Mechanics 1: Space Curves

If a particle is in motion we can imagine it tracing out a curve in space. In order to give a mathematical description of this motion we need to develop some mathematical tools for describing a curve in three dimensional space, or a space curve.

Space Curves, or Paths. Consider the following vector that depends on a scalar variable $t$ ("time"):  
\[ r(t) = x(t)i + y(t)j + z(t)k \]  
(1)

For each $t$ this vector locates a point in space. As $t$ varies, the tip of the vector traces out a curve. We say that (1) defines a space curve, see Fig. 1. We imagine that this curve is the path of a particle.

**Velocity.** We define the velocity of the particle at the point $P$ (also called instantaneous velocity) as: 
\[ v(t) = \lim_{\Delta t \to 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}, \quad \text{if the limit exists.} \]  
(2)

Geometrically, it is a vector tangent to the path at the point $P$. In coordinates, we have:

\[ v(t) = \frac{dr}{dt}(t) = \frac{dx}{dt}(t)i + \frac{dy}{dt}(t)j + \frac{dz}{dt}(t)k. \]

The magnitude of the velocity is called the speed, and is given by:

\[ |v(t)| = \left| \frac{dr}{dt}(t) \right| = \sqrt{\left( \frac{dx}{dt}(t) \right)^2 + \left( \frac{dy}{dt}(t) \right)^2 + \left( \frac{dz}{dt}(t) \right)^2} = \frac{ds}{dt}(t), \]

where $s$ is the arclength along the curve as measured from some initial point $P$. More precisely, we imagine the components of the curve, $(x(t), y(t), z(t))$ as given, then this is a differential equation whose solution is the arclength along the curve (but you must choose a starting and ending point in order to do the integral).

**Acceleration.** Acceleration of the particle at the point $P$ (or, sometimes called instantaneous acceleration) is the derivative of the velocity at the point $P$, i.e.,

\[ a(t) = \frac{dv}{dt}(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}, \quad \text{if the limit exists.} \]

In cartesian coordinates this is expressed as:

\[ a(t) = \frac{dv}{dt}(t) = \frac{d^2x}{dt^2}(t)i + \frac{d^2y}{dt^2}(t)j + \frac{d^2z}{dt^2}(t)k, \]
and the magnitude of the acceleration is given by:
\[ |a(t)| = \left| \frac{dv}{dt}(t) \right| = \sqrt{\left( \frac{d^2x}{dt^2}(t) \right)^2 + \left( \frac{d^2y}{dt^2}(t) \right)^2 + \left( \frac{d^2z}{dt^2}(t) \right)^2}. \]

A “Moving Coordinate System” Along a Space Curve. We will now construct a coordinate system (i.e., a set of three, orthogonal unit vectors) at each point of a space curve. This can be viewed in two ways. If we imagine a particle moving along the curve then our coordinate system could be viewed as a moving coordinate system, moving with the particle (with the particle always at the origin of the coordinate system). Alternately, we could view the curve as being fixed in space and the coordinate system at each point could be used to describe the shape of the curve. Both points of view occur in many applications. For example, in biology DNA molecules are modeled as curves with a complicated folded structure. In physics and chemistry polymers are also modeled a curves with a complicated spatial structure. In pure mathematics the subject of knot theory is concerned with describing the structure of closed curves in three dimensions.

We begin by considering a curve, which we refer to as \( C \), that is traced out by the vector function \( r = r(t) \), as \( t \) varies through all possible values. We consider a particular point on that curve, called \( P \), see Fig. 2.

**Notation:** Make sure you distinguish vectors (boldface type) from scalars (regular type) in the following. Also, all quantities are defined at a specific point on the curve, i.e., they are functions of \( t \). Remember this, because for the sake of less cumbersome notation we are going to omit the explicit dependence on \( t \).

![Figure 2](image)

First, we define the **unit tangent vector** \( T \), tangent to \( C \) at \( P \). This is defined as:
\[ T = \frac{dr}{ds}, \] (3)

where \( s \) is arclength measured from some initial point on the curve (which is arbitrary). So where does this come from, and why is it of unit length?

First, the velocity, \( v(t) \equiv \frac{dr}{dt}(t) \) is, by definition, tangent to \( C \). Now recall the definition of arclength given in an earlier lecture:

**Key point:**
\[ |v(t)| = \left| \frac{dr}{dt}(t) \right| = \sqrt{\left( \frac{dx}{dt}(t) \right)^2 + \left( \frac{dy}{dt}(t) \right)^2 + \left( \frac{dz}{dt}(t) \right)^2} \equiv \frac{ds}{dt}. \] (4)

A good candidate for a unit tangent vector to \( C \) at \( P \) would be
\[ \frac{v(t)}{|v(t)|}. \] (5)

Now if you remember the chain rule from calculus, it follows from (4) and the definition of velocity, that
Key point:
\[
\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{d\mathbf{r}}{dt} \frac{dt}{ds} = \frac{d\mathbf{r}}{ds} = \mathbf{T},
\]  
(6)

which is the same as (3).

