these methods also come with a relatively high computational cost and it has long been recognised in

Q. Jl Mech. Appl. Math. (????) ?? (??), 1–17

practical coastal engineering applications that accurate solutions of the exact linear problem (itself already an approximation to the full non-linear problem) are not always necessary and it can often be sufficient to obtain good, less numerically expensive, approximations.

A number of approaches have been developed over the years with this goal in mind. This is typically done either by simplifying the governing Laplace equation or the boundary or both. One such method, termed the 'step approximation', developed about 30 years ago (e.g. (**6**), (**7**), (**8**), (**9**), (**10**), (**11**), (**12**), (**13**)), involves approximating the continuous function representing the depth, $h(x)$, by piecewise constants and the bed is therefore represented by a finite number of discrete sections of constant depth connected by vertical steps. Besides this approximation to the geometry, a further approximation is usually introduced to simplify the calculation of the scattering process which is that the influence of evanescent waves generated at each step on neighbouring steps is neglected. An exception was (**10**) who extended the basic method to include the influence of a finite number of evanescent waves. In addition, the solution to the canonical problem that remains, that of the scattering by single isolated step, is required and this is typically approximated also. Again there are exceptions: (**13**) solved the two-dimensional problem in a mapped domain of uniform height in which the step manifested itself as an abrupt change in one of the lateral boundary condition conditions allowing the solution to be found without approximation. Combining all three approximations described above results in a fast numerical scheme in which the scattering process over a local change in depth can be represented by the multiplication of N two-by-two matrices (where N is the number of steps describing $h(x)$) whose entries are given explicitly. Numerical results (e.g. (**13**)) have shown the method can provide a good approximation to independently-derived exact or accurate computational results. More recent applications of the step approximation include (**14**) and (**15**).

However, until now it has been unclear whether the step approximation method converges to the solution of the original problem in the limit $N \to \infty$, as the step size is reduced to zero. In this paper we consider this limit and show it does not. In fact, it will be shown that the limiting step-approximation calculation of scattering can be recast in terms of solutions of a 2nd order ODE whose coefficients are related to the solution of the canonical problem of scattering across a single step. When the single step scattering is approximated using shallow-water (or long-wave) assumptions, we will show that the corresponding limiting 2nd order ODE is the Shallow Water Equation (or SWE – see, e.g. (**16**)). When scattering at a step is approximated by a variational principle involving a plane-wave solution (an approximation due to (**17**)) we will show that corresponding limiting 2nd order ODE is a new version of the Mild-Slope Equation (MSE) recently derived by (**18**).

The SWE and MSE are themselves popular approximate models of wave scattering which have, prior to this work, been considered independent to the step-approximation approach. In particular, the bathymetry is regarded as continuous whilst simplifying assumptions are made about the structure of the solution in the domain. Classified as depth-averaged models, their popularity arises from the simultaneous removal of the complication of the variable depth and reduction in the order of the underlying equations. Both the SWE and MSE models are underpinned by an assumption that the bed gradients are small when compared to h/λ, λ being the wavelength; the SWE distinguishes itself from the MSE in that it also requires h/λ to be small. A consequence of this connection made in this paper is that the restriction on bed gradients needs to apply to step approximations also. This conclusion is supported by the following comment found in (**13**): "Thus, according to a referee, although the Devillard method works surprisingly well for the sinusoidal bottom profile considered by OHare & Davies (1992) (sic) it does not work for any rapidly varying bottom profiles."

The SWE can be derived in many different ways. Stoker's (**16**) account describes the derivation as the leading order equation that results from an expansion, in the small parameter h/λ, of velocities, pressures and surface elevations in the governing mass and (linearised) momentum equations. The derivation reveals that the velocity field is independent of depth at leading order. When this assumption is forced upon the approximation to the velocity field in a variational principle, as in (**18**), the Shallow Water Equations result also. The MSE results from the same variational principle as the SWE but using a depth dependence whose functional form coincides with that for waves propagating over a locally-flat bed.

The MSE has a long history, with the earliest models attributed to (**19**) and (**20**). (**21**) provides an account of the rich background to the MSE and builds from the paper of (**22**) who introduced the so-called Modified MSE (MMSE). At the time of writing a new 'fundamental' version of the MSE has emerged (**18**) which is established by adopting a more general variational principle, relaxing constraints placed on earlier derivations including those of (**22**) and (**21**). Thus the underlying variational principle is capable of reproducing both the MMSE of (**22**) and the Complementary MSE (CMSE) of (**23**) and (**24**), in each case by introducing additional constraints on the approximation.

The structure of the paper is as follows. Sections 2 & 3 provide the background and details to the step approximation method. Much of this material can be found in (**17**) and elsewhere. However, these details are required to set up the new theory developed in Section 4 in which the step approximation is considered in the limit as the step size tends to zero using an approach which the author believes to be new. Section 5 summarises the work and suggests how it might be extended or developed to help shed light on other problems.

2. Setting up the step approximation

Cartesian coordinates (x, z) are used with $z = 0$ coinciding with the mean free surface and z pointing upwards. The fluid bed is described by $z = -h(x)$ where h is a continuous function which tends to h_1 as $x \to -\infty$ and to h_{N+1} as $x \to +\infty$. Waves are incident from $x = -\infty$ and are partially reflected back to $-\infty$ and partially transmitted to $x = \infty$. It is supposed that we define the points x_n, $1 \le n \le N$ and make an approximation $H_N(x)$, say, to $h(x)$ of the form

$$H_N(x) = h(\tfrac{1}{2}(x_n + x_{n+1})) = h_{n+1}, \qquad x_n < x < x_{n+1}$$

for $n = 1, \ldots, N-1$ with $H_N(x) = h_1$ for $x < x_1$ and $H_N(x) = h_{N+1}$ for $x > x_N$. We do not elaborate on strategies for choosing x_n here, but assume whatever strategy is adopted ensures that $\lim_{N\to\infty} H_N(x)$ approaches $h(x)$ under some reasonable measure.

Wave scattering over the piecewise-constant fluid depth, $H_N(x)$, is considered under the step-approximation method in which propagating waves across each section of constant depth are connected by discrete scattering processes at the discontinuities, $x = x_n$, of $H_N(x)$. It is also assumed that the calculation of scattering at $x = x_n$ is made in isolation of all other discontinuities. Thus, the canonical problem at the heart of the step-approximation method is to determine the relation between incoming and outgoing propagating wave amplitudes as waves pass across a vertical step from one constant depth to another. This problem is considered in the next section.

