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A note on convergence of expansion formulae for wave-structure interaction problems

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Highlights

- Convergence of the expansion formulae for problems of gravity wave-interaction with floating flexible structure are demonstrated using Green's function approach in both the cases of finite and infinite water depth.
- In the present approach, the Green's function for the differential operator associated with the wave-structure interaction problems is integrated over the complex plane to derive the spectral representation. This approach is independent of whether the spectrum for the differential operator is discrete or continuous.

1. Introduction

In recent decades, emphasis is given to the dynamic analysis of large class of problems in the field of hydroelasticity and hydroacoustics involve higher order conditions on a wavy boundary. Apart from the development of expansion formulae and associated mode-coupling relations as in Manam et al. [1], emphasis is given to study the convergence of the various infinite series and integrals (see Evans & Porter [2]). Lawrie [3], [4] proved the point-wise convergence of a class of problems associated with Helmholtz equation satisfying higher order boundary conditions arising in hydroacoustics in two and three dimensions. Later, Mondal et al. [5] demonstrated the convergence of the expansion formulae in case of infinite depth for a class of wave-structure interaction problems associated with Laplace equation satisfying higher order boundary conditions in the broad field of hydroelasticity. In the present study, convergence of the expansion formulae for wave-structure interaction problems arising in hydroelasticity are demonstrated in both the cases of finite and infinite water depth. In the present approach, the Green's functions for the differential operator associated with the wave-structure interaction problems along the water depth is integrated over the complex plane to derive the spectral representation. This approach is independent of water depth.

2. Mathematical formulation

Under the assumption of the linearized theory of water waves, the wave-structure interaction problem is formulated in a two-dimensional channel in both the cases of water of finite and infinite depth. The physical problem is studied in the Cartesian co-ordinate system with x axis being in the horizontal direction and y-axis being positive in the vertically downward direction. Assuming that a flexible plate acting under the action of uniform compression being infinitely extended along the positive direction of x-axis, is floating on the mean free surface. Thus, the fluid domain occupies the region $0 < x < \infty$ and $0 < y < h$ in case of finite water depth ($0 < x < \infty$ and $0 < y < \infty$ in case of infinite water depth). The fluid is assumed to be incompressible, inviscid with the fluid motion being irrotational and simple harmonic in time with angular frequency ω. Thus, there exists a velocity potential $\Phi(x, y, t)$ of the form $\Phi(x, y, t) = Re\{\phi(x, y)e^{-\mathrm{i}\omega t}\}$ with Re being the real part. Hence, in the fluid region, the spatial velocity potential $\phi(x, y)$ satisfies the Laplace equation

$$\nabla^2\phi = 0, \tag{1}$$

along with the linearized plate covered boundary condition on the mean free surface

$$(D\partial_{yyyyy} - Q\partial_{yyy} + \partial_y + K)\phi = 0, \quad \text{on } y = 0, \tag{2}$$

with D, Q, K being known positive constants as in Manam et al.[1], and the bottom boundary condition

$$\left.\begin{array}{ll} \phi_y = 0 & \text{on} \quad y = h \text{ in case of finite water depth,} \\ \phi, |\phi_y| \to 0 & \text{on } y \to \infty \text{ in case of infinite water depth.} \end{array}\right\} \tag{3}$$

The spatial velocity potentials ϕ satisfying Eq.(1) along with the boundary conditions in Eqs.(2) - (3) are expressed as (see Manam et al. [1])

$$\phi(x, y) = \sum_{n=0,I}^{IV} A_n(x)\psi_n(y) + \sum_{n=1}^{\infty} A_n(x)\psi_n(y), \quad \text{in case of finite water depth} \tag{4}$$

$$\phi(x,y) = \sum_{n=0,I}^{IV} A_n(x)\psi_n(y) + \frac{2}{\pi}\int_0^\infty \frac{A(k,x)M(k,y)}{T^2+K^2}dk, \quad \text{in case of infinite water depth} \tag{5}$$

where $A_n(x)$ and $A(k,x)$ are of the forms $A_n(x) = A_n \mathrm{e}^{\mathrm{i}k_n x}$ and $A(k,x) = A(k)\mathrm{e}^{-kx}$ and are given by

$$A_n(x) = \frac{1}{D_n}\left[\int_0^h \phi(x,y)\psi_n(y)dy - \frac{Q}{K}\phi_y(x,0)\psi_{ny}(0) + \frac{D}{K}\{\phi_y(x,0)\psi_{nyyy}(0) + \psi_{ny}(0)\phi_{yyy}(x,0)\}\right], \tag{6}$$