**Technical Point:** There is an issue here that we have side-stepped. What happens if \(v(t) = 0\)? Dividing by zero is (usually) not allowed. In this course we will not explicitly deal with this issue. When you learn more about curves and surfaces you will learn how to understand this point.

Next we define the *unit principal normal* \(\mathbf{N}\) at \(P\). This is defined as:

Key point:
\[
\mathbf{N} \equiv \frac{\mathbf{dT}}{ds} \left/ \frac{d\mathbf{T}}{ds} \right|.
\]  
(7)

Clearly, \(\mathbf{N}\) is of unit length (at least it should be “clear”, or else you have missed something really fundamental and should go back and find out what it is), but it should also be perpendicular to \(\mathbf{T}\). This is (probably) not obvious, so we will prove it. We’ve made the argument (by intimidation) that \(\mathbf{T}\) is of unit length. Therefore,

\[
\mathbf{T} \cdot \mathbf{T} = 1.  
\]  
(8)

Now let’s differentiate this expression with respect to \(s\):

\[
\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} + \frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = 2\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0,
\]  
(9)

from which it is immediate that:

\[
\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} = 0.  
\]  
(10)

From this expression, and the definition of \(\mathbf{N}\) given in (7), it follows that \(\mathbf{N} \cdot \mathbf{T} = 0\). At this point we introduce some notation and terminology. It is standard to define:

Key point:
\[
\left| \frac{d\mathbf{T}}{ds} \right| \equiv \kappa,  
\]  
(11)

to be the *curvature of* \(C\) at \(P\). Moreover,

Key point:
\[
R = \frac{1}{\kappa}  
\]  
(12)

is defined to be the *radius of curvature of* \(C\) at \(P\). With these definitions we can write:

\[
\mathbf{N} \equiv R \frac{d\mathbf{T}}{ds}.
\]  
(13)

We complete the construction of our coordinate system by defining the *unit binormal* \(\mathbf{B}\) to \(C\) at \(P\). To do this, we desire to construct a third unit vector that is perpendicular to the two that we have already constructed. How would we do this? You should already have a good idea. We merely need to take the cross product of the two unit vectors that we have already constructed:
Key point: \[ B = T \times N. \] (14)

Technical Point: Why did we take \( B = T \times N \), rather than \( B = N \times T \)? Because we wanted a right-handed coordinate system. Go back and convince yourself that this is the type of coordinate system we have constructed.

It follows from our construction (see (3) and (6)) that the velocity of a particle on this curve is given by:

Key point: \[ v(t) = vT, \quad \text{where} \quad v \equiv |v|. \] (15)

What is the acceleration of a particle on this curve? We can compute this by differentiating (15):

\[
a = \frac{dv}{dt} = d\left( vT \right) = \frac{dv}{dt}T + v\frac{dT}{dt}.
\] (16)

Now

\[
\frac{dT}{dt} = \frac{dT}{ds} \frac{ds}{dt}.
\] (17)

From (4) it follows that

\[
\frac{ds}{dt} = v,
\] (18)

and from (7) and (11) it follows that:

\[
\frac{dT}{ds} = \kappa N.
\] (19)

Therefore

\[
\frac{dT}{dt} = \kappa N \frac{ds}{dt} = \kappa v N = \frac{vN}{R}.
\] (20)

Putting this all together gives the following expression for the acceleration:

Key point: \[
a = \frac{dv}{dt}T + v \left( \frac{vN}{R} \right) = \frac{dv}{dt}T + \frac{v^2}{R}N,
\] (21)

where the first and second terms on the right hand side are called the tangential acceleration and normal or centripetal acceleration, respectively.

Key point:

- For a space curve, \( r(t) \) (which we can think of as the position vector of a particle) we have defined an orthonormal coordinate system at each point on the curve, with the vectors \( T \), \( N \), and \( B \). These three vectors (generally) vary from point to point on this curve, so we say that \( T \), \( N \), and \( B \) define a moving coordinate system.

- The space curve can be parametrized by \( t \) or \( s \), and the relation between these parameters is given by (4). This is a point that often gives students difficulties, at first. So think about what (4) means.

- The velocity and the acceleration of the particle located by this position vector \( r(t) \) can be described at each point in space defined by \( r(t) \) in terms of the (moving) coordinate system defined by \( T \), \( N \), and \( B \).