Hydroelastic response of a floating thin plate due to a surface-piercing load*

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1. INTRODUCTION

A thin elastic plate floating on an inviscid fluid is an ideal model for the very large floating structure in offshore engineering [1] and the homogeneous ice sheet in the polar region [2]. A fundamental problem for this model is the flexural response due to a moving concentrated load [3–7], in which the Euler–Bernoulli plate was commonly employed. Recently, the effect of compression of the plate on the hydroelastic dynamics were considered [8]. A general model the effect of lateral stress is presented here for the elastic plate floating on an infinitely deep fluid. A special case of this model is capillary–gravity waves on an inertial surface. The wave response and the wave resistance are analytically investigated for steadily moving, suddenly starting and suddenly stopping concentrated loads on the surface of the floating plate. For the purpose of analytical study, two-dimensional problems are considered.

2. GENERAL MATHEMATICAL FORMULATION

We consider an inviscid, incompressible and homogeneous fluid of infinite depth, being covered by a thin elastic plate of infinite extent. As a starting point for the analytical study, a two-dimensional problem is addressed here. The Cartesian coordinates oxz are chosen in such a way that the z axis points vertically upwards. The fluid occupies the domain $(-\infty < x < \infty,\ -\infty < z \leqslant 0)$ with $z = 0$ being the undisturbed plate–fluid interface. Under the assumption that the motion is irrotational, the velocity potential $\phi(x, z, t)$ for the fluid satisfies the Laplace equation $\nabla^2\phi = 0$. For an infinitely deep fluid, we have $\partial\phi/\partial z = 0$ as $z \to -\infty$.

Let $\zeta(x, t)$ represent the vertical plate deflection subjected to a downward external load $-P_{\rm ext}(x, t)$. It is assumed that the wave amplitudes generated are small in comparison with the wavelengths. Thus the linearized boundary conditions will be applied on the plate–fluid interface ($z = 0$). The kinematic boundary condition on $z = 0$ reads $\partial\zeta/\partial t = \partial\phi/\partial z$, which implies that there is no cavitation between the plate and the fluid and the fluid particles once on the interface will always remain there. The dynamic boundary condition on $z = 0$ reads

$$D\nabla^4\zeta + Q\nabla^2\zeta + M\frac{\partial^2\zeta}{\partial t^2} = -\rho\left(\frac{\partial\phi}{\partial t} + g\zeta\right) - P_{\rm ext}, \tag{1}$$

where the flexural rigidity of the plate D is determined by Young's modulus E, Poisson's ratio ν and the plate thickness d as $D = Ed^3/12(1-\nu^2)$; Q is related to the lateral stress of the plate (with compression at $Q > 0$ and stretch at $Q < 0$) [8]; $M = \rho_{\rm e}d$; $\rho_{\rm e}$ and ρ denote the densities of the plate and the fluid, respectively; and g is the acceleration due to gravity.

Obviously, Eq. (1) indicates the balance among the elastic, inertial, hydrodynamic forces and the downward external load. Equation (1) is a general linear model for a floating elastic plate. In particular, as $D = 0$ and $Q = -T$, Eq. (1) is for the capillary–gravity waves on an inertial ($M \neq 0$) or a free ($M = 0$) surface, where T with $T > 0$ is the coefficient of the surface tension.

To have a formal solution, we introduce the Fourier transforms as $\{\widetilde{\phi}(\alpha, z, t), \widetilde{\zeta}(\alpha, t), \widetilde{P}_{\rm ext}(\alpha, t)\} = \int_{-\infty}^{\infty}\{\phi(x, z, t), \zeta(x, t), P_{\rm ext}(x, t)\}\exp(-{\rm i}\alpha x){\rm d}x$. Upon some mathematical derivation, we obtain, for $z = 0$,

$$\frac{\partial^2\widetilde{\zeta}}{\partial t^2} + \omega^2\hat{\zeta} = -\frac{\hat{P}_{\rm ext}k}{\rho(1+\sigma k)}, \tag{2}$$

where

$$\omega^2 = \frac{gk(\Gamma k^4 - \Lambda k^2 + 1)}{1+\sigma k}, \tag{3}$$

$$\Gamma = D/\rho g, \quad \Lambda = Q/\rho g, \quad \sigma = M/\rho, \tag{4}$$

and $k = |\alpha|$ is the wave number.