$$A(k,x) = \frac{1}{D_n}\left[\int_0^\infty \phi(x,y)M(k,y)dy - \frac{Q}{K}\phi_y(x,0)M_y(k,0) + \frac{D}{K}\{\phi_y(x,0)M_{yyy}(k,0) + M_y(k,0)\phi_{yyy}(x,0)\}\right], \tag{7}$$

with $h = \infty$ in case of infinite water depth and $D_n = \mathcal{G}'(k)\psi_y(0)/\{2k_nK\}$, $M(k,y) = T\cos ky - K\sin ky$ with $T = (Dk^5 - Qk^3 + k)$. In Eqs. (4) - (5), the eigenfunctions $\psi_n(y)$s are given by

$$\psi_n(y) = \begin{cases} \dfrac{\cosh k_n(h-y)}{\cosh k_n h}, & 0<y<h, \quad n = 0, I, II, III, IV, 1, ..., \quad \text{(for finite water depth)},\\ e^{k_n y}, & 0<y<\infty, \quad n = 0, I, II, III, IV, \quad \text{(for infinite water depth)}, \end{cases} \tag{8}$$

with k_ns satisfying the dispersion relation

$$\mathcal{G}(k) = 0, \tag{9}$$

where $\mathcal{G}(k) = K - (Dk^4 - Qk^2 + 1)k\tanh kh$ for finite water depth and $\mathcal{G}(k) = K - (Dk^4 - Qk^2 + 1)k$ for infinite water depth. We assume that the dispersion relation in Eq. (9) has one real positive roots $k = k_0$, two complex conjugate pairs of the form $k_I, k_{II}(=\bar{k}_I)$ and $k_{III}, k_{IV}(=\bar{k}_{III})$ and infinitely many imaginary roots of the form $\pm \mathrm{i}k_n$ for $n = 1, 2, ...$, whilst no imaginary root exists for infinite water depth. Further, the eigenfunctions $\psi_n(y)$s in Eq. (8) satisfy the orthogonal mode-coupling relation (as in Manam et al. [1])

$$\langle\psi_m(y), \psi_n(y)\rangle = \int_0^h \psi_m(y)\psi_n(y)dy - \left\{\frac{Q}{K}\psi_{my}\psi_{ny} + \frac{D}{K}(\psi_{my}\psi_{nyyy} + \psi_{myyy}\psi_{ny})\right\}\Big|_{y=0} = D_n\delta_{mn}, \tag{10}$$

with δ_{mn} being the Kroneckar delta, D_n being the same as in Eqs. (6)-(7) and $h = \infty$ in case of infinite water depth.

2.1. Characteristics of the eigenfunctions in case of finite water depth

In order to understand various characteristics of the eigenfunctions $\psi_n(y)$, consider the boundary value problem

$$\psi_{yy} - k^2\psi = 0, \quad \text{for} \quad 0<y<h, \tag{11}$$

subject to the boundary conditions as in Eqs. (2) - (3) in terms of ψ. It can be easily observed that $\psi_n(y)$ as in Eq. (8) satisfies the boundary value problem defined in Eqs. (2), (3) and (11). Next, the Green's function $G(y, y_0; k)$ associated with the boundary value problem in $\psi(y)$, where y is the field point and y_0 is the source point, is derived followed by various characteristics of the eigenfunctions which are expressed in terms of certain lemmas.

Lemma 1 *The Green's function $G(y, y_0; k)$, satisfying*

$$G_{yy} - k^2 G = \delta(y-y_0), \quad \textit{for} \quad 0<y, y_0<h, \tag{12}$$

subject to the conditions in Eqs. (2) - (3) with the continuity and jump conditions in water of finite depth,

$$G\big|_{y=y_{0+}} - G\big|_{y=y_{0-}} = 0 \quad \textit{and} \quad G_y\big|_{y=y_{0+}} - G_y\big|_{y=y_{0-}} = 1 \quad \textit{at} \quad y = y_0 \tag{13}$$

is given by

$$G = \begin{cases} \{\cosh ky(Dk^5 - Qk^3 + k) + K\sinh ky\}\cosh k(h-y_0)/\{k\mathcal{G}(k)\}, & 0<y<y_0,\\ \{\cosh ky_0(Dk^5 - Qk^3 + k) + K\sinh ky_0\}\cosh k(h-y)/\{k\mathcal{G}(k)\}, & y_0<y<h. \end{cases} \tag{14}$$

Lemma 2 *The eigenfunctions $\psi_n(y)$ in water of finite depth have the following spectral representation*

$$\delta(y-y_0) = \sum_{n=0,I,II,1}^{\infty} Y_n\psi_n(y)\psi_n(y_0), \quad 0<y, y_0<h, \tag{15}$$

where $\delta(y)$ is the Dirac delta function and Y_n is given by $Y_n = -(\sinh 2k_nh)/(2D_n)$.