3. Transfer matrix for a single step

In $x < 0$ suppose the fluid depth is h_1 and in $x > 0$ it is h_2. To consider a general step at $x = x_n$ we can replace h_1 by h_n and h_2 by h_{n+1} and x by $x - x_n$ throughout. For now, we assume $h_2 < h_1$ and will comment on the reversal of this case later. A vertical wall runs over $-h_1 < z < -h_2$ on $x = 0$. Linearised time-harmonic assumptions apply to the governing equations. That is the fluid is assumed to be incompressible and inviscid and its motion irrotational of angular frequency ω and of small amplitude. Thus the two-dimensional fluid velocity is $\mathbf{u}(x, z, t) = \Re\{[\nabla\phi(x, z)]\mathrm{e}^{-\mathrm{i}\omega t}\}$ where

$$\nabla^2\phi = 0 \tag{3.1}$$

in the fluid domain and

$$\phi_z - (\omega^2/g)\phi = 0 \tag{3.2}$$

where g is gravity. Where the fluid connects to rigid boundaries with outward normal $\mathbf{n}$

$$\mathbf{n} \cdot \nabla\phi = 0. \tag{3.3}$$

Far away from both the left and right of the step, we assume incoming and outgoing waves. We write

$$\phi(x, z) \sim \begin{cases} (A_1\mathrm{e}^{\mathrm{i}k_1 x} + B_1\mathrm{e}^{-\mathrm{i}k_1 x})Z_0(h_1, z), & x \to -\infty \\ (A_2\mathrm{e}^{\mathrm{i}k_2 x} + B_2\mathrm{e}^{-\mathrm{i}k_2 x})Z_0(h_2, z), & x \to \infty \end{cases} \tag{3.4}$$

where $k_i = k(h_i)$, $i = 1, 2$, are defined as the real positive roots of the dispersion relation $\omega^2/g = k(h)\tanh(k(h)h)$ and

$$Z_0(h, z) = C_0(h)\cosh[k(h)(h + z)] \tag{3.5}$$

where $C_0(h)$ is an arbitrary real scale factor whose definition affects the scaling of the surface amplitudes given by $\Re\{(\mathrm{i}\omega/g)\phi(x, 0)\mathrm{e}^{-\mathrm{i}\omega t}\}$.

3.1 *Formulation of integral equations*

We follow (**17**) and (**25**). Solutions in $x < 0$ and $x > 0$ are expressed in terms of separation series. In $x < 0$ we write

$$\phi(x, z) = (A_1\mathrm{e}^{\mathrm{i}k_1 x} + B_1\mathrm{e}^{-\mathrm{i}k_1 x})Z_0(h_1, z) + \sum_{n=1}^{\infty} a_{1,n}\mathrm{e}^{\gamma_{1,n}x}Z_n(h_1, z) \tag{3.6}$$

and in $x > 0$

$$\phi(x, z) = (A_2\mathrm{e}^{\mathrm{i}k_2 x} + B_2\mathrm{e}^{-\mathrm{i}k_2 x})Z_0(h_2, z) - \sum_{n=1}^{\infty} a_{2,n}\mathrm{e}^{-\gamma_{2,n}x}Z_n(h_2, z) \tag{3.7}$$

where

$$Z_n(h_i, z) = \cos[\gamma_{i,n}(h_i + z)] \tag{3.8}$$

for $i = 1, 2$ and $\gamma_{i,n}$, $n \geq 1$, are the real roots of $\omega^2/g = -\gamma_{i,n}\tan(\gamma_{i,n}h_i)$. We note the orthogonality condition

$$\frac{1}{h}\int_{-h}^{0} Z_n(h,z)Z_m(h,z)\,dz = \delta_{mn}N_n(h) \tag{3.9}$$

for $n, m \geq 0$ where

$$N_0(h) = \frac{1}{2}\left(1 + \frac{\sinh(2k(h)h)}{2k(h)h}\right)C_0^2(h) \tag{3.10}$$

and

$$N_n(h_i) = \frac{1}{2}\left(1 + \frac{\sin(2\gamma_{i,n}h_i)}{2\gamma_{i,n}h_i}\right) \tag{3.11}$$

for $i = 1, 2$. We let $U(z) = \phi_x(0,z)$, $-h_2 < z < 0$ noting that $\phi_x(0,z) = 0$ for $-h_1 < z < -h_2$ and apply the orthogonality result to the two expansions to give

$$\mathrm{i}k_ih_iN_0(h_i)(A_i - B_i) = \int_{-h_2}^{0} U(z)Z_0(h_i,z)\,dz \tag{3.12}$$

and

$$\gamma_{i,n}h_iN_n(h_i)a_{i,n} = \int_{-h_2}^{0} U(z)Z_n(h_i,z)\,dz \tag{3.13}$$

each for $i = 1, 2$. Matching expressions for $\phi(0,z)$ from (3.6) and (3.7) across $-h_2 < z < 0$ and reusing (3.13) then results in

$$(\mathcal{K}U)(z) \equiv \int_{-h_2}^{0} U(z')K(z,z')\,dz' = -(A_1 + B_1)Z_0(h_1,z) + (A_2 + B_2)Z_0(h_2,z), \tag{3.14}$$

over $-h_2 < z < 0$ where the kernel of the integral operator $\mathcal{K}$ is defined by

$$K(z,z') = \sum_{n=1}^{\infty}\left\{\frac{Z_n(h_1,z)Z_n(h_1,z')}{N_n(h_1)\gamma_{1,n}h_1} + \frac{Z_n(h_2,z)Z_n(h_2,z')}{N_n(h_2)\gamma_{2,n}h_2}\right\}. \tag{3.15}$$

Linearity of (3.14) allows us to write

$$U(z) = -(A_1 + B_1)U_1(z) + (A_2 + B_2)U_2(z) \tag{3.16}$$

where $U_i(z)$, $i = 1, 2$ satisfy

$$(\mathcal{K}U_i)(z) = Z_0(h_i,z), \qquad \text{on } -h_2 < z < 0. \tag{3.17}$$

We next define

$$\mathsf{S}_{ij} = \int_{-h_2}^{0} U_j(z)Z_0(h_i,z)\,dz \tag{3.18}$$

for $i, j = 1, 2$ and find, using (3.16) in (3.12) with (3.18), that

$$\mathrm{i}k_ih_iN_0(h_i)(A_i - B_i) = -(A_1 + B_1)\mathsf{S}_{i1} + (A_2 + B_2)\mathsf{S}_{i2}. \tag{3.19}$$

