Non-linear problem on unsteady free surface flow forced by submerged cylinder

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A fully nonlinear problem on unsteady water waves generated by submerged circular cylinder is considered semi-analytically. Main purpose is to evaluate impact of non-linearity acting at early stage of non-stationary wave motion when the cylinder accelerates impulsively near the free surface. Effect of non-linearity was originally studied by Tuck [1] who used the Wehausen scheme [2] by constructing power expansion on radius of the cylinder for the solution which describes stationary wave train past horizontally moving body. We apply here analytical method developed by Ovsyannikov [3] for a class of initial boundary value problems on unsteady free surface flows. By this way, the mathematical formulation reduces to an integral-differential system of equations for the functions defining the free surface shape and the normal and tangential components of fluid velocity. This method was extended by Makarenko [4] to the problem on unsteady water waves forced by circular cylinder, as well as the problem on elliptic cylinder moving under free surface [5] or under ice cover [6] was considered. Small-time solution expansions were obtained systematically starting from the papers by Tyvand & Miloh [7, 8] devoted to the case of unsteady motion of a circular cylinder. We revisit here this problem in order to accent the role of non-linearity in the mechanism of formation of finite amplitude surface waves.

Statement of the problem

The plane irrotational flow of a heavy inviscid deep fluid is considered in the coordinate system Oxy with a vertical y–axis. The circular cylinder of nondimensional radius r centered at $(x_{cyl}(t), y_{cyl}(t))$ moves totally submerged in the deep fluid with free surface $y = \eta(x,t)$, having the equillibrium level $y = 0$. Dimensionless variables use the initial depth of submergence h as the length scale, characteristic speed of cylinder u_0 as the velocity scale, ρu_0^2 as the pressure scale and h/u_0 as the time scale. The Euler equations for the fluid velocity $\mathbf{u} = (U, V)$ and pressure p are

$$\begin{cases} U_t + UU_x + VU_y + p_x = 0, \\ V_t + UV_x + VV_y + p_y = -\lambda, \\ U_x + V_y = 0, \quad U_y - V_x = 0. \end{cases} \tag{1}$$

Here $\lambda = gh/u_0^2$ is the square of the inverse Froude number. The fully non-linear kinematic and dynamic free-surface boundary conditions

$$\eta_t + U\eta_x = V, \quad p = 0, \quad (y = \eta(x,t)), \tag{2}$$

together with the exact rigid body surface condition

$$(\mathbf{u} - \mathbf{u}_{cyl}) \cdot \mathbf{n} = 0, \qquad (x - x_{cyl}(t))^2 + (y - y_{cyl}(t))^2 = r^2 \tag{3}$$

are employed. Here $\mathbf{n}$ is the unit normal to the cross-section of the cylinder. We suppose that the fluid is at rest at infinity $(U, V \to 0, \quad \eta \to 0, \quad |\mathbf{x}| \to \infty)$ and initial velocity field satisfies compatibility conditions, i.e. it is potential and irrotational.

We reduce equations (1)–(3) to an equivalent system of boundary integral-differential equations which are one-dimensional with respect to spatial variables. Let $u = U + \eta_x V$ and $v = V - \eta_x U$ be tangential and normal fluid velocitites at the free surface $y = \eta(x,t)$. Excluding the pressure p from momentum equations (1) we obtain under conditions (2) an evolution system for η, u, v

$$\eta_t = v, \qquad u_t + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{u^2 - 2\eta_x uv - v^2}{1 + \eta_x^2}\right) + \lambda\eta_x = 0. \tag{4}$$

Differential equations (4) are complemented by the integral equation which follows from the representation of complex velocity $F = U - iV$ using the boundary integrals on the free surface only

$$2\pi i F(z,t) = \int\limits_{\Gamma} \frac{F(\zeta,t)d\zeta}{\zeta - z} + \frac{r^2}{(z - z_{cyl})^2}\overline{\int\limits_{\Gamma} \frac{F(\zeta,t)d\zeta}{\zeta - z_*}} + \frac{\gamma}{z - z_{cyl}} + \frac{2\pi i r^2 z'_{cyl}}{(z - z_{cyl})^2}. \tag{5}$$