Equation (3) is the dispersion relation between the frequency $\omega(k)$ and the wave number k for the flexural–gravity wave motion on the elastic plate floating on the inviscid fluid of infinite depth. Three parameters, Γ, Λ and σ, are associated with the effects of flexural rigidity, lateral stress and the inertia of the thin plate. For the capillary–gravity waves on

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an inertial ($\sigma \neq 0$) or a free ($\sigma = 0$) surface, the corresponding dispersion relation follows from Eq. (3) by setting $\Gamma = 0$ and $\Lambda = -\tau$, where $\tau = T/\rho g > 0$.

3. STEADILY TRANSLATING LOADS

Let $\boldsymbol{e}_x$ be the unit vector along the positive x-axis. We consider a concentrated load steadily moving with a constant velocity $-U\boldsymbol{e}_x$ and a constant magnitude of strength P_0, namely $P_{\rm ext} = P_0\delta(x - x_0)$, where $\delta(\cdot)$ is the Dirac delta function and $x_0 = -Ut$ is the source point namely the location of the concentrated load. In this case the right-hand side of Eq. (2) reads $-P_0 k \exp(\mathrm{i}\alpha U t)/[\rho(1+\sigma k)]$. Neglecting the transient effect, we have the particular solution of Eq. (2) with $P_{\rm ext} = P_0\delta(x - x_0)$, denoted by $\widetilde{\zeta}^{\rm S}(\alpha, t)$, for the ultimately steady-state plate deflection as follows

$$\widetilde{\zeta}(\alpha, t) = \widetilde{\zeta}^{\rm S}(\alpha, t) = -\frac{P_0}{\rho\Delta}\exp(\mathrm{i}\alpha U t), \tag{5}$$

where

$$\Delta(k) = k(1 + \sigma k)(c^2 - U^2), \tag{6}$$

and $c(k) = \omega/k$ is the phase speed.

By the inverse Fourier transform for Eq. (5), the steady plate deflection, denoted by $\zeta^{\rm S}$, due to a steadily translating load is given by

$$\zeta^{\rm S}(X, U) = -\frac{P_0}{2\pi\rho}\int_{-\infty}^{\infty}\frac{\exp(\mathrm{i}\alpha X)}{\Delta}\mathrm{d}\alpha, \tag{7}$$

where $X = x - x_0 = x + Ut$. Taking X as a new coordinate, one can see from Eq. (7) that the wave is time-independent, which can be seen as a system of steady waves in a reference frame steadily moving with the load. It is noted that the denominator of the integrand in Eq. (7) is an even function with respect to α since $k = |\alpha|$. To have an explicit expression for the plate deflection, the Jordan lemma will be used. The contribution to the integral comes from the pole of the integrand, namely the roots of the equation

$$c^2 - U^2 = 0. \tag{8}$$

One can find that there is a minimal phase speed $c_{\rm min}$ for the flexural–gravity waves in the floating plate with a given d. $c_{\rm min}$ is usually referred to as the critical speed of the moving load. The critical wave number $k_{\rm cr}$ at which the minimal phase speed occurs satisfies

$$\frac{\mathrm{d}c}{\mathrm{d}k} = \frac{1}{k}(c_{\rm g} - c) = 0, \tag{9}$$

where $c_{\rm g}(k) = \mathrm{d}\omega/\mathrm{d}k$ is the group speed of the wave generated. Equation (9) implies $c_{\rm g} = c = c_{\rm min}$ at $k = k_{\rm cr}$. For $k < k_{\rm cr}$, $c_{\rm g} < c$ while for $k > k_{\rm cr}$, $c_{\rm g} > c$.

The nature of the real roots of Eq. (8) depends crucially on the relation between U and $c_{\rm min}$. As $U < c_{\rm min}$, Eq. (8) has no real roots. As $U = c_{\rm min}$, Eq. (8) has one real root $k_{\rm cr}$. As $U > c_{\rm min}$, Eq. (8) has two real roots, denoted by k_1 and k_2 with $k_1 < k_{\rm cr} < k_2$.

