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A comparative study of the GN-3 and Boussinesq equations for nonlinear wave propagation

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Highlights:

- We set the level of the GN model to Level III, i.e., GN-3 model, and use it to compare the results with the results of a program we developed based on the theory of BOUSS-2D.
- For nonlinear shallow water waves, we increase the wave amplitude; both the GN-3 results and the Boussinesq results are presented and compared with the stream function theory.

1 Introduction

Nwogu's (1993, 1996) Boussinesq-type equations are widely used to study wave-current interaction, wave breaking, run-up (Nwogu and Demirbilek, 2001, 2010), among others. Nwogu's (1993) Boussinesq-type equations are based on the assumption that the wave heights are much smaller than the water depth. This may limit the ability of the equations to describe highly nonlinear waves in shallow water, and therefore this led Wei et al. (1995) to derive a fully nonlinear form of the equations. Wei et al. (1995) derived the equations from the dynamic free-surface boundary condition by retaining all nonlinear terms, up to the order of truncation of the dispersive terms. Nwogu (1996) derived a more compact form of the equations. The computer program BOUSS-2D is based on the Boussinesq-type equations derived by Nwogu (1993, 1996). The equations are depth-integrated equations for the conservation of mass and momentum for nonlinear waves propagating in shallow and intermediate water depths.

The Green-Naghdi approach (see e.g., Demirbilek and Webster, 1992) is fundamentally different from the perturbation method which is used in deriving the Boussinesq model. The GN model only introduces an assumption on the velocity variation in the vertical direction across the fluid layer or sheet. No restriction is placed on the wave amplitude. Following the development of different polynomial orders for the description of velocity in the vertical direction, the GN theory can be of different levels, such as I (GN-1), II (GN-2), III (GN-3), and so forth. Zhao et al. (2014) applied the GN-3, GN-5 and GN-7 models to some wave transformation problems, and they showed that high-level GN models can simulate strongly dispersive and strongly nonlinear waves.

In this study, we set the level of the GN model to Level III. And use the GN-3 model to compare the results with a computer program we developed based on the theory of the BOUSS-2D model to study nonlinear wave propagation for periodic and solitary waves.

2 The BOUSS-2D model

In the BOUSS-2D model, the vertical profile of the flow field is obtained by expanding the velocity potential, $\Phi(\boldsymbol{x}, z, t)$, in a Taylor series about an arbitrary elevation, z_α, in the water column. For waves of length, L, much longer than the water depth, h, the series is truncated at the second order resulting in a quadratic variation of the velocity potential over depth:

$$\begin{aligned}\Phi(\boldsymbol{x}, z, t) = \varphi_\alpha + \mu^2(z_\alpha - z)[\nabla\varphi_\alpha \cdot \nabla h] \\ +\tfrac{1}{2}\mu^2\left[(z_\alpha + h)^2 - (z + h)^2\right]\nabla^2\varphi_\alpha\end{aligned} \tag{1}$$

where $\varphi_\alpha = \Phi(\boldsymbol{x}, z_\alpha, t)$, $\nabla = (\partial/\partial x, \partial/\partial y)$, and $\mu = h/L$is a measure of frequency dispersion. The horizontal and vertical velocities are obtained from the velocity potential as:

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x}, z, t) = \nabla\Phi = \boldsymbol{u}_\alpha + (z_\alpha - z)[\nabla(\boldsymbol{u}_\alpha \cdot \nabla h) + (\nabla \cdot \boldsymbol{u}_\alpha)\nabla h] \\ +\tfrac{1}{2}\left[(z_\alpha + h)^2 - (z + h)^2\right] \nabla(\nabla \cdot \boldsymbol{u}_\alpha)\end{aligned} \tag{2}$$

$$w(\boldsymbol{x}, z, t) = \frac{\partial \Phi}{\partial z} = -[\boldsymbol{u}_\alpha \cdot \nabla h + (z + h)\nabla \cdot \boldsymbol{u}_\alpha] \tag{3}$$