Here $z_* = z_{cyl} + r^2/\overline{z}$ is the inversion image of $z = x + iy$ with respect to the circle centered at the point $z_{cyl}(t) = x_{cyl}(t) + iy_{cyl}(t)$. The constant γ is the velocity circulation around the cylinder. Combining real and imaginary parts of the formula (5) with $z = x + i\eta(x,t)$ taken on the free surface gives the real-valued form of boundary integral equation as follows:

$$\pi v(x) + v.p.\int_{-\infty}^{\infty}(A_f(x,s)+r^2A_r(x,s))v(s)ds = v.p.\int_{-\infty}^{+\infty}(B_f(x,s)+r^2B_r(x,s))u(s)ds + v_{curl}(x) + v_{dip}(x), \quad (6)$$

where the kernels of integral operators are given by

$$A_f + iB_f = \frac{i[1+i\eta'(x)]}{x - s + i[\eta(x)-\eta(s)]}, \quad A_r + iB_r = \frac{i[1-i\eta'(x)]}{[x-i\eta(x)][r^2-(x-i\eta(x))(s-i\eta(s))]}.$$

The functions v_{curl} and v_{dip} are the normal velocities induced at the free surface by vortex and dipole:

$$v_{curl}(x) = \gamma \mathrm{Re}\Big[\log(x+i\eta(x)-z_{cyl}(t))\Big]_x, \quad v_{dip}(x) = \mathrm{Re}\left[\frac{2iz'_{cyl}(t)}{x+i\eta(x)-z_{cyl}(t)}\right]_x.$$

The time variable t was omitted in (6) because it appears in this integral equation only as a parameter. It should be noted that the kernels A_f and B_f correspond to the problem on free waves in deep water without cylinder. The terms A_r and B_r describe the interaction between the cylinder and free surface.

Small-time asymptotic solution

We consider the unsteady flow which starts from the rest and is caused by the motion of the circular cylinder along the trajectory $z_{cyl}(t) = -i + e^{i\theta}t^2$. The angle θ of the motion direction relative to the horizon remains constant. We look for a solution in the form of power series

$$\eta(x,t) = t^2\eta_2(x) + t^3\eta_4(x) + \dots, \quad u(x,t) = t^3u_3(x) + t^4u_4(x) + \dots, \quad v(x,t) = tv_1(x) + t^2v_2(x) + \dots$$

It is easy to see that the coefficients η_n for $n \geq 1$ and u_n for $n \geq 3$ may be evaluated via v_n by recursive formulas following from equations (4)

$$\eta_{n+1} = \frac{1}{n+1}v_n, \quad u_3 = \frac{1}{6}(v_1^2 - \lambda v_1)_x, \quad u_4 = \frac{1}{4}(v_1v_2)_x - \frac{1}{12}\lambda v_{2x}, \quad u_5 = \frac{1}{10}(2v_1v_3 + v_2^2)_x - \frac{1}{20}\lambda v_{3x}. \quad (7)$$

Using the expansion of free surface elevation η one can determine the power series for integral operators

$$A_f = t^2A_f^{(2)} + t^3A_f^{(3)} + \dots, \quad B_f = B_f^{(0)} + t^2B_f^{(2)} + \dots, \quad A_r = A_r^{(0)} + t^2A_r^{(2)} + t^3A_r^{(3)} + \dots, \quad B_r = B_r^{(0)} + t^2B_r^{(2)} + \dots$$

The operators $B_f^{(0)}$ and $A_r^{(0)}$ are important for the solution construction. First of them is the Hilbert transform $H = B^{(0)}$

$$Hu(x) = v.p.\int_{-\infty}^{+\infty}\frac{u(s)ds}{x-s}, \qquad A_r^{(0)}v(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(1-r^2p(x))q'(x) + (r^2q(x)-s)p'(x)}{(1-r^2p(x))^2 + (r^2q(x)-s)^2}v(s)ds$$

and the operator $A_r^{(0)}$ is nonlinear with respect to the Poisson kernels

$$p(x) = \frac{1}{1+x^2}, \quad q(x) = \frac{x}{1+x^2}.$$

The integral equation (6) for the normal velocity v leads to a set of equations for coefficients v_n $(n \geq 1)$

$$\pi v_n(x) + r^2\int_{-\infty}^{+\infty}A_r^{(0)}(x,s)v_n(s)ds = \varphi_n(x) \qquad (n = 1, 2, ...), \quad (8)$$

where the functions φ_n can be evaluated via the coefficients v_1, v_2, $\dots$, v_{n-1} by the formulas

$$\varphi_1 = v_{dip}^{(1)}, \quad \varphi_2 = 0, \quad \varphi_3 = v_{dip}^{(3)} + Hu_3 + r^2\Big(B_r^{(0)}u_3 - A_r^{(2)}v_1\Big) - A_f^{(2)}v_1.$$

Here $v_{dip}^{(n)}$ are the coefficients of small-time expansion for normal velocity v_{dip} generated by dipole.

