Mechanics 1: Week 16 Problem Solutions

1. The starting point is Newton’s equations, which are given by:

\[ m \frac{d^2 x}{dt^2} = F, \quad x(0) = 0, \dot{x}(0) = v_0. \]

We integrate once (with respect to time) to get speed and velocity:

\[ \int_0^t \frac{d}{d\tau} \left( \frac{dx}{d\tau} (\tau) \right) d\tau = \frac{F}{m} \int_0^t d\tau, \]

or

\[ \frac{dx}{dt}(t) = \left( v_0 + \frac{F}{m} t \right) \hat{i}, \]

which gives the velocity as a function of time. The speed is the magnitude of velocity:

\[ \frac{dx}{dt}(t) = v_0 + \frac{F}{m} t. \]  \hspace{1cm} (1)

To get distance we integrate the expression for velocity (with respect to time):

\[ \int_0^t \frac{dx}{d\tau} (\tau) d\tau = \int_0^t \left( v_0 + \frac{F}{m} \tau \right) d\tau, \]

or

\[ x(t) = v_0 t + \frac{F}{2m} t^2. \]  \hspace{1cm} (2)

Finally, we solve for speed as a function of position. Start with (2). This is a quadratic equation for \( t \) that we can solve for \( t \):

\[ t = -\frac{mv_0}{F} \pm \frac{m}{F} \sqrt{v_0^2 + \frac{2F x(t)}{m}}. \]

There are two choices of sign here. Which one do we take? Now \( t \) is positive (we start from \( t = 0 \) and \( t \) increases). The constants \( m, F, \) and \( v_0 \) are all positive, which implies that \( x(t) \) is positive (look at (2)). Therefore for \( t \) positive we must have:

\[ t = -\frac{mv_0}{F} + \frac{m}{F} \sqrt{v_0^2 + \frac{2F x(t)}{m}}. \]

Substituting this into (1) (and writing \( \frac{dx}{dt}(t) = v(t) \)) gives:

\[ v(t) = \sqrt{v_0^2 + \frac{2F x(t)}{m}}, \]  \hspace{1cm} (3)

or

\[ (v(t))^2 = v_0^2 + \frac{2F x(t)}{m}. \]
2. We denote the position vector of the object by \( \mathbf{r} = z \mathbf{k} \). The Newton’s equations become:

\[
md^2z/dt^2 \mathbf{k} = -mg \mathbf{k}, \quad z(0) = 0, \quad \dot{z}(0) = v_0 > 0,
\]

or,

\[
d^2z/dt^2 \mathbf{k} = -g \mathbf{k}, \quad z(0) = 0, \quad \dot{z}(0) = v_0 > 0,
\]

Now these equations are identical to those of the previous problem with

\[
F/m = -g.
\]

So, using (2), we have

\[
z(t) = v_0 t - \frac{g}{2} t^2.
\]

Next we need to compute the time taken to reach the highest point. We must ask ourselves, “what characterizes the highest point”? The object goes up, stops “instantaneously”, and falls back down. So the highest point is reached at the time when the speed vanishes.

Using (1), we have:

\[
\frac{d}{dt}(t) = v_0 - gt.
\]

Setting the left-hand-side of this equation to zero gives:

\[
t = \frac{v_0}{g}.
\]

What is the maximum height? We merely substitute this time into (4) to get:

\[
z_{\text{max}} = \frac{v_0^2}{2g}.
\]

To get the speed as a function of distance from the origin we use (3 to obtain:

\[
v(t) = \sqrt{v_0^2 - 2gz(t)}.
\]

3. First, we write down Newton’s equations:

\[
m \frac{d^2z}{dt^2} \mathbf{k} = -mg \mathbf{k} - \beta \frac{dz}{dt} \mathbf{k}, \quad z(0) = h, \quad \dot{z}(0) = 0,
\]

or

\[
\dot{w} + \frac{\beta}{m} w = -g, \quad w(0) = 0,
\]

where \( w \equiv \frac{dz}{dt} \). As discussed in class, this is a linear, inhomogeneous first order equation for \( w \). We solve for \( w \), then integrate the result to get the height.

To find the general solution of (7), we find a solution to the homogeneous equation:
\[ \dot{w} + \frac{\beta}{m} w = 0, \]

a particular solution to the inhomogeneous equation:

\[ \dot{w} + \frac{\beta}{m} w = -g. \]  \hspace{1cm} (8)

then add the two together, and evaluate the unknown constant in the homogeneous solution by satisfying the initial condition.

The solution to the homogeneous equation is given by:

\[ w(t) = Ce^{-\frac{\beta}{m} t}, \]

where \( C \) is a constant.

Now we need to obtain a particular solution to the inhomogeneous problem. There is a general method for this. But this problem has a particular structure that makes it simple. Look at the right-hand-side of (8). It is a constant. The derivative of a constant is zero. Now look at the left-hand-side of (8). It has a term that is a derivative of \( w \), plus a constant times \( w \). Hence, it follows that we can find a solution of the form \( w = \) constant. In this case:

\[ w_p = -\frac{mg}{\beta}. \]

Then the general solution is:

\[ w(t) = Ce^{-\frac{\beta}{m} t} - \frac{mg}{\beta}. \]

Now \( w(0) = 0 \), so we have:

\[ w(0) = C - \frac{mg}{\beta} = 0, \]

or

\[ C = \frac{mg}{\beta}, \]

and therefore:

\[ w(t) = \frac{mg}{\beta} e^{-\frac{\beta}{m} t} - \frac{mg}{\beta}. \]

or

\[ \frac{dz}{dt} = w(t) = \frac{mg}{\beta} \left( e^{-\frac{\beta}{m} t} - 1 \right). \]