As $U < c_{\rm min}$, the plate deflection profile can be numerically calculated by the fast Fourier transform [4]. According to Schulkes and Sneyd [4], there is no wave propagation. As $U < c_{\rm min}$ the deflection profile is similar to a static one. As $U > c_{\rm min}$, it is noted that the poles of the integrand $\pm k_1$ and $\pm k_2$ lie on the real α axis, which is due to the use of potential theory for an inviscid fluid. For the viscous fluid, the poles are automatically off from the axis since the viscosity coefficient appears in the imaginary part, as shown by Lu and Chwang [9]. To perform the α integration for the wave motion in an inviscid fluid, an artificial viscosity is necessary to move the poles off the axis. According to Lighthill's method [10], the artificial viscosity, denoted by ϵ with $\epsilon > 0$, can be introduced as $\alpha = \alpha_0 - \mathrm{i}\epsilon/(c_{\rm g} - U)$, where $\alpha_0 = \pm k_1, \pm k_2$ is the original pole. For $\alpha_0 = k_1 < k_{\rm cr}$, we have $c_{\rm g} < c = U$. For $\alpha_0 = k_2 > k_{\rm cr}$, we have $c_{\rm g} > c = U$. Therefore, with the aid of the artificial viscosity, $\pm k_1$ and $\pm k_2$ are moved into the upper and lower half α-plane, respectively.

According to the Jordan lemma, infinite semicircles in the upper and lower half α-planes are chosen for $X > 0$ and $X < 0$, respectively. Thus the wave profile for $U > c_{\rm min}$ is given by

$$\zeta^{\rm S}(X, U) = \begin{cases} \zeta_1^{\rm S}, & (X > 0), \\ \zeta_2^{\rm S}, & (X < 0), \end{cases} \tag{10}$$

where $\zeta_j^{\rm S} = 2(-1)^{j+1}P_0\sin(k_j X)\mathrm{H}(U - c_{\rm min})/\rho\Delta_j'$, $\Delta_j' = \mathrm{d}\Delta(k_j)/\mathrm{d}k$, and $\mathrm{H}(\cdot)$ is the Heaviside step function. k_1 is the wave number of the long gravity-dominated wave trailing the moving object ($x_0 < x$), while k_2 is the wave number of the short elasticity-dominated (capillarity-dominated) wave leading the moving object ($x < x_0$). This theoretical perdition is in agreement with the experimental observatories conducted by Squire et al. [5]

According to the formula of wave resistance given by Kim and Webster [11], the wave resistance $\boldsymbol{R} = R^{\rm S}\boldsymbol{e}_x$ for a moving load can be given by

$$R^{\rm S}(U) = \int_{-\infty}^{\infty} P_{\rm ext}\frac{\partial\zeta}{\partial x}\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}\alpha\widetilde{P}_{\rm ext}^{*}\widetilde{\zeta}\mathrm{d}\alpha, \tag{11}$$

where $\widetilde{P}_{\rm ext}^{*}(\alpha, t)$ is the conjugate function of $\widetilde{P}_{\rm ext}(\alpha, t)$. It follows from Eq. (5) that the steady-state wave resistance reads

$$R^{\rm S}(U) = -\frac{P_0^2}{2\pi\rho}\int_{-\infty}^{\infty}\frac{\mathrm{i}\alpha}{\Delta}\mathrm{d}\alpha. \tag{12}$$

By the residue theorem developed by Lighthill [10] for the dispersive waves, the far-field wave resistance for $V > c_{\rm min}$ can be analytically given by

$$R^{\rm S}(U) = \frac{2P_0^2}{\rho}\sum_{j=1}^{2}\frac{k_j}{\Delta_j'}\mathrm{H}(U - c_{\rm min}). \tag{13}$$