This pair of equations for $i = 1, 2$ can be organised as

$$\begin{pmatrix} -B_1 \\ A_2 \end{pmatrix} = (\mathsf{D} + \mathrm{i}\mathsf{S})^{-1}(\mathsf{D} - \mathrm{i}\mathsf{S}) \begin{pmatrix} -A_1 \\ B_2 \end{pmatrix} \tag{3.20}$$

in which vectors representing outgoing wave amplitudes are related to those representing incoming wave amplitudes via a 2×2 scattering matrix (as in (**17**)) where

$$\mathsf{D} = \begin{pmatrix} k_1 h_1 N_0(h_1) & 0 \\ 0 & k_2 h_2 N_0(h_2) \end{pmatrix}, \qquad \mathsf{S} = \begin{pmatrix} \mathsf{S}_{11} & \mathsf{S}_{12} \\ \mathsf{S}_{21} & \mathsf{S}_{22} \end{pmatrix}. \tag{3.21}$$

Since $K(z, z')$ is symmetric and real and $Z_0(h_i, z)$ is real then so are S_{ij}, whilst the symmetry relation $\mathsf{S}_{ij} = \mathsf{S}_{ji}$ is also readily confirmed. Using just these properties and the particular structure of the matrix equation (3.21) alone allows us to prove energy conservation, implying it is automatically satisfied by any approximation to S_{ij} which retains those properties. Key to this is showing that

$$(\mathsf{D} - \mathrm{i}\mathsf{S})^{-1}\mathsf{D}(\mathsf{D} + \mathrm{i}\mathsf{S})^{-1} = [(\mathsf{D} + \mathrm{i}\mathsf{S})\mathsf{D}^{-1}(\mathsf{D} - \mathrm{i}\mathsf{S})]^{-1} \tag{3.22}$$

$$= [(\mathsf{D} - \mathrm{i}\mathsf{S})\mathsf{D}^{-1}(\mathsf{D} + \mathrm{i}\mathsf{S})]^{-1} = (\mathsf{D} + \mathrm{i}\mathsf{S})^{-1}\mathsf{D}(\mathsf{D} - \mathrm{i}\mathsf{S})^{-1} \tag{3.23}$$

from which

$$(-B_1^*, A_2^*)\mathsf{D} \begin{pmatrix} -B_1 \\ A_2 \end{pmatrix} = (-A_1^*, B_2^*)\mathsf{D} \begin{pmatrix} -A_1 \\ B_2 \end{pmatrix} \tag{3.24}$$

follows from (3.20) and the properties of S previously stated. This identity equates, once (3.21) is used, to

$$k_1 h_1 N_0(h_1)(|A_1|^2 + |B_1|^2) = k_2 h_2 N_0(h_2)(|A_2|^2 + |B_2|^2) \tag{3.25}$$

and this expresses conservation of energy. Note the presence of the factors $N_0(h)$ which depend quadratically on the amplitude scaling factor $C_0(h)$ according to (3.10).

We have been diverted temporarily from the principal focus of the paper and are actually concerned with a different arrangement of the equations (3.19) in which amplitudes to the left of the step connect to those on the right of the step through a transfer matrix (also sometimes referred to as a transition matrix). Thus, with some work, (3.19) can be arranged into the form

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} c_1 & d_1 \\ d_1^* & c_1^* \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \tag{3.26}$$

where the asterisk denotes complex conjugation and

$$c_1 = \frac{\Delta\mathsf{S} + \mathrm{i}k_1 h_1 N_0(h_1)\mathsf{S}_{22} + \mathrm{i}k_2 h_2 N_0(h_2)\mathsf{S}_{11} - k_1 h_1 k_2 h_2 N_0(h_1) N_0(h_2)}{2\mathrm{i}\mathsf{S}_{12} k_2 h_2 N_0(h_2)} \tag{3.27}$$

and

$$d_1 = \frac{\Delta\mathsf{S} - \mathrm{i}k_1 h_1 N_0(h_1)\mathsf{S}_{22} + \mathrm{i}k_2 h_2 N_0(h_2)\mathsf{S}_{11} + k_1 h_1 k_2 h_2 N_0(h_1) N_0(h_2)}{2\mathrm{i}\mathsf{S}_{12} k_2 h_2 N_0(h_2)} \tag{3.28}$$

with $\Delta\mathsf{S} = \det\{\mathsf{S}\} = \mathsf{S}_{11}\mathsf{S}_{22} - \mathsf{S}_{12}\mathsf{S}_{21}$.

Again, we observe that the scattering process is therefore determined by the evaluation of the three real quantities S_{11}, $\mathsf{S}_{12} = \mathsf{S}_{21}$ and S_{22}.

3.2 *Variational approximation*

In what follows we continue to assume $h_2 < h_1$. Consider the definition of the functional

$$\mathcal{S}_{ij}(v,w) = \langle Z_j, v\rangle + \langle w, Z_i\rangle - \langle \mathcal{K}w, v\rangle \tag{3.29}$$

for $i,j = 1,2$ where Z_j represents the function $Z_0(h_j,z)$, v, w are real functions defined over $-h_2 < z < 0$ and

$$\langle v,w\rangle \equiv \int_{-h_2}^{0} v(z)w(z)\,dz \tag{3.30}$$

represents the real inner product. Then

$$\delta\mathcal{S}_{ij} = \mathcal{S}_{ij}(v+\delta v, w+\delta w) - \mathcal{S}_{ij}(v,w) = \langle Z_j, \delta v\rangle + \langle \delta w, Z_i\rangle - \langle \mathcal{K}w, \delta v\rangle - \langle \delta w, \mathcal{K}v\rangle \tag{3.31}$$

to first order in the variations δv, δw and using the self-adjointness of the integral operator $\mathcal{K}$ (its kernel is real and symmetric). This shows that the stationary values of $\mathcal{S}_{ij}$ coincide with solutions of the integral equations; that is when $v = U_i$, $w = U_j$. At these stationary values, $\mathcal{S}_{ij}(U_i,U_j) = \mathsf{S}_{ij}$. Moreover, approximations to exact values of S_{ij} are second-order in deviations of the functions v, w from U_i, U_j.

Let us consider using $v = \alpha f$, $w = \beta f$ where α, β are real coefficients and f is a prescribed real function, in $\mathcal{S}_{ij}$ and setting $\partial\mathcal{S}_{ij}/\partial\alpha = 0$ and $\partial\mathcal{S}_{ij}/\partial\beta = 0$, gives $\alpha = \langle f, Z_i\rangle/\langle \mathcal{K}f, f\rangle$ and $\beta = \langle f, Z_j\rangle/\langle \mathcal{K}f, f\rangle$ which allows us to infer that

$$\mathsf{S}_{ij} \approx \frac{\langle f, Z_i\rangle\langle f, Z_j\rangle}{\langle \mathcal{K}f, f\rangle}. \tag{3.32}$$

The accuracy of this approximation depends upon how close f is to the exact functional form of U_i, $i = 1,2$, with errors proportional to terms like $\langle f - U_i, f - U_j\rangle$; that is in an averaged, not local, sense.