Thus nonlinearity realizes in two different ways if the solution expansion is constructed. Firstly, nonlinear terms arise from the integral operators depending linearly on the coefficients u_n which are nonlinear with respect to v_1, v_2, $\ldots$, v_{n-1} due to the recursive formulas (7). In addition, nonlinearity is also presented by the kernels of integral operators depending on the coefficients η_n. The Table 1 illustrates the leading order terms of integral equation (8) collected by the powers of the time variable t and the cylinder radius r.

Table 1. Coefficients in the expansion of the functions φ_n from equations (8).

	t^2	t^4	$\ldots$
r^2	$v_{dip}^{(1)}$	$v_{dip}^{(3)}$, Hu_3	$\ldots$
r^4	$A_r^{(0)}v_1$	$A_r^{(0)}v_3$, $B_r^{(0)}u_3$, $A_f^{(2)}v_1$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$

Analytic solution of the equation (8) can be constructed explicitly by using the Neumann series of integral operator $A_r^{(0)}$. It is important here that the leading-order coefficient v_1 results as linear combination of the Poisson kernels $p(x)$, $q(x)$ and their derivatives $p'(x)$, $q'(x)$. Subsequently, calculation of higher order coefficients v_n involves nonlinear combinations of derivatives $p^{(k)}(x)$ and $q^{(k)}(x)$ with $k = 1, \ldots, n$. This version of multi-pole expansion procedure can be simplified essentially by using special identities such as follows:

$$p'(x)q'(x) = -\frac{1}{12}p'''(x), \quad p'^2(x) = \frac{1}{4}p(x) + \frac{1}{4}q'(x) + \frac{1}{12}q'''(x), \quad q'^2(x) = \frac{1}{4}p(x) + \frac{1}{4}q'(x) - \frac{1}{12}q'''(x), \quad \ldots$$

The main difficulty appears by evaluation of integral terms like $A_f^{(2)}v_1$ that can be rewritten as follows:

$$A_f^{(2)}v_1(x) = \text{v.p.}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\eta_2(x) - \eta_2(s)}{(x-s)^2}v_1(s)ds - \eta_2'(x)\,\text{v.p.}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{v_1(s)ds}{x-s}. \tag{9}$$

As will readily be observed the first integral in the equation (9) is a commutator of the Hilbert transform H with some differential operator. To be exact:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\eta_2(x) - \eta_2(s)}{(x-s)^2}v_1(s)ds = \eta_2 H v_{1x} - H(\eta_2 v_1)_x.$$

Finally combining all the terms of integral equation (8) we obtain under recursive formulas (7) the power expansion for the leading-order solution coefficients as follows:

$$\eta_2(x) = 2\left(r^2 - r^4\right)\Big(q'(x)\sin\theta - p'(x)\cos\theta\Big) + O(r^6), \tag{10}$$

$$\eta_4(x) = r^2\Big(p''(x)\cos 2\theta - q''(x)\sin 2\theta\Big) + \frac{\lambda(r^2 - r^4)}{6}\Big(p''(x)\sin\theta + q''(x)\cos\theta\Big) + \\ + \frac{r^4}{9}\Big(p''''(x)\cos 2\theta - q''''(x)\sin 2\theta\Big) + \frac{r^4}{3}\Big(p''(x) - q'(x)\Big) + O(r^6).$$

