Mechanics 1: Week 17 Problem Solutions

1. We did this example in class without friction. In this case, in addition to the forces \( W \) and \( N \) acting on \( P \), there is a frictional force \( f \) directed up the incline (in a direction opposite to the motion) and with magnitude:

\[
\mu N = \mu mg \cos \alpha, \quad \text{or} \\
\mu = \mu mg \cos \alpha e_1.
\]

Using this to modify Newton’s equations that we derived in class, you should readily see that:

\[
m\frac{d^2(se_1)}{dt^2} = W + N + f, \\
= mg \sin \alpha e_1 - \mu mg \cos \alpha e_1.
\]

The acceleration is given by:

\[
\frac{d^2s}{dt^2} e_1 = g(\sin \alpha - \mu \cos \alpha) e_1, \quad (1)
\]

where, recall, \( s \) is the distance from the top of the incline. It should be clear that we must have \( \sin \alpha > \mu \cos \alpha \) or the frictional force is so great that the particle does not move at all.

Next we compute the velocity. Replacing \( \frac{d^2s}{dt^2} \) by \( \frac{dv}{dt} \) in (1), using the fact that the particle starts from rest (i.e., \( v(0) = 0 \)), and integrating from 0 to \( t \) gives the velocity:

\[
v e_1 = g(\sin \alpha - \mu \cos \alpha) t e_1.
\]

Finally, we compute the distance traveled after time \( t \). Replacing \( v \) with \( \frac{ds}{dt} \) in the above equation, using \( s(0) = 0 \), and integrating from 0 to \( t \) gives:

\[
s = \frac{g}{2}(\sin \alpha - \mu \cos \alpha) t^2,
\]

where we have dropped \( e_1 \) since we are only interested in displacement.

2. In class, we solved for the motion of the projectile in the absence of the incline. We found that the position vector at any time \( t \) was given by:

\[
r = (v_0 \cos \beta) t \mathbf{j} + \left( (v_0 \sin \beta) t - \frac{g}{2} t^2 \right) \mathbf{k},
\]

or, in components,

\[
y = (v_0 \cos \beta) t, \quad z = (v_0 \sin \beta) t - \frac{g}{2} t^2. \quad (2)
\]

The equation for the incline (which is a line in the \( y - z \) plane) is given by:

\[
z = y \tan \alpha. \quad (3)
\]

Substituting (2) into (3), it follows that the projectiles path and the incline intersect at those values of \( t \) for which:
\[(v_0 \sin \beta) t - \frac{g t^2}{2} = ((v_0 \cos \beta) t) \tan \alpha,\]
i.e.,
\[
t = 0, \quad \text{or} \quad t = \frac{2v_0 (\sin \beta \cos \alpha - \cos \beta \sin \alpha)}{g \cos \alpha} = \frac{2v_0 \sin (\beta - \alpha)}{g \cos \alpha}.\]
The value \(t = 0\) gives the intersection point \(O\). The second value of \(t\) gives point \(A\), which is the required point. Using this value of \(t\) in the first equation of (2), the range of the projectile up the incline is given by:
\[
R = y \sec \alpha = (v_0 \cos \beta) \left(\frac{2v_0 \sin (\beta - \alpha)}{g \cos \alpha}\right) \sec \alpha = \frac{2v_0^2 \sin (\beta - \alpha) \cos \beta}{g \cos^2 \alpha}.
\]
3. Three forces are acting on the object: the weight, \(\mathbf{W} = -mg \mathbf{k}\), the normal force \(\mathbf{N}\) of the surface on the object, and the frictional force \(\mathbf{f}\). Hence, Newton’s equations have the form:
\[
m \frac{dv}{dt} = \mathbf{W} + \mathbf{N} + \mathbf{f}.
\]
But \(\mathbf{N} = -\mathbf{W}\), and the magnitude of \(\mathbf{f}\) is \(f = \mu N = \mu mg\) so that \(\mathbf{f} = -\mu mg \mathbf{i}\). Then Newton’s equations are written as:
\[
m \frac{dv}{dt} = -\mu mg \mathbf{i}, \quad \text{or} \quad \frac{dv}{dt} = -\mu g. \quad (4)
\]
Integrating this equation, and using \(v = v_0\) at \(t = 0\) gives:
\[
v = v_0 - \mu gt, \quad \text{or} \quad \frac{dx}{dt} = v_0 - \mu gt. \quad (5)
\]
Integrating again, using \(x = 0\) at \(t = 0\) gives:
\[
x = v_0 t - \frac{1}{2} \mu gt^2. \quad (6)
\]
From (5), we see that the object comes to rest (i.e. \(v = 0\)) when:
\[
v_0 - \mu gt = 0 \quad \text{or} \quad t = \frac{v_0}{\mu g}.
\]
Substituting this time into (6), and noting that \(x = x_0\) at this time gives:
\[
x_0 = \frac{v_0^2}{\mu g} - \frac{1}{2} \mu g \left(\frac{v_0}{\mu g}\right)^2,
\]
or
\[
\mu = \frac{v_0^2}{2gx_0).
\]
4. \(z(t)\) negative is a perfectly valid solution of the differential equation governing the dynamics of the projectile. However, a difficulty arises if we want to use the differential equation to model a particular physical situation. For example, if \(z = 0\) is the ground (i.e. the “flat Earth”) then we cannot consider situations in which \(z(t)\) becomes negative.
5. Using the expression for the position of \( z \) as a function of time from the first example of the Week 18 Notes, we have:

\[-H = (v_0 \sin \alpha)t - \frac{g}{2}t^2,\]

or

\[t^2 - \frac{2v_0 \sin \alpha}{g}t + \frac{2H}{g} = 0.\]

Solving this quadratic equation for \( t \) gives:

\[t = \frac{v_0 \sin \alpha}{g} \pm \frac{1}{2} \sqrt{\frac{4v_0^2 \sin^2 \alpha}{g^2} + \frac{8H}{g}}.\]

Of the two roots, we take the "plus sign" since the other is negative (and the minus sign, which is perfectly valid from the point of view of the differential equation, is not valid for the physical situation we are modelling):

\[t = \frac{v_0 \sin \alpha}{g} + \sqrt{\frac{4v_0^2 \sin^2 \alpha}{g^2} + \frac{2H}{g}}.\]

Now there is a detail we need to check. If the projectile is to go over the "side of the cliff" (and therefore hit the bottom at \( z = -H \)), the horizontal distance that it travels must be larger than \( d \).

6. (a) \( t = \frac{d}{v_0 \cos \alpha} \).

(b) The height that it reaches after this time is:

\[T = d \tan \alpha - \frac{gd^2}{2v_0^2 \cos^2 \alpha}.\]

For the correct physical interpretation, the right hand side of this expression must be positive, i.e., we must have

\[\tan \alpha > \frac{gd}{2v_0^2 \cos^2 \alpha},\]

or

\[d < \frac{2v_0^2 \sin \alpha \cos \alpha}{g}.\]

(c) The equation to solve for \( d \) is:

\[T = d \tan \alpha - \frac{gd^2}{2v_0^2 \cos^2 \alpha}.\]

Using the values of the parameters given in the statement of the problem, we have:

\[10 = d - 0.00098d^2,\]

There are two possible values:

\[d = 10.2m \quad \text{and} \quad d = 1010.308m.\]

Does it make sense for there to be two possible values of \( d \)? If you think about the shape of the path of the projectile (a parabola) it does make sense, but only if the two values for \( d \) are smaller than the range (without the wall being present). Using the parameters given, and the formula for the range from the lecture notes, we compute that \( R = 1020.4m.\)