Of course, we can improve on the approximation to S_{ij} by expanding v and w in a set of (more than one) basis functions and the variational approximation outlined above leads to systems of equations which coincide with an application of Galerkin's method to the integral equations. This is pursued in (**25**), but is not immediately relevant to the current work.

Under the step approximation we envisage small step heights ($h_2 - h_1 \ll h_2$) and thus weak scattering at isolated steps. Indeed, as the discretisation of the topography becomes increasingly refined then the scattering at each step, whose height is reduced in proportion, becomes weaker. Of course, simultaneously the steps become shorter in length too, but the propagation of information from one step to another is a separate issue, dealt with by scattering matrices. Moreover, throughout most of the depth of the fluid and away from the step itself, the principle underlying fluid behaviour will be that associated with a propagating wave, with some local correction to account for the step. In relation to the remarks made after (3.32) it is reasonable to suppose that choice $f = Z_0(h_2,z)$ is an obvious candidate for the approximation. This is the choice made by (**17**) albeit for steps of arbitrary height not just small step heights. We will not need the details of the calculation of S_{ij} according to (3.18) and this can be found in (**17**). A different choice could have been to use the function $f = 1/(h_2^2 - z^2)^{1/3}$ advocated by (**25**); this is the first term in a set of functions used within Galerkin's method. This choice sacrifices the representation of the fluid motion through the depth in order to capture its local behaviour close to the corner of the step and is accurate for large steps rising close to the surface but is not as accurate as the choice of (**17**) for the shallow steps under consideration here.

3.3 *Shallow water*

Under a long wave/shallow water assumption approximations to the governing equations can be made, the consequence of which means the potential in $x < 0$ where the depth is h_1 can be written as

$$\phi(x, z) \approx \varphi(x) = (A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}) \tag{3.33}$$

and in $x > 0$ where the depth is h_2,

$$\phi(x, z) \approx \varphi(x) = (A_2 e^{ik_2 x} + B_1 e^{-ik_2 x}) \tag{3.34}$$

where, now, k_i are connected to the depth h_i by the shallow water relation $k_i^2 h_i = \omega^2/g$.

The formal assumptions of shallow water theory are violated near the step, but (**26**) provides the justification for the application of matching conditions at the step itself. Across $x = 0$, the requirement that the free surface (proportional to $\varphi(x)$) be continuous gives $\varphi(0^-) = \varphi(0^+)$ whilst continuity of mass flux requires $h_1\varphi'(0^-) = h_2\varphi'(0^+)$. Application of these two conditions to (3.33) and (3.34) results in a transfer matrix with the same structure as in (3.26) but with (3.27), (3.28) replaced by the simpler expressions

$$c_1 = \tfrac{1}{2}(1 + k_1h_1/k_2h_2), \qquad d_1 = \tfrac{1}{2}(1 - k_1h_1/k_2h_2). \tag{3.35}$$

4. Scattering across variable bathymetry under the step approximation

In the previous section we have put in place a transfer matrix for a single step from depth h_1 to h_2 expressed in the form (3.26) which is established either by an exact integral equation formulation paired with a variational approximation or by using shallow water/long wave theory. We now use this transfer matrix to determine overall scattering by the piecewise constant depth $H_N(x)$ as described at the beginning of §2.

Over the section $x_{n-1} < x < x_n$ where $H_N(x) = h_n$ we assign A_n and B_n to amplitudes of right and left-propagating waves (a temporary extension is made to existing notation so that $x_0 = -\infty$ and $x_{N+1} = \infty$.) Consider the step at $x = x_n$ and letting $X = x - x_n$ we have, sufficiently far away from the vertical step at $X = 0$ joining depths h_n to h_{n+1}, the conditions

$$\phi(x, z) \sim (A_n e^{ik_n x_n} e^{ik_n X} + B_n e^{-ik_n x_n} e^{-ik_n X}) Z_0(h_n, z), \qquad x_{n-1} \ll x \ll x_n \tag{4.1}$$

and

$$\phi(x, z) \sim (A_{n+1} e^{ik_{n+1} x_n} e^{ik_{n+1} X} + B_{n+1} e^{-ik_{n+1} x_n} e^{-ik_{n+1} X}) Z_0(h_{n+1}, z), \qquad x_n \ll x \ll x_{n+1} \tag{4.2}$$

apply. Using the transfer matrix for the step, established in the previous section, it follows that

$$\begin{pmatrix} A_{n+1} e^{ik_{n+1} x_n} \\ B_{n+1} e^{-ik_{n+1} x_n} \end{pmatrix} = \begin{pmatrix} c_n & d_n \\ d_n^* & c_n^* \end{pmatrix} \begin{pmatrix} A_n e^{ik_n x_n} \\ B_n e^{-ik_n x_n} \end{pmatrix} \tag{4.3}$$

in which c_n and d_n depend on h_n and h_{n+1} in a natural extension of the definitions of c_1 and d_1 in terms of h_1 and h_2. We apply this relation recursively to give

$$\begin{pmatrix} A_{n+1}\mathrm{e}^{\mathrm{i}k_{n+1}x_n} \\ B_{n+1}\mathrm{e}^{-\mathrm{i}k_{n+1}x_n} \end{pmatrix} = \mathsf{Q}_n \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \tag{4.4}$$