4. SUDDENLY STARTING LOADS

We assume that the concentrated load suddenly starts from rest at $t = 0$ and then moves with a constant velocity $-U\boldsymbol{e}_x$, then we have $P_{\text{ext}} = P_0\delta(x - x_0)\text{H}(t)$. The initial conditions for Eq. (2) read

$$\widetilde{\zeta}|_{t=0} = 0, \quad \frac{\partial\widetilde{\zeta}}{\partial t}|_{t=0} = 0. \tag{14}$$

The solution for Eq. (2) with $P_{\text{ext}} = P_0\delta(x - x_0)\text{H}(t)$ and (14) can readily be given by

$$\widetilde{\zeta}(\alpha, t) = \widetilde{\zeta}^{\text{S}}(\alpha, t) + \widetilde{\zeta}^{\text{T}}(\alpha, t), \tag{15}$$

where

$$\widetilde{\zeta}^{\text{T}}(\alpha, t) = \frac{P_0}{\rho\Delta}\Big[\cos(\omega t) + \frac{\text{i}\alpha U}{\omega}\sin(\omega t)\Big], \tag{16}$$

and $\widetilde{\zeta}^{\text{S}}(\alpha, t)$ is given by Eq. (5). One can see that $\widetilde{\zeta}^{\text{S}}(\alpha, t)$ and $\widetilde{\zeta}^{\text{T}}(\alpha, t)$ are the steady-state and transient responses due to a suddenly starting load, respectively.

The analysis on $\hat{\zeta}^{\text{S}}(\alpha, t)$ follows Section 3. Equation (16) can be rewritten as

$$\widetilde{\zeta}^{\text{T}}(\alpha, t) = A\exp(\text{i}\alpha U t)\sum_{\pm}\frac{\exp(\mp\text{i}\Omega_{\pm}t)}{\Omega_{\pm}}, \tag{17}$$

where $A(k) = P_0/2\rho(1 + \sigma k)c$ and $\Omega_{\pm}(\alpha) = \omega \pm \alpha U$. The transient plate deflection, denoted by ζ^{T}, due to a suddenly starting load is given by

$$\zeta^{\text{T}} = \frac{1}{2\pi}\sum_{\pm}\int_{-\infty}^{\infty} A\exp(\text{i}\alpha X)\frac{\exp(\mp\text{i}\Omega_{\pm}t)}{\Omega_{\pm}}\text{d}\alpha. \tag{18}$$

Equation (18) with large t will be performed by means of the method of stationary phase. Ω_- with $\alpha > 0$ and Ω_+ with $\alpha < 0$ have the same stationary points, which make the main contribution to the integral for ζ^{T}. Equation (18) can be rewritten as

$$\zeta^{\text{T}} = \sum_{n=1}^{2}\int_0^{\infty}\frac{A}{\Omega}\exp\big[(-1)^{n+1}\text{i}(kX + \Omega t)\big]\text{d}k, \tag{19}$$

where $\Omega(k) = \omega - kU$. The stationary point of $\Omega(k)$ is determined by

$$\frac{\text{d}\Omega}{\text{d}k} = c_{\text{g}} - U = 0. \tag{20}$$

There exists a minimal group velocity c_{gmin}. The nature of the real roots of Eq. (20) depends crucially on the relation between U and c_{gmin}. Equation (20) has no roots for $U < c_{\text{gmin}}$, one real root (denoted by κ_{gm}) for $U = c_{\text{gmin}}$, and two real roots (denoted by κ_1 and κ_2) for $U > c_{\text{g}}$. Obviously, $\kappa_1 < \kappa_{\text{gm}} < \kappa_2$.

5. SUDDENLY STOPPING LOADS

We consider a concentrated load steadily moving with a constant velocity $-U\boldsymbol{e}_x$ and a constant magnitude of strength P_0 for $t < 0$. The load suddenly stops at $t = 0$ and keeps at rest for $t > 0$. The governing equation is Eq. (2) with $\hat{P}_{\text{ext}} = 0$. The initial values at $t = 0$ for the plate defection and the velocity are taken as those for the steady-state solution. The initial conditions are given by

$$\Big\{\widetilde{\zeta}|_{t=0}, \frac{\partial\widetilde{\zeta}}{\partial t}|_{t=0}\Big\} = -\frac{P_0}{\rho\Delta}\Big\{1, \text{i}\alpha U\Big\}. \tag{21}$$

The corresponding solution reads $\widetilde{\zeta} = -\widetilde{\zeta}^{\text{T}}(\alpha, t)$, where $\widetilde{\zeta}^{\text{T}}(\alpha, t)$ is given in Eq. (16). It should be noted that $\widetilde{\zeta} = -\widetilde{\zeta}^{\text{T}}(\alpha, t)$ is similar to the transient part due to a suddenly starting load.