where $\boldsymbol{u}_\alpha = \nabla\Phi|_{z_\alpha}$is the horizontal velocity at $z = z_\alpha$. Given a vertical profile for the flow field, the continuity and Euler (momentum) equations can be integrated over depth, reducing the three-dimensional problem to two dimensions. Nwogu and Demirbilek (2001) gave the revised form of the fully nonlinear equations as

$$\eta_{,t} + \nabla \cdot \boldsymbol{u}_f = 0 \tag{4}$$

$$\begin{aligned}&\boldsymbol{u}_{\alpha,t} + g\nabla\eta + (\boldsymbol{u}_\eta \cdot \nabla)\boldsymbol{u}_\eta + w_\eta \nabla w_\eta + \\ &(z_\alpha - \eta)[\nabla(\boldsymbol{u}_{\alpha,t} \cdot \nabla h) + (\nabla \cdot \boldsymbol{u}_{\alpha,t})\nabla h] \\ &+\tfrac{1}{2}\left[(z_\alpha + h)^2 - (\eta + h)^2\right] \nabla(\nabla \cdot \boldsymbol{u}_{\alpha,t}) \\ &-[\boldsymbol{u}_{\alpha,t} \cdot \nabla h + (\eta + h)\nabla \cdot \boldsymbol{u}_{\alpha,t}]\nabla\eta \\ &+[\nabla(\boldsymbol{u}_\alpha \cdot \nabla h) + (\nabla \cdot \boldsymbol{u}_\alpha)\nabla h + (z_\alpha + h)\nabla(\nabla \cdot \boldsymbol{u}_\alpha)] z_{\alpha,t} = 0\end{aligned} \tag{5}$$

where z_α is now a function of time and is given by $z_\alpha + h = 0.465(h + \eta)$. The volume flux density $\boldsymbol{u}_f$is given by:

$$\boldsymbol{u}_f = (h + \eta)\left\{\begin{array}{c}\boldsymbol{u}_\alpha + \left[(z_\alpha + h) - \frac{(h+\eta)}{2}\right][\nabla(\boldsymbol{u}_\alpha \cdot \nabla h) + (\nabla \cdot \boldsymbol{u}_\alpha)\nabla h] \\ +\left[\frac{1}{2}(z_\alpha + h)^2 - \frac{1}{6}(h + \eta)^2\right] \nabla(\nabla \cdot \boldsymbol{u}_\alpha)\end{array}\right\} \tag{6}$$

3 The GN-3 model

In the GN-3 model, the horizontal velocity along the water column also changes as a quadratic polynomial. It is

$$u(x, z, t) = u_0(x, t) + u_1(x, t)z + u_2(x, t)z^2 \tag{7}$$

The GN-3 equations are as follows:

$$\frac{\partial \beta}{\partial t} = \sum_{n=0}^{K} \beta^n \left(w_n - \frac{\partial \beta}{\partial x} u_n\right) \tag{8}$$

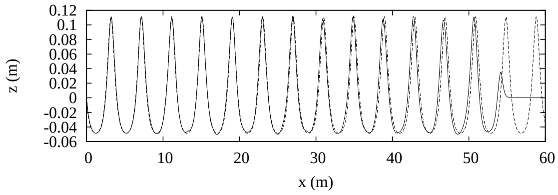
$$\frac{\partial}{\partial x}\left(G_n + gS1_n\right) + nE_{n-1} - \alpha^n \frac{\partial}{\partial x}\left(G_0 + gS1_0\right) = 0 \quad \text{for} \quad n = 1, 2, 3, \cdots, K \tag{9}$$

where $K = 3$ in this work. For more details on the GN-3 model, the reader is referred to Demirbilek and Webster (1992), Webster et al. (2011) and Zhao et al. (2014).