(for mathematical convenience we now redefine the auxiliary point, $x_0 = 0$) where

$$\mathsf{Q}_n = \mathsf{P}_n \mathsf{P}_{n-1} \dots \mathsf{P}_1 \tag{4.5}$$

and

$$\mathsf{P}_n = \begin{pmatrix} c_n & d_n \\ d_n^* & c_n^* \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}k_n\delta_n} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k_n\delta_n} \end{pmatrix} \tag{4.6}$$

with $\delta_n = x_n - x_{n-1}$. On account of the particular structure of P_n we can write

$$\mathsf{Q}_n = \begin{pmatrix} \beta_n & \gamma_n \\ \gamma_n^* & \beta_n^* \end{pmatrix} \tag{4.7}$$

where the entries satisfy the recurrence relation

$$\beta_n = c_n \mathrm{e}^{\mathrm{i}k_n\delta_n}\beta_{n-1} + d_n \mathrm{e}^{-\mathrm{i}k_n\delta_n}\gamma_{n-1}^* \tag{4.8}$$

and

$$\gamma_n = c_n \mathrm{e}^{\mathrm{i}k_n\delta_n}\gamma_{n-1} + d_n \mathrm{e}^{-\mathrm{i}k_n\delta_n}\beta_{n-1}^* \tag{4.9}$$

with $\beta_1 = c_1\mathrm{e}^{\mathrm{i}k_1x_1}$ and $\gamma_1 = d_1\mathrm{e}^{-\mathrm{i}k_1x_1}$. Making the following change of variable

$$\Gamma_n = \beta_n + \gamma_n^*, \qquad \text{and} \qquad \Upsilon_n = \mathrm{i}(\beta_n - \gamma_n^*) \tag{4.10}$$

allows us to express the pair of recurrence relations above as

$$\Gamma_n = (c_n + d_n)(\cos k_n\delta_n \Gamma_{n-1} + \sin k_n\delta_n \Upsilon_{n-1}) \tag{4.11}$$

and

$$\Upsilon_n = (c_n - d_n)(\cos k_n\delta_n \Upsilon_{n-1} - \sin k_n\delta_n \Gamma_{n-1}) \tag{4.12}$$

with $\Gamma_1 = (c_1 + d_1^*)\mathrm{e}^{\mathrm{i}k_1x_1}$ and $\Upsilon_1 = \mathrm{i}(c_1 - d_1^*)\mathrm{e}^{\mathrm{i}k_1x_1}$.

For a wave incident of unit amplitude from $x = -\infty$, partially reflected back to $x = -\infty$ with reflection coefficient R and partially transmitted with transmission coefficient T, then we set $A_1 = 1$, $B_1 = R$, $A_{N+1} = T$, $B_{N+1} = 0$, so that from (4.4), (4.7) with $n = N$

$$R = -\frac{\gamma_N^*}{\beta_N^*}, \qquad T = \mathrm{e}^{-\mathrm{i}k_{N+1}x_N}(\beta_N + R\gamma_N). \tag{4.13}$$

In terms of the transformed variables (4.10), we have

$$R = -\frac{\Gamma_N + \mathrm{i}\Upsilon_N}{\Gamma_N^* + \mathrm{i}\Upsilon_N^*}, \qquad T = \mathrm{e}^{-\mathrm{i}k_{N+1}x_N}(\Gamma_N - \mathrm{i}\Upsilon_N + R(\Gamma_N^* - \mathrm{i}\Upsilon_N^*)). \tag{4.14}$$

4.1 *The zero step-size limit: shallow water assumption*

For the shallow water case the expressions c_1 and d_1 in (3.35) give us

$$c_n + d_n = 1 \tag{4.15}$$

and

$$c_n - d_n = \frac{k_n h_n}{k_{n+1} h_{n+1}} = \frac{\sqrt{h_n}}{\sqrt{h_{n+1}}} \tag{4.16}$$

since $k_n^2 h_n = \omega^2/g$. But $h_{n+1} = h(x_n + \frac{1}{2}\delta_n) \approx h(x_n) + \frac{1}{2}\delta_n h'(x_n)$. This leads to

$$c_n - d_n \approx 1 - \tfrac{1}{4}(\delta_{n-1} + \delta_n)\frac{h'(x_n)}{h(x_n)} \tag{4.17}$$

to leading order in δ_n. For simplicity let us assume x_n are equally spaced so that $\delta_n = \delta$ for all n, and then

$$c_n - d_n \approx 1 - \tfrac{1}{2}\delta\frac{h'(x_n)}{h(x_n)}. \tag{4.18}$$

We return to (4.11), (4.12) which we consider in the limit $\delta_n = \delta \to 0$ in conjunction with the results established above so that, to leading order, we have

$$\Gamma_n - \Gamma_{n-1} \approx k_n \delta \Upsilon_{n-1}, \qquad \text{and} \qquad \Upsilon_n - \Upsilon_{n-1} \approx -\tfrac{1}{2}\delta\frac{h'(x_n)}{h(x_n)}\Upsilon_{n-1} - k_n\delta\Gamma_{n-1}. \tag{4.19}$$

Additionally it is assumed that $\Gamma_n = \Gamma(x_n)$, $\Upsilon_n = \Upsilon(x_n)$, $k_n = k(h(x_n))$ form discrete evaluations of continuous functions and so, under the limit $\delta \to 0$ (4.19) becomes

$$\Gamma'(x) \approx k(h(x))\Upsilon(x) \qquad \text{and} \qquad \Upsilon'(x) \approx -\tfrac{1}{2}\frac{h'(x)}{h(x)}\Upsilon(x) - k(h(x))\Gamma(x). \tag{4.20}$$

The second of these equations can be written

$$(\sqrt{h(x)}\Upsilon(x))' \approx -k(h(x))\sqrt{h(x)}\Gamma(x) \tag{4.21}$$

and when combined with the first equation, with the local shallow water dispersion relation $\omega^2/g = k^2 h$, gives

$$(h(x)\Gamma'(x))' + (\omega^2/g)\Gamma(x) = 0 \tag{4.22}$$

which may also be expressed as

$$(k^{-2}\Gamma'(x))' + \Gamma(x) = 0. \tag{4.23}$$

This ODE is the well-known Shallow Water Equation (SWE).

4.2 *The zero step-size limit: non-shallow assumption*

We now consider the case of a more general depth in which the one-term variational approximation has been used to approximate scattering at a step. We have already advocated the use of the function $f = Z_0(h_2, z)$ in the approximation (3.32) of S_{ij}. The first thing to notice is that the definition (3.32) implies that $\Delta\mathsf{S} = 0$ and so from (3.27), (3.28) we have

$$c_n + d_n = \frac{\mathsf{S}_{11}}{\mathsf{S}_{12}} \tag{4.24}$$

and

$$c_n - d_n = \frac{k_n h_n N_0(h_n)}{k_{n+1}h_{n+1}N_0(h_{n+1})}\left(\frac{\mathsf{S}_{22}}{\mathsf{S}_{12}} + \mathrm{i}\frac{k_{n+1}h_{n+1}N_0(h_{n+1})}{\mathsf{S}_{12}}\right) \tag{4.25}$$

where, now, the elements of S are calculated assuming h_n to the left of the step and h_{n+1} to its right.