This gives the speed as a function of time. We easily see that there is a limiting speed since:

\[ \lim_{t \to \infty} w(t) = \lim_{t \to \infty} \frac{mg}{\beta} \left( e^{-\frac{\beta}{m} t} - 1 \right) = -\frac{mg}{\beta}. \]
We could quickly get the acceleration as a function of time by differentiating the expression for the velocity as a function of time:

\[ \ddot{z} = -ge^{-\frac{a}{m}t}. \]

To obtain the position as a function of time we integrate the expression for the velocity as a function of time:

\[
\int_0^t \frac{dz}{d\tau}(\tau)d\tau = \int_0^t \left( \frac{mg}{\beta} \left( e^{-\frac{a}{m} \tau} - 1 \right) \right) d\tau,
\]

which gives:

\[ z(t) = h - \frac{mg}{\beta}t - \frac{m^2g}{\beta^2} \left( e^{-\frac{a}{m}t} - 1 \right). \]

4. Substitute the proposed solution into the ODE and see if it indeed satisfies the ODE.

(a) We need to show that:

\[
m \frac{d^2(k_1s_1)}{dt^2} - (a_0 + a_1(t))(k_1s_1) - (b_0 + b_1(t)) \frac{d}{dt}(k_1s_1) = 0.
\]

but this is the same as:

\[ k_1 \left( m \frac{d^2s_1}{dt^2} - (a_0 + a_1(t))s_1 - (b_0 + b_1(t))s_1 \right) = 0. \]

and we know that the expression in parentheses is zero since \( s_1(t) \) is a solution.

(b) We need to show that:

\[
m \frac{d^2(k_1s_1 + k_2s_2)}{dt^2} - (a_0 + a_1(t))(k_1s_1 + k_2s_2) - (b_0 + b_1(t)) \frac{d}{dt}(k_1s_1 + k_2s_2) = 0.
\]

but this is the same as:

\[ k_1 \left( m \frac{d^2s_1}{dt^2} - (a_0 + a_1(t))s_1 - (b_0 + b_1(t))s_1 \right) + k_2 \left( m \frac{d^2s_2}{dt^2} - (a_0 + a_1(t))s_2 - (b_0 + b_1(t))s_2 \right) = 0. \]

and we know that the expressions in parentheses are zero since \( s_1(t) \) and \( s_2(t) \) are solutions.

(c) No.

(d) What do you know? You know that \( \tilde{s}(t) \) is a solution of Newton’s equations satisfying \( \tilde{s}(0) = 12 \) and \( \frac{d\tilde{s}}{dt}(0) = 0 \). What we would like to find is a solution of Newton’s equations, \( \hat{s}(t) \) satisfying \( \hat{s}(0) = 24 \) and \( \frac{d\hat{s}}{dt}(0) = 0 \). It is very tempting to appeal to the linear properties of the equation by setting \( \hat{s}(t) \equiv 2\tilde{s}(t) \), then \( \hat{s}(0) = 2\tilde{s}(0) = 24 \) and \( \frac{d\hat{s}}{dt}(0) = 2\frac{d\tilde{s}}{dt}(0) = 0 \). However, this is not correct since these properties only apply to linear homogeneous equations. So we have:

\[
\hat{s}(t) = \tilde{s}(0) - \frac{1}{2}gt^2
\]

\[
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\]

The term \( \frac{1}{2}gt^2 \) is due to the inhomogeneous term in Newton’s equations.
5. (a) linear,  
   (b) nonlinear,  
   (c) linear,  
   (d) linear,  
   (e) nonlinear.

6. The general solution of Newton’s equation in one dimension for a constant force is:

\[ s(t) = s_0 + v_0(t - t_0) + \frac{F}{2m} (t - t_0)^2. \]

So for this problem we have:

\[ s(t) = s_0 + \frac{g}{2m} t^2. \]

7. The general solution of Newton’s equation in one dimension for a purely time-dependent force is:

\[ s(t) = s_0 + v_0(t - t_0) + \frac{1}{m} \int_{t_0}^{t} \int_{t_0}^{\tau'} F(\tau) d\tau d\tau'. \]

So for this problem we have:

\[ s(t) = s_0 + \frac{1}{m} (t - \sin t). \]

Does this result make sense? The force is bounded, and it’s average value is zero. Yet, according to the solution for the position as a function of time, the particle moves to infinity as \( t \to \infty \).

8. We define:

\[ V(s) = -\int_c^s (s' - s'^2) ds', \]

where \( c \) is a “conveniently chosen” constant. Choosing \( c = 0 \) we have:

\[ V(s) = -\frac{s'^2}{2} + \frac{s^3}{3}. \]

Then we showed that all solutions must satisfy:

\[ \frac{m}{2} \ddot{s}^2 - \frac{s^2}{2} + \frac{s^3}{3} = \text{constant}. \]

What do we mean by “all solutions”? Where are the initial conditions? You will see plenty of this later in the course.

9. With \( t = \sqrt{m} \tau \) we have:

\[ \frac{d}{dt} = \frac{d}{d\tau} \frac{d\tau}{dt} = \frac{1}{\sqrt{m}} \frac{d}{d\tau}, \]

and

\[ \frac{d^2}{dt^2} = \frac{1}{m} \frac{d^2}{d\tau^2} \]

from which the result easily follows.

10. No. This should be a trivial calculation. In general, superposition does NOT hold for nonlinear ODE’s. This is ONE of the major differences. (However, there are certain exceptional situations where nonlinear ODEs can be said to obey a superposition principle.)