Proof: Proceeding in a similar manner as in Friedman [6], it can be proved that

$$\lim_{R\to\infty} \frac{1}{\pi \mathrm{i}} \oint G(y, y_0; k)kdk = -\delta(y - y_0), \tag{16}$$

where $G(y, y_0, k)$ is the Green's function as in Eq. (14), R being the radius of a closed semi-circular contour of large radius in the upper half of the complex $k-$plane. Thus, to prove the Lemma, it is necessary to evaluate the contour integral as $R \to \infty$. It may be noted that $k = 0$ is neither a pole nor a branch point of $G(y, y_0; k)$. Further, it is found that the poles of the integrand $G(y, y_0; k)$ in Eq. (16) are the simple zeros of the relation as in Eq. (9), with one of these zeros k_0 being on the real axis, $k_n, n = I, II, III, IV$, are four complex roots lie on in four quadrants and the other infinity number of zeros $\pm k_n$, $k_n > 0, n = 1, 2, 3, ...,$ being on the imaginary axis. For the sake of boundedness of the Green's function, k_{III} and k_{IV} have been neglected. The path of integration of the closed contour is deformed onto a semicircular arc (Γ) of large radius $R(\to \infty)$ on upper half plane, the line segments $[-R, k_0 - \epsilon]$, a semi circle (γ_ϵ) from $-\epsilon$ to ϵ and the line segment $[k_0 + \epsilon, R]$ which contains all the poles in the upper half plane. Now

$$\frac{1}{\pi \mathrm{i}} \oint G(y, y_0; k)kdk = \frac{1}{\pi \mathrm{i}} \oint kdk\Big\{G_1(y, y_0; k)H(y_0 - y) + G_2(y, y_0; k)H(y - y_0)\Big\}, \tag{17}$$

where G_1 (for $y < y_0$) and G_2 (for $y > y_0$) are the Green's functions given in Eq. (14) with $H(y)$ being the Heaviside step function. Now using Jordan's lemma and applying Cauchy residue theorem one can find

$$\frac{1}{\pi \mathrm{i}} \oint G_1(y, y_0; k)H(y_0 - y)kdk = -\sum_{n=0,I,II,1}^{\infty} Y_n\psi_n(y)\psi_n(y_0)H(y_0 - y), \tag{18}$$

with $Y_n = -(\sinh 2k_nh)/(2D_n)$. Proceeding in a similar manner, it can be proved that

$$\frac{1}{\pi \mathrm{i}} \oint G_2(y, y_0; k)H(y - y_0)kdk = -\sum_{n=0,I,II,1}^{\infty} Y_n\psi_n(y)\psi_n(y_0)H(y - y_0), \quad \text{for} \quad y_0 < y. \tag{19}$$

Thus, Eqs. (17) - (19) yields

$$\frac{1}{\pi \mathrm{i}} \oint G(y, y_0; k)kdk = -\sum_{n=0,I,II,1}^{\infty} Y_n\psi_n(y)\psi_n(y_0)\Big\{H(y_0 - y) + H(y - y_0)\Big\}. \tag{20}$$

Eliminating the integral in Eqs. (16) and (20), the spectral representation given in Eq. (15) is obtained. □
It may be noted that Lawrie [3]) derived similar spectral representations for the eigenfunctions associated with acoustic wave structure interaction problems following a different approach.

2.2. Characteristics of the eigenfunctions in case of infinite water depth

In this subsection, the spectral representation of the eigenfunctions and the associated results of convergence for the velocity potential is discussed in the case of infinite water depth using a similar approach as discussed in case of finite water depth in the previous Section.