Now, with $h_2 \equiv h_{n+1}$ and $f = Z_0(h_{n+1}, z)$, we have

$$\langle f, Z_0(h_{n+1}, z)\rangle = h_{n+1}N_0(h_{n+1}) \tag{4.26}$$

by the orthogonality definition, (3.9). If we assume equally-spaced points $\{x_n\}$ separated by δ, a sufficiently small number, then

$$Z_0(h_n, z) \approx Z_0(h_{n+1} - \delta h'(x_n), z) \approx Z_0(h_{n+1}, z) - \delta h'(x_n)\frac{\partial Z_0}{\partial h}(h_{n+1}, z) \tag{4.27}$$

to leading order in δ. This means that

$$\langle f, Z_0(h_n, z)\rangle \approx h_{n+1}N_0(h_{n+1}) - \delta h'(x_n)W_1(h_{n+1}) \tag{4.28}$$

where

$$W_1(h) = \int_{-h}^{0} \frac{\partial Z_0}{\partial h}(h, z)Z_0(h, z)\, dz. \tag{4.29}$$

The explicit calculation of this term is not needed in this paper. We can now return to (4.24) and determine, to leading order in δ, that

$$c_n + d_n = \frac{\langle f, Z_0(h_n, z)\rangle}{\langle f, Z_0(h_{n+1}, z)\rangle} \approx 1 - \delta\frac{h'(x_n)}{h(x_n)}\frac{W_1(h(x_n))}{N_0(h_{n+1})}. \tag{4.30}$$

For the second term in (4.30) we also need to consider the leading order contribution from $\langle \mathcal{K}f, f\rangle$ which is calculated using methods similar to above and found to be $O(\delta^2)$. That is to say, evanescent effects do not contribute at leading order in the step length δ. Thus, (4.25) is represented to leading order as

$$c_n - d_n \approx \frac{k_n h_n N_0(h_n)}{k_{n+1}h_{n+1}N_0(h_{n+1})}\frac{\langle f, Z_0(h_{n+1}, z)\rangle}{\langle f, Z_0(h_n, z)\rangle}. \tag{4.31}$$

The second part of this fraction is just the reciprocal of the expression for $c_n + d_n$ and, for example, the denominator in the first part of the fraction can be expanded using Taylor series about the nth state to give, ultimately,

$$c_n - d_n \approx \left(1 - \delta\left.\frac{(khN_0(h))'}{khN_0(h)}\right|_{x=x_n}\right)\left(1 + \delta\frac{h'(x_n)}{h(x_n)}\frac{W_1(h(x_n))}{N_0(h_n)}\right) \tag{4.32}$$

to leading order in δ. Now we use the two expressions (4.30) and (4.32) in (4.11) and (4.12) to give

$$\Gamma_n - \Gamma_{n-1} \approx -\delta \frac{h'(x_n)}{h(x_n)} \frac{W_1(h(x_n))}{N_0(h_n)} \Gamma_n + k_n \delta \Upsilon_{n-1}, \tag{4.33}$$

and

$$\Upsilon_n - \Upsilon_{n-1} \approx \delta \left(\frac{h'(x_n)}{h(x_n)} \frac{W_1(h(x_n))}{N_0(h_n)} - \left. \frac{(khN_0(h))'}{khN_0(h)} \right|_{x=x_n} \right) \Upsilon_{n-1} - k_n \delta \Gamma_{n-1}. \tag{4.34}$$

Taking the limit $\delta \to 0$ in the manner described in §4.1 transforms these equations into the coupled ODEs

$$\Gamma'(x) \approx -\frac{h'(x)W_1(h)}{h(x)N_0(h)} \Gamma(x) + k(h(x))\Upsilon(x) \tag{4.35}$$

and

$$\Upsilon'(x) \approx \left(\frac{h'(x)W_1(h)}{h(x)N_0(h)} - \frac{(k(h(x))h(x)N_0(h))'}{k(h(x))h(x)N_0(h)} \right) \Upsilon(x) - k(h(x))\Gamma(x). \tag{4.36}$$

This latter equation can be arranged as

$$(khN_0\Upsilon)' \approx kh'W_1\Upsilon - k^2 hN_0 \Gamma \tag{4.37}$$

(removing arguments to make it read more easily) and the former equation as

$$(hN_0\Gamma')' \approx -(h'W_1\Gamma)' + (khN_0\Upsilon)' \tag{4.38}$$

which is enough to eliminate Υ and leaves the equation, after tidying up terms,

$$(hN_0\Gamma')' + \left(k^2 hN_0 + (h'W_1)' - \frac{(h'W_1)^2}{hN_0} \right) \Gamma = 0. \tag{4.39}$$

This can also be written as

$$(W_0(h)\Gamma')' + \left(k^2 W_0(h) + W_1(h)h''(x) + W_2(h)(h'(x))^2\right)\Gamma = 0 \tag{4.40}$$

where

$$W_0(h) = \int_{-h}^{0} Z_0^2(h,z)\,dz \equiv hN_0(h) \tag{4.41}$$

and

$$W_2(h) = \dot{W}_1 - \frac{W_1^2}{W_0} \tag{4.42}$$

where $\dot{W}_1 \equiv dW_1/dh$. We also note that we can write

$$W_1(h) = \frac{d}{dh} \int_{-h}^{0} \frac{1}{2} Z_0^2(h,z)\,dz - \frac{1}{2} Z_0^2(h,-h) = \tfrac{1}{2}(\dot{W}_0 - C_0^2(h)). \tag{4.43}$$

This is a good point at which to take stock. We have shown that (4.40) is the 2nd order ODE that is derived from taking the limit of step size tends to zero under the plane-wave variational approximation of (**17**). We note that (4.40) is very nearly, but not quite, the

Modifield Mild-Slope Equation of (**22**): there is a difference in the definition of the final term defining $W_2(h)$. We come back to this point in a moment.

We recall from that the coefficients in the ODE (4.40) are defined in terms of the factor $C_0(h)$ which scales $Z_0(h,z)$, introduced in (3.5). It is an arbitrary real function. Thus we can use the definition of $C_0(h)$ to reshape the ODE and we aim to simplify the equation as far as possible. This process is reminiscent of the rescaling of the Modified Mild-Slope Equation in (**21**).