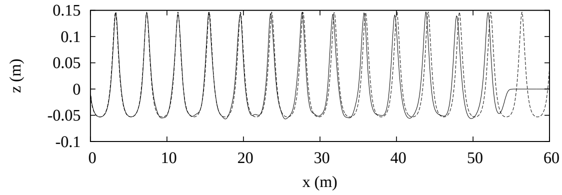
4 Test cases

In this section, we simulate periodic nonlinear regular waves in shallow water generated through the stream function theory at the wavemaker through the computer programs we developed. The water depth is $h = 0.4$m and the wave period is $T = 2.02$s. We increase the wave height H from 0.16m to 0.20m, 0.24m and 0.28m. This means that the nonlinearity parameter, H/h, changes from 0.4 to 0.5, 0.6 and 0.7. The results of the Boussinesq equations are shown in Figure 1. The GN-3 results are shown in Figure 2.

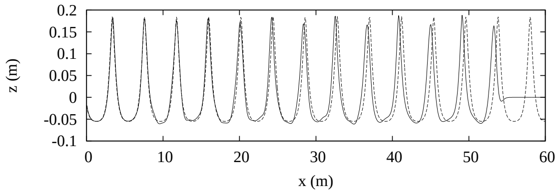
For this case, the dimensionless depth is around $kh = 0.63$. Both the Boussinesq model and the GN-3 model should be able to simulate waves with this dispersive property. But we see that the results from Figures 1 and 2 show that neither of them can simulate successfully the largest-amplitude wave when $H/h = 0.7$. For waves when $H/h = 0.5$ and 0.6, we observe that the GN-3 results agree with the stream function wave theory better than the Boussinesq model.



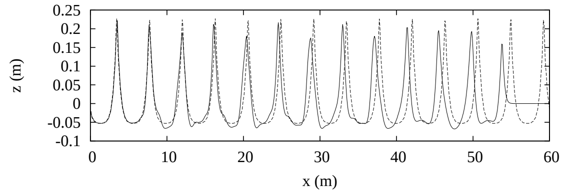
(a) $H/h = 0.4$



(b) $H/h = 0.5$

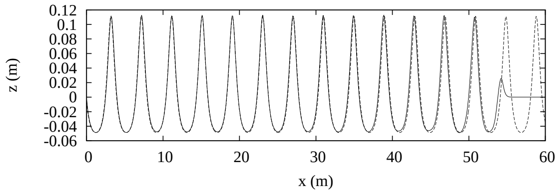


(c) $H/h = 0.6$

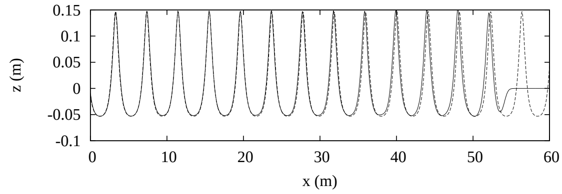


(d) $H/h = 0.7$

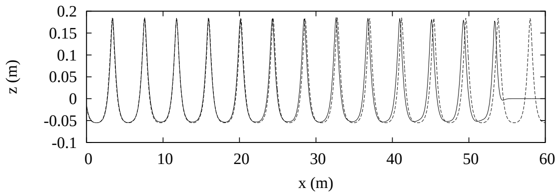
Figure 1: Snapshots at t=40s, solid line: Boussinesq model, dashed line: stream function wave theory.



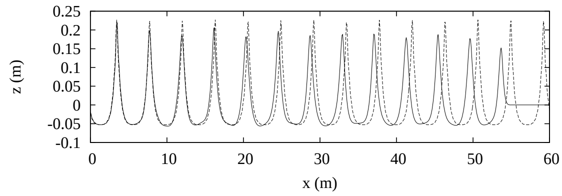
(a) $H/h = 0.4$



(b) $H/h = 0.5$



(c) $H/h = 0.6$



(d) $H/h = 0.7$

Figure 2: Snapshots at t=40s, solid line: GN-3 model, dashed line: stream function wave theory.