6. DISCUSSION

6.1. Flexural–gravity waves

In this Subsection, we consider the flexural–gravity waves on an elastic surface with $\Gamma > 0$ and $\Lambda \neq 0$. The general case with $\sigma \neq 0$ is discussed at first. Then the special case with $\sigma = 0$ is of interest since the inertial effect of the thin plate can be neglected in comparison with the effects of the elastic force and lateral stress of the plate and with the fluid inertia. This assumption is justified since the wavelength of the plate deflection is usually much large than the plate thickness [3].

From the dispersion relation of the flexural–gravity waves on an elastic surface, some remarkable characteristics can be found due to the presence of lateral stress ($\Lambda \neq 0$). As $Q = Q_{\max} = 2\sqrt{D\rho g}$ (namely $\Lambda = 2\sqrt{\Gamma}$), we have $c_{\min} = 0$ at $k = k_{\text{p}0} = (\rho g/D)^{1/4}$. Therefore $Q < Q_{\max}$ is a necessary condition for the wave propagation. Close examination on c_{g} shows that there exists a critical value of Q (denoted by $Q_{\text{g}0}$) at which $c_{\text{gmin}} = 0$, and $c_{\text{g}} > 0$ holds if and only if $Q < Q_{\text{g}0}$. As $Q_{\text{g}0} < Q < Q_{\text{c}0}$, we have $c_{\text{g}} < 0$ and $c > 0$. As $\sigma \neq 0$, the values of $Q_{\text{g}0}$ and the wave number $k_{\text{g}0}$ at which $c_{\text{g}} = 0$ holds satisfy quintic equations which can be solved numerically. As $\sigma = 0$, we have the analytical expressions $Q_{\text{g}0} = 2\sqrt{5D\rho g}/3$ (namely $\Lambda = 2\sqrt{5\Gamma}/3$) and $k_{\text{g}0} = (\rho g/25D)^{1/4}$.

As $\sigma \neq 0$, the solutions for k_{cr} and κ_{gm}, which satisfy respectively $\text{d}c/\text{d}k = 0$ and $\text{d}c_{\text{g}}/\text{d}k = 0$, can be obtained numerically. As $\sigma = 0$, exactly analytical solutions for k_{cr} and κ_{gm} can be given as

$$k_{\text{cr}} = \left(\frac{\sqrt{F} + \sqrt{12 + F}}{6\sqrt{\Gamma}}\right)^{1/2}, \tag{22}$$

$$\kappa_{\text{gm}} = \left(\frac{\sqrt{E_1} - \sqrt{D_1} + 11\sqrt{F}}{30\sqrt{\Gamma}}\right)^{1/2}, \tag{23}$$

where $F = \Lambda^2/\Gamma$, $E_1 = 363F - D_1 - 90(10+F) + 8(900-209F)\sqrt{F/D_1}$, $D_1 = 15C_1 + 121F - 30(10+F) + 15(80-24F+F^2)/C_1$, $C_1 = (8\sqrt{B_1} + 1600 - 272F - 36F^2 + F^3)^{1/3}$, and $B_1 = (4-F)^2(2000 + 600F - 19F^2)$. Accordingly, the analytical expressions for the minimal phase and group speeds are given by $c_{\min} = c(k_{\rm cr})$ and $c_{\rm gmin} = c_{\rm g}(\kappa_{\rm gm})$, respectively.