Lemma 3 *The Green's function $G(y, y_0; k)$ satisfying*

$$G_{yy} + k^2G = \delta(y - y_0), \quad \textit{for} \quad 0 < y, y_0 < \infty, \tag{21}$$

subject to the boundary conditions in Eqs. (2) - (3) and the continuity and jump conditions as in Eq. (13) in case of infinite water depth, is given by

$$G = \begin{cases} M(k, y)/\{k\mathcal{G}(\mathrm{i}k)\}e^{-\mathrm{i}ky_0}, & 0 < y < y_0, \\ M(k, y_0)/\{k\mathcal{G}(\mathrm{i}k)\}e^{-\mathrm{i}ky}, & y_0 < y < \infty, \end{cases} \tag{22}$$

where $M(k, y)$ is same as in Eq. (5).

Lemma 4 *The eigenfunctions $\psi_n(y)$ and $M(k, y)$ defined in Eqs. (8) and (5) have the following spectral representation*

$$\delta(y - y_0) = \sum_{n=0,I}^{II} Z_n\psi_n(y)\psi_n(y_0) + \frac{2}{\pi}\int_0^{\infty} \frac{M(k, y)M(k, y_0)}{T^2 + K^2}dk, \tag{23}$$

with $Z_n = -1/(2D_n)$ and D_n being the same as in Eqs. (6)-(7).

Proof: As discussed in case of finite water depth in Eq. (16), in case of infinite water depth

$$\frac{1}{\pi \mathrm{i}}\oint G(y,y_0;k)kdk = -\delta(y-y_0), \tag{24}$$

where $G(y,y_0,k)$ is the Green's function as in Eq. (22) and the integration to be performed in the complex $k-$ plane as in Eq. (16). In this case, since G has a branch point singularity at $k=0$, a branch cut is introduced in the complex plane along the positive real axis and the closed contour is chosen as a large circle not crossing the branch cut as in Friedman [6]. Using the fact that the poles of the integrand $G(y,y_0;k)$ are the simple zeros of the relation as in Eq. (9), with one of these zeros k_0 being on the real axis and $k_n, n=I,II$, the two complex roots contributing to the boundedness of the Green's function. Thus, using complex function theory and proceeding in a similar manner as in Lemma 2, it can be easily derived that

$$\begin{aligned}\frac{1}{\pi \mathrm{i}}\oint G(y,y_0;k)kdk = \quad & -Z_n\psi_n(y)\psi_n(y_0) - \frac{1}{\pi \mathrm{i}}\int_0^\infty \Big[\big\{G_1(k)-G_1(-k)\big\}H(y_0-y) \\ & +\big\{G_2(k)-G_2(-k)\big\}H(y-y_0)\Big]kdk,\end{aligned} \tag{25}$$

where G_1 (for $y<y_0$) and G_2 (for $y>y_0$) are the Green's functions given in Eq. (22) with $H(y)$ being the Heaviside function as in Lemma 2. Now substituting G_1 and G_2 one can find that

$$\frac{1}{\pi \mathrm{i}}\oint G(y,y_0;k)kdk = -\sum_{n=0,I}^{II} Z_n\psi_n(y)\psi_n(y_0) - \frac{2}{\pi}\int_0^\infty \frac{M(k,y)M(k,y_0)dk}{T^2+K^2}. \tag{26}$$

Now, eliminating the integral in Eqs. (24) and (26), the spectral representation given in Eq. (23) is obtained. □
It may be noted that the relation in Eq. (23) was derived in Mondal et al. [5] using a different approach.

Theorem 1 *Given the coefficients $A_n(x)$ and $A(k,x)$ in Eqs. (6) and (7) where the velocity potential $\phi(x,y)$ satisfies the Laplace equation as in Eq. (1) along with the boundary conditions in Eqs. (2) - (3), the sums*

$$S(x,y) = \begin{cases} \displaystyle\sum_{n=0,I}^{II} A_n(x)\psi_n(y) + \sum_{n=1}^{\infty} A_n(x)\psi_n(y), & \textit{in case of finite water depth,} \\ \displaystyle\sum_{n=0,I}^{II} A_n(x)\psi_n(y) + \frac{2}{\pi}\int_0^\infty \frac{A(k,x)M(k,y)}{T^2+K^2}dk, & \textit{in case of infinite water depth} \end{cases} \tag{27}$$

converge to $\phi(x,y)$ in water of finite and infinite depth, as appropriate.

Proof: Using the spectral representation of the eigenfunctions as in Eqs. (15) and (23), the expression for the unknowns $A_n(x)$ and $A(k,x)$ and the orthogonal mode-coupling relation defined in Eq. (10), it can be easily shown that $S(x,y)$ converges to $\phi(x,y)$ in finite and infinite water depth. □

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