Calculations and visualization of the flow

Non-linear theory contribute to the analytical solution (10) by the terms of the order $O(r^4)$, so the correction to linear theory becomes essential at the time scales when the cylinder approaches the free surface closely. The Fig. 2 shows that constructed solution gives the correction not only in the elevation of the free surface but also in its formation rate. In addition, asymptotic solution (10) allows one to calculate velocity field in the whole fluid domain. Subsequent flow can be constructed effectively by the representation of complex velocity (5), this solution has the form

$$F(z,t) = -2r^2\left(e^{i\theta}\frac{t}{(z-i)^2} + 2e^{2i\theta}\frac{t^3}{(z-i)^3}\right) + +2r^2\left(e^{i\theta}\frac{t}{(z+i)^2} + 2e^{2i\theta}\frac{t^3}{(z+i)^3}\right) + \\ + 2r^4\left(e^{i\theta}\frac{t}{(z-i)^2} + 2e^{2i\theta}\frac{t^3}{(z-i)^5}\right) + \frac{r^4}{2}\left(e^{-i\theta}\frac{t}{(z+i)^2} + ie^{-2i\theta}\frac{t^3}{(z+i)^2} + 2\frac{t^3}{(z+i)^3}\right) + O(r^6). \tag{11}$$

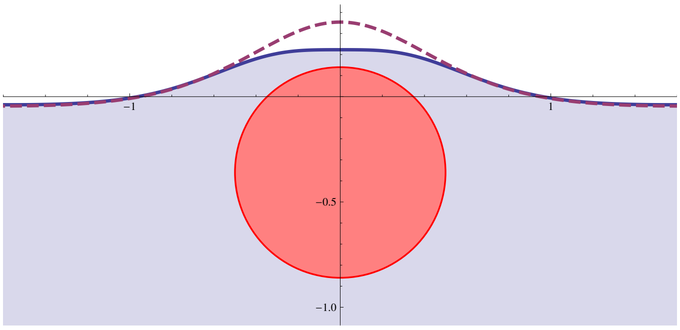


Fig 1: The shape of free surface at $t = 0.7$ ($r = 0.5$, $\lambda = 5$) predicted by the non-linear approximation (10) (solid line) and by the linear approximation [4] (dashed line).

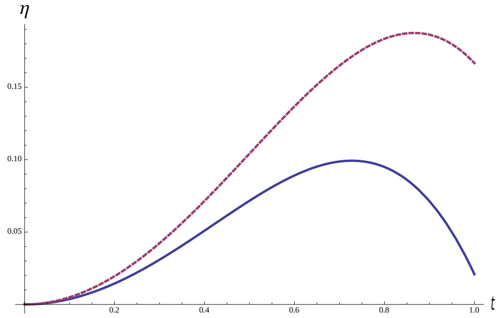
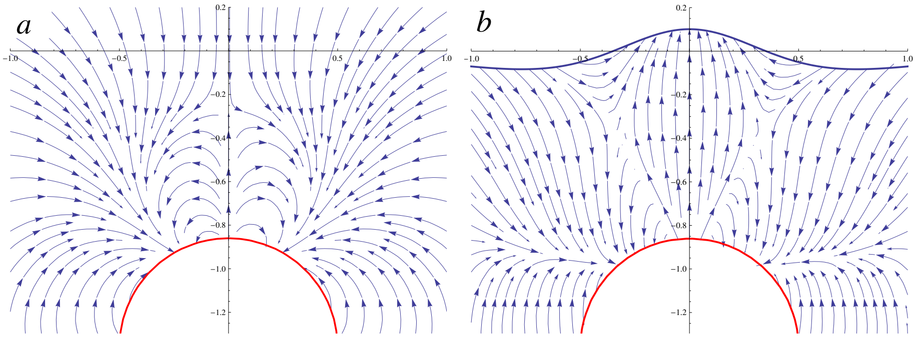


Fig 2: Free surface elevation at $x = 0$ ($r = 0.5$, $\lambda = 5$) predicted by the non-linear approximation (10) (solid line) and by the linear approximation [4] (dashed line).

Fig 3: Velocity field around the cylinder of radius $r = 0.5$ at the time $t = 0.7$ ($\lambda = 10$) moving in an infinite fluid (a) and downwards under the free surface (b).



From relation (11) we see that in the leading order as $r \to 0$ the flow is determined by the two poles located at the points $z = \pm i$ symetrically with respect to undisturbed free surface $y = 0$. The effect of self-induced dipole that takes into account non-linear effects appears only when cylinder moves close to the free surface.

In this paper the nonlinear problem of free surface flow in the presence of a submerged circular cylinder has been studied analytically. The leading-order solution with the accuracy $O(r^4)$ was constucted in explicit form. The effect of non-linearity was clarified for the case of circular cylinder moving with constant acceleration from the rest.

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