As $\sigma \neq 0$, we have, for the steady-state flexural–gravity wave elevation and wave resistance,

$$\Delta_j' = g(4\Gamma k_j^3 - 2L_2 k_j - L), \tag{24}$$

where $L_2 = \Lambda + \sigma L$ and $L = U^2/g$. The exact solutions for the wave numbers k_1 and k_2 of the steady-state response can be given by

$$k_j = \frac{\sqrt{K_2}}{2} + \frac{(-1)^j}{2}\sqrt{-K_2 + \frac{2L}{\sqrt{K_2}\Gamma} + \frac{2L_2}{\Gamma}}, \tag{25}$$

where $K_2 = [l_2/2^{1/3} + 2^{1/3}(12\Gamma + L_2^2)/l_2 + 2L_2]/3\Gamma$, $l_2 = [L_6 + \sqrt{L_6^2 - 4(12\Gamma + L_2^2)^3}]^{1/3}$, $L_6 = 27L^2\Gamma + 72L_2\Gamma - 2L_2^3$. As $\sigma = 0$, the solutions for Δ_j' and k_j can readily be obtained from Eqs. (24) and (25) by setting $\sigma = 0$.

As $\sigma \neq 0$ or $\sigma = 0$, the solutions for the wave numbers κ_1 and κ_2 of the transient flexural–gravity waves can be calculated numerically.

6.2. Capillary–gravity waves

Another special case with $\Gamma = 0$ and $\Lambda = -\tau < 0$ corresponds to the capillary–gravity waves on an inertial ($\sigma \neq 0$) or a free ($\sigma = 0$) surface. There is a maximal phase and group speeds for the capillary–gravity waves on an inertial surface, namely

$$\lim_{k\to+\infty} c = \lim_{k\to+\infty} c_{\rm g} = \sqrt{g\tau/\sigma}. \tag{26}$$

The analytical solutions for $k_{\rm cr}$ and $\kappa_{\rm gm}$ are given as

$$k_{\rm cr} = \frac{1}{\sqrt{\tau}}[\sqrt{C} + \sqrt{C+1}], \tag{27}$$

$$\kappa_{\rm gm} = \frac{\sqrt{D_2}}{2} + \frac{1}{2}\sqrt{2D_2 - 3C_2 + \frac{E_2}{4\sqrt{D_2}} - \frac{\sigma}{B_2}}, \tag{28}$$

where $E_2 = 32\sigma(1/B_2\tau - 2\sigma^2/B_2 + 3)/B_2^2$, $D_2 = 4(\sigma^2/B_2 - 1)/B_2 + C_2$, $C_2 = 4(C+1)^{2/3}/B_2$, $B_2 = \tau(4C+3)$, and $C = \sigma^2/\tau$.

For the the steady-state capillary–gravity wave elevation and wave resistance, we have

$$\Delta_j' = g[2(\tau - \sigma L)k_j - L], \tag{29}$$

where $k_j = [L + (-1)^j\sqrt{L^2 - 4\tau + 4\sigma L}]/2(\tau - \sigma L)$.

As $\sigma \neq 0$, the solutions for the wave numbers κ_1 and κ_2 of the transient capillary–gravity waves can be calculated numerically. For $\sigma = 0$, the exact solution for κ_j is

$$\kappa_j = \frac{\sqrt{C_3}}{2} + \frac{L}{9\tau} + \frac{(-1)^j}{2}\sqrt{\frac{D_3}{4\sqrt{C_3}} + \frac{4L^2}{27\tau^2} - \frac{4}{3\tau} - C_3},$$

where $D_3 = 4(L^2 - 9\tau)/27\tau^2$, $C_3 = 4(3^{4/3}B_3^2 + L^2B_3 - 9B_3\tau - 3^{5/3}L^2\tau + 3^{8/3}\tau^2)/81B_3\tau^2$, and $B_3 = \{9L^2\tau^2 + L\tau[3\tau(L^4 + 18L^2\tau - 27\tau^2)]^{1/2} - 9\tau^3\}^{1/3}$.

7. CONCLUSIONS

Analytical solutions are explicitly derived for the wave response and the wave resistance due to steadily moving, suddenly starting and suddenly stopping concentrated loads on the surface of the floating plate, taking the effects of lateral stress into consideration. It is found that there is a critical value for the compression effect, at which the phase speed of flexural–gravity waves is zero.

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