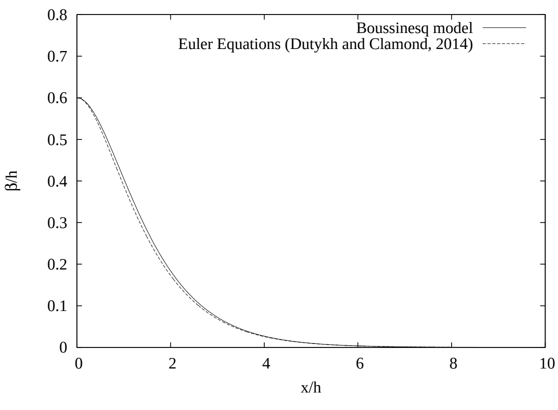


Figure 3: $H/h = 0.6$

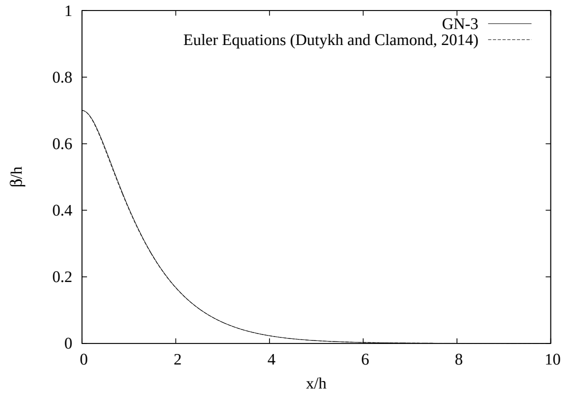


Figure 4: $H/h = 0.7$

We also studied the steady solitary wave solution from the GN-3 and Boussinesq models. Figure 3 shows that the Boussinesq model shows some differences compared with the Euler solution when $H/h = 0.6$. Figure 4 shows that the GN-3 model can simulate large amplitude solitary wave even when $H/h = 0.7$. More results will be presented at the workshop.

5 Conclusions

In this paper, we studied the GN-3 and Boussinesq models comparatively. We determined that the GN-3 model is more suitable to simulate strongly nonlinear solitary waves. For periodic waves, it appears that both models are incapable of simulating very large waves that are near breaking.

Acknowledgments

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References

1. Dutykh, D. and Clamond, D. (2014). Efficient computation of steady solitary gravity waves, *Wave Motion*, Vol. 51, pp. 86-99.

2. Nwogu, O., & Demirbilek, Z. (2010). Infragravity wave motions and runup over shallow fringing reefs. *J. waterway, port, coastal, and ocean engineering*, 136(5), 295-305.

3. Nwogu, O. (1993). Alternative form of Boussinesq equations for nearshore wave propagation. *J. waterway, port, coastal, and ocean engineering*, 119(5), 618-638.

4. Nwogu, O., & Demirbilek, Z. (2001). BOUSS-2D: A Boussinesq wave model for coastal regions and harbors. *ERDC. CHL TR-01-25, US Army Engineer Research and Development Center, Vicksburg, MS.*

5. Wei, G., Kirby, J. T., Grilli, S. T., & Subramanya, R. (1995). A fully nonlinear Boussinesq model for surface waves. Part 1. Highly nonlinear unsteady waves. *J. Fluid Mech.*, 294, 71-92.

6. Webster, W. C., Duan, W.Y., & Zhao, B.B. (2011). Green-Naghdi theory, part A: Green-Naghdi (GN) equations for shallow water waves. *J. Marine Science and Application*, 10(3), 253-258.

7. Zhao, B. B., Duan, W. Y., & Ertekin, R. C. (2014). Application of higher-level GN theory to some wave transformation problems. *Coastal Engineering*, 83, 177-189.

8. Demirbilek, Z., & Webster, W. C. (1992). Application of the Green-Naghdi theory of fluid sheets to shallow-water wave problems. *US Army Corps. of Eng. Waterways Experiment Station, Rep. No. CERC-92-11, Vicksburg, MS.*