Mechanics 1: Week 15 Problem Solutions

1. We have:
   \[ r = \cos \omega t \mathbf{i} + \sin \omega t \mathbf{j}, \]
   from which it follows that:
   \[ v = -\omega \sin \omega t \mathbf{i} + \omega \cos \omega t \mathbf{j}, \]
   and
   \[ a = -\omega^2 \cos \omega t \mathbf{i} - \omega^2 \sin \omega t \mathbf{j}. \]

   (a) This is a trivial calculation.
   (b) \( a = -\omega^2 r \)
   (c) \( r \times v = \omega \mathbf{k}. \)

2. \( m \ddot{r} \cdot \dot{r}. \)

3. \( \nabla V(r) \cdot \dot{r}. \)

4. Use the previous problem:
   \[
   \frac{dV}{dt} = (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \cdot (-\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j} + \mathbf{k}),
   \]
   \[
   = (\cos \omega t \mathbf{i} + \sin \omega t \mathbf{j} + \mathbf{k}) \cdot (-\sin \omega t \mathbf{i} + \cos \omega t \mathbf{j} + \mathbf{k}) = t.
   \]

5. First, you should have verified that the two points are on the curve. Then recall the definition of arclength, \( s \):
   \[
   \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}(t)\right)^2 + \left(\frac{dy}{dt}(t)\right)^2 + \left(\frac{dz}{dt}(t)\right)^2}.
   \]
   So for this problem we have:
   \[
   \text{length} = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2} dt = \frac{\pi}{\sqrt{2}}.
   \]
   It is crucial that you understand the reason for the choice of the limits in the integral.

6. Use the indefinite integral from the previous problem to compute arclength as a function of \( t \):
   \[
   s = \int_0^t ds = \int_0^t \sqrt{2} dt = \sqrt{2} t.
   \]
   (Why were the limits on the integrals chosen as above?) Then we have:
   \[
   \mathbf{r}(t) = \cos \omega t \mathbf{i} + \sin \omega t \mathbf{j} + \mathbf{k} = \cos \frac{1}{\sqrt{2}} \mathbf{i} + \sin \frac{1}{\sqrt{2}} \mathbf{j} + \frac{1}{\sqrt{2}} \mathbf{k} = \mathbf{r}(s) = \mathbf{r}(s).
   \]
   (Make sure you understand what is meant by the four equality signs in the expression above.)
7. \[
\frac{dr}{dt} = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}.
\]

\[
\frac{dr}{ds} = \frac{1}{\sqrt{2}} \left( -\sin \frac{1}{\sqrt{2}} s \mathbf{i} + \cos \frac{1}{\sqrt{2}} s \mathbf{j} + \mathbf{k} \right).
\]

\[
ds = \sqrt{2}.
\]

8. First,
\[
\int_C \mathbf{A} \cdot d\mathbf{r} = \int_C (3x^2 - 6yz)\mathbf{i} + (2y + 3xz)\mathbf{j} + (1 - 4xyz^2)\mathbf{k} \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}),
\]

\[
= \int_C (3x^2 - 6yz)dx + (2y + 3xz)dy + (1 - 4xyz^2)dz.
\]

(a) If \( x = t, y = t^2, z = t^3 \), then the points (0,0,0) and (1,1,1) correspond to \( t = 0 \) and \( t = 1 \), respectively. Then we have
\[
\int_C \mathbf{A} \cdot d\mathbf{r} = \int_{t=0}^{t=1} (3t^2 - 6t^5)dt + (2t^2 + 3t^4)dt(t^2) + (1 - 4t^0)dt(t^2),
\]

\[
= \int_{t=0}^{t=1} (3t^2 - 6t^5)dt + (4t^3 + 6t^5)dt + 3(t^2 - 4t^1)dt,
\]

\[
= \left( t^3 - t^6 + t^4 + t^3 - t^{12} \right)^{1}_{0} = 2.
\]

(b) Along the straight line joining (0,0,0) to (1,1,1) we have \( x = t, y = t, z = t \). Then since \( dx = dy = dz = dt \), we have:
\[
\int_C \mathbf{A} \cdot d\mathbf{r} = \int_C (3x^2 - 6yz)dx + (2y + 3xz)dy + (1 - 4xyz^2)dz,
\]

\[
= \int_{0}^{1} (3t^2 - 6t^2)dt + (2t + 3t^2)dt + (1 - 4t^4)dt,
\]

\[
= \left( -t^3 + t^2 + t^3 - t^4 \right)^{1}_{0} = 6.
\]

9. \[
\frac{\partial \phi}{\partial x} = 3x^2 + y + y \cos xy - \frac{2x}{z} \sin \frac{x^2}{z},
\]

\[
\frac{\partial \phi}{\partial y} = z + x + x \cos xy.
\]

\[
\frac{\partial \phi}{\partial z} = y + \frac{x^2}{z^2} \sin \frac{x^2}{z}.
\]
10. Show that $\nabla \times \mathbf{A} = 0$.

\[
\frac{\partial A_1}{\partial x} = 2y, \quad \frac{\partial A_1}{\partial y} = 2x, \quad \frac{\partial A_1}{\partial z} = 3z^2.
\]

\[
\frac{\partial A_2}{\partial x} = 2x, \quad \frac{\partial A_2}{\partial y} = 2, \quad \frac{\partial A_2}{\partial z} = 0.
\]

\[
\frac{\partial A_3}{\partial x} = 3z^2, \quad \frac{\partial A_3}{\partial y} = 0, \quad \frac{\partial A_3}{\partial z} = 6xz.
\]

Now it is easy to verify that:

\[
\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = 0.
\]

11. First, note the each vector field is the gradient of a scalar valued function, $\mathbf{A} = \nabla V$. Therefore, the line integral of the vector along a path between two points is the difference of the scalar valued function evaluated at the two points.

(a) $V = \sin x \sin y \sin z$. $V(1,1,1) - V(0,0,0) = (\sin 1)^3$.

(b) $V = xyz$. $V(1,1,1) - V(0,0,0) = 1$.

(c) $V = \frac{z^2}{x}$. $V(1,1,1) - V(0,0,0) = \frac{1}{2}$. 

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