Consider the definition $C_0^2 = 2/(k(2kh + \sinh 2kh))$. Then it follows from (4.41), using the definition (3.9) that

$$W_0(h) = \frac{1}{2k^2(h)}. \tag{4.44}$$

It then follows that

$$\dot{W}_0(h) = -\frac{2\dot{k}}{k^3} = \frac{2}{k(2kh + \sinh 2kh)} \tag{4.45}$$

after using the dispersion relation to determine $\dot{k}$. Now from (4.43) we find that $W_1(h) = 0$ and this implies from (4.42) that $W_2(h) = 0$ also. So the under this particular scaling the ODE (4.40) has been reduced to

$$(k^{-2}\Gamma')' + \Gamma = 0. \tag{4.46}$$

This equation is once again very nearly, but not quite, the transformed Modified Mild-Slope Equation of (**21**) which includes an additional term of small magnitude proportional to h'^2 multiplying Γ. Instead, however, (4.46) does coincide with a new, simpler, variant of the Mild-Slope Equations developed recently by (**18**). Porter describes this new variant as the 'fundamental Mild Slope Equation' since it is established within a more general framework than previous versions of the Mild-Slope Equations that removes bias from the underlying governing equations and constraints on how they are approximated. It is consequently of no surprise that the scaling $C_0(h)$ used above coincides precisely with that used by (**18**). Since the fundamental MSE, (4.46), is derived from the variational principle of (**18**) with the fewest constraints it should be expected to be the least accurate of the variants discussed in (**18**) including the Modified Mild-Slope Equation of (**21**). On the other hand, both (**18**) and (**21**) highlight the relative insignificance on the results of the additional term in the latter model.

It is also noteworthy that (4.46) is exactly the same as the version of the Shallow Water Equation expressed by (4.23). The difference, of course, is that $k(h)$ in (4.23) is defined by the shallow water limit of the full dispersion relation which applies to the definition of $k(h)$ in (4.46).

We finish this section by commenting on how to recover the reflection and transmission coefficients from the solution of the ODE. We must solve (4.46) subject to initial conditions which are determined by the initial values Γ_1 and Υ_1 originating from the discrete system and determined in the line after (4.12). Assuming $x_n = n\delta$ and that $\delta = L/N$, in the limit as $N \to \infty$, the ODE (4.46) holds over $0 < x < L$ with $k_0 = k(h(0))$ and $k_L = k(h(L))$. The initial conditions are deduced from (4.30), (4.31) in the limit $\delta \to 0$ and with $W_1(h) = 0$ in (4.35) to give the relation $k(x)\Upsilon(x) = \Gamma'(x)$ and leads to

$$\Gamma(0) = 1, \qquad \Gamma'(0) = \mathrm{i}k_0. \tag{4.47}$$

	SWE		MSE	
N	$\|R\|$	error	$\|R\|$	error
8	0.19755	0.00809	0.18753	0.01032
16	0.20151	0.00413	0.19253	0.00532
32	0.20356	0.00208	0.19516	0.00268
64	0.20460	0.00104	0.19649	0.00136
128	0.20512	0.00052	0.19717	0.00067
256	0.20538	0.00026	0.19751	0.00034
ODE	0.20564		0.19784	

Table 1 Convergence of step approximation implementation with number of steps, N, to the solution of the corresponding ODE in the case of a linear ramp of length l between depths h_1 and h_{N+1} with $l/h_1 = 2$, $h_{N+1}/h_1 = \frac{1}{3}$ and at a wavenumber $k_1 h_1 = \frac{1}{2}$.

Also from (4.13), (4.14) we readily find

$$R = -\frac{k_L \Gamma(L) + \mathrm{i}\Gamma'(L)}{(k_L \Gamma(L) - \mathrm{i}\Gamma'(L))^*} \tag{4.48}$$

whilst

$$T = \mathrm{e}^{-\mathrm{i}k_L L}(\Gamma(L) - \mathrm{i}\Gamma'(L)/k_L + R(\Gamma(L) + \mathrm{i}\Gamma'(L)/k_L)^*). \tag{4.49}$$

5. Remarks on numerical methods

In Tab. 1 and Fig. 1 numerical computations have been made of $|R|$ for a bed comprised of a plane shoaling ramp over $0 < x < l$ between two depths h_1 and h_{N+1}. In both cases, $l/h_1 = 2$ and $h_{N+1}/h_1 = \frac{1}{3}$, but these are not special values and similar conclusions can be drawn from other parameters and bed shapes. In Tab. 1, the step-approximation has been implemented under the two different approximation using: (a) the shallow water variables (3.35) and (b) the variational method of (**17**) (3.27), (3.28). The results confirm that the step approximation tends to the results derived from solving the shallow water and mild slope equation ODEs as the number of steps, N, are increased. Moreover the convergence rate is commensurate with the use of Euler's method for integrating ODEs. From this point of view, the step approximation is an inefficient method for solving wave scattering by variable beds, being slow to converge and requiring an unnecessary level of detailed computation (e.g. (3.27), (3.28)) compared to a higher-order solver applied to the much simpler ODE (e.g. (4.46)).

Fig. 1 provides a typical comparison of the MSE of (4.46) and the SWE (4.23) with precise numerical computations of (**5**).

6. Conclusion

This paper has determined the mathematical link between two previously unconnected approximations for calculating wave scattering over variable beds. The step approximation involves describing the bed as a piecewise constant representation of the continuous function $h(x)$ and neglecting the evanescent wave field in interactions between neighbouring steps. By approximating the scattering at each step under the shallow water assumption, it has

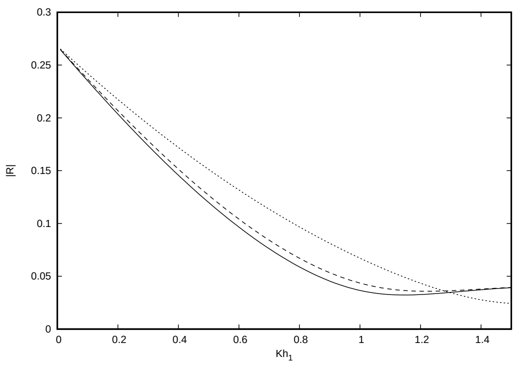

Fig. 1 Computations of $|R|$ against frequency parameter, Kh_1, for a linear ramp of length l between depths h_1 and h_{N+1} with $l/h_1 = 2$, $h_{N+1}/h_1 = \frac{1}{3}$: exact formulation (solid line), MSE (dashed line), SWE (dotted line).

been shown the Shallow Water Equation (a depth averaged 2nd order ODE) results in the limit as the step size tends to zero. If, instead, the scattering at each step is calculated under the plane-wave variational approximation of (**17**) it has been shown that a different depth-averaged model ODE results, being a new simpler variant of the Mild-Slope Equation (MSE) derived recently by (**18**).

It is not clear why the particular variant of the MSE, described by (**18**) is selected. Nor is it clear how changes to the approximation may lead to different, perhaps more sophisticated 2nd order ODEs describing wave scattering over variable beds. For example, an extended transfer matrix which includes evanescent effects in the interactions between neighbouring steps such as that used in (**10**) may lead to a coupled ODE system similar to that described in the extended MSE of (**27**). It should also be possible to apply the ideas of this paper to the particular embodiment of the step approximation considered by (**13**); it would not be surprising if the limiting step-size analysis resulted in something very close to the ODE developed in (**29**).

Beyond the present topic, there are possibilities to extend the current idea to other settings. Other scatterers in different physical settings can be treated similarly. Indeed, the current work was stimulated by a toy problem considered in (**28**) involving waves on a long string connected to a finite number of perpendicular strings of finite length. There it was shown that the scattering process could be replaced, for small step lengths between neighbouring junctions, by an approximate 2nd order ODE which could *not* have been derived by alternative means. The solutions of that ODE turn out to be very useful in identifying certain features of the solution to the toy problem which would be hard to access otherwise. One future direction of this work will be to consider how wave propagation by small broken ice floes might be approximated by ODEs and to use this to shed light on wave attenuation in large regions of broken ice. Current theoretical models are not able to capture the attenuation rates measured experimentally; see (**30**), (**31**).

References

1. M. Roseau, *Asymptotic Wave Theory*, Vol. 2 of North-Holland Series in Applied Mathematics and Mechanics. Amsterdam, The Netherlands (1976).
2. J.W. Kim & K.J. Bai, A finite element method for two-dimensional water-wave problems, *Int. J. Numer. Meth. Fluids*, **30**, (1999), 105–121.
3. M. Penalba, T. Kelly & J.V. Ringwood, Using NEMOH for Modelling Wave Energy Converters: A Comparative Study with WAMIT. *Proceedings of the 12th European Wave and Tidal Energy Conference, Cork, Ireland*, (2017).
4. D.J. Staziker, D. Porter & D.S.G. Stirling, The scattering of surface waves by local bed elevations, *Appl. Ocean Res.*, **18**, (1996), 283–291.
5. R. Porter & D. Porter, Wave scattering by a step of arbitrary profile, *J. Fluid Mech.*, **411**, (2000) 131–164.
6. P. Devillard, F. Dunlop & B. Souillard, Localization of gravity waves on a channel with a random bottom, *J. Fluid Mech.*, **186**, (1988), 521–538.
7. E.R. Johnson, The low-frequency scattering of Kelvin waves by stepped topography, *J. Fluid Mech.*, **215**, (1990), 23–44.
8. F. Mattioli, Resonant reflection of surface waves by non-sinusoidal bottom undulations, *Appl. Ocean. Res.*, **13**, (1991) 49–53.
9. T.J. O'Hare & A.G. Davies, A new model for surface wave propagation over undulating topography, *Coastal Engng.*, **18**, (1991), 251–266.
10. E. Guazzelli, V. Rey & M. Belzons, Higher-order Bragg reflection of gravity surface waves by periodic beds, *J. Fluid Mech.*, **245**, (1992), 301–317.
11. V. Rey, Propagation and local behaviour of normal incident gravity waves over varying topography, *Eur. J. Mech. B: Fluids*, **11**, (1992), 213–232.
12. T.J. O'Hare & A.G. Davies, A comparison of two models for surface-wave propagation over rapidly varying topography, *Appl. Ocean Res.*, **15**(1), (1993), 1–11.
13. D.V. Evans & C.M. Linton, On step approximations for water-wave problems, *J. Fluid Mech.*, **278**, (1994), 229–249.
14. C.-C. Tsai, T.-W. Hsu & Y.-T. Lin, On Step Approximation for Roseau's Analytical Solution of Water Waves, *Mathematical Problems in Engineering* (2011) 607196 (20 pages) `http://dx.doi.org/10.1155/2011/607196`
15. C.-C. Tsai, W. Tai, T.-W. Hsu & S.-C. Hsiao, Step approximation of water wave scattering caused by tension-leg structures over uneven bottoms, *Ocean Engng.* **166** (2018) 208–225.
16. J.J. Stoker, *Water Waves.* Interscience, New York (1957).
17. J.W. Miles, Surface wave scattering matrix for a shelf, *J. Fluid Mech.*, **28**, (1967) 755–767.
18. D. Porter, The Mild Slope Equations: A unified theory, *Accepted for publication in Journal of Fluid Mechanics.*
19. J.C.W. Berkhoff, Computation of combined refraction-diffraction, *Proc. 13th Conf. Coastal Engng., Vancouver, vol. 2*, (1972), 471–490. ASCE.
20. R. Smith & T. Sprinks, Scattering of surface waves by a conical island, *J. Fluid Mech.*, **72**, (1975), 373–384.
21. D. Porter, The mild-slope equations, *J. Fluid Mech.*, **494**, (2003), 51–63.
22. P.G. Chamberlain & D. Porter, The modified mild-slope equation, *J. Fluid Mech.*, **291**,

(1995), 393–407.

23. J.W. Kim & K.J. Bai, A new complementary mild-slope equation, *J. Fluid Mech.*, **511**, (2004), 25–40.

24. Y. Toledo & Y. Agnon, A scalar form of the complementary mild-slope equation, *J. Fluid Mech.*, **656**, (2010), 407–416.

25. R. Porter, *Complementary methods and bounds for linear water waves*, PhD. Thesis, University of Bristol, UK (1995).

26. C.C. Mei, M. Stiassnie & D.K.-P. Yu, *Theory and Applications of Ocean Surface Waves Part 1: Linear Aspects.* World Scientific Publishing Co. Pte. Ltd, Singapore, (2005).

27. D. Porter & D.G. Staziker, Extensions of the mild-slope equation, *J. Fluid Mech.*, **300**, (1995), 367–382.

28. R. Porter, *N* strings on a string, *Online report*, (2018).

29. D, Porter & R. Porter, Approximations to water wave scattering by steep topography, *J. Fluid Mech.*, **562**, (2006), 279–302.

30. M.J. Doble, G. De Carolis, M.H. Meylan, J.-R. Bidlot & P. Wadhams, Relating wave attenuation to pancake ice thickness, using field measurements and model results, *Geophys. Res. Lett.*, **42**, (2015), doi:10.1002/2015GL063628.

31. M.H. Meylan, L.G. Bennetts, J.E.M. Mosig, W.E. Rogers, M.J. Doble, & M.A. Peter, Dispersion relations, power laws, and energy loss for waves in the marginal ice zone, *J. Geophys. Res.: Oceans*, **123**, (2018), 3322–3335.