Mechanics 1: Week 18 Problem Solutions

1. \( \mathbf{F} \) and the displacement, \( \mathbf{r} \) would be proportional, i.e. lie along the same line, if \( \mathbf{r} \) and \( \frac{d^2 \mathbf{r}}{dt^2} \) were proportional. However, we know that this is not generally the case (although it could be true in special cases, see problem 4 below).

2. A force of this particular form does no work since it is perpendicular to the velocity (think about this in the context of the question above).

3. An issue with both of these questions is how to translate "common language" into mathematical formulae.

   (a) We have proven that the work done by the net forces acting on a particle of constant mass \( m \) in moving a particle from a point \( P_1 \) to a point \( P_2 \) is the kinetic energy of the particle at \( P_2 \) minus the kinetic energy of the particle at \( P_1 \).

   (b) If we equate motion to nonzero velocity then if there is no motion, there is no velocity (of the particle), and therefore it has no kinetic energy, and therefore no change in kinetic energy is possible.

4. (a) The position vector is:

   \[
   \mathbf{r} = x \mathbf{i} + y \mathbf{j} = a \cos \omega t \mathbf{i} + b \sin \omega t \mathbf{j},
   \]

   or

   \[
   x = a \cos \omega t, \quad y = b \sin \omega t,
   \]

   which are just the parametric equations of an ellipse having semi-major axis of length \( a \) and semi-minor axis of length \( b \). Alternately, since

   \[
   \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = \cos^2 \omega t + \sin^2 \omega t = 1,
   \]

   we also obtain the "other" equation for an ellipse that we usually learn:

   \[
   \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
   \]

   (b) Assuming that the particle has constant mass, the force acting on it is:

   \[
   \mathbf{F} = m \frac{d^2 \mathbf{r}}{dt^2} = m \frac{d^2}{dt^2} (a \cos \omega t \mathbf{i} + b \sin \omega t \mathbf{j}),
   \]

   \[
   = m (-\omega^2 a \cos \omega t \mathbf{i} - \omega^2 b \sin \omega t \mathbf{j}),
   \]

   \[
   = -m \omega^2 (a \cos \omega t \mathbf{i} + b \sin \omega t \mathbf{j}) = -m \omega^2 \mathbf{r},
   \]

   from which it follows immediately that the force is directed towards the origin.

   (c) The velocity is given by:

   \[
   \mathbf{v} = -\omega a \sin \omega t \mathbf{i} + \omega b \cos \omega t \mathbf{j}.
   \]

   Therefore the kinetic energy is given by:

   \[
   \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m \left( \omega^2 a^2 \sin^2 \omega t + \omega^2 b^2 \cos^2 \omega t \right).
   \]

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So we have:

Kinetic energy at A (where \( \cos \omega t = 1, \sin \omega t = 0 \)) \( = \frac{1}{2}m\omega^2b^2 \).

Kinetic energy at B (where \( \cos \omega t = 0, \sin \omega t = 1 \)) \( = \frac{1}{2}m\omega^2a^2 \).

(d) Work done \( = \int_A^B \mathbf{F} \cdot d\mathbf{r} \),

\[ = \int_0^{\frac{\pi}{2}} (-m\omega^2 (a \cos \omega t \mathbf{i} + b \sin \omega t \mathbf{j})) \cdot (-\omega a \sin \omega t \mathbf{i} + \omega b \cos \omega t \mathbf{j}) \, dt, \]

\[ = \int_0^{\frac{\pi}{2}} m\omega^3(a^2 - b^2) \sin \omega t \cos \omega t \, dt, \]

\[ = \frac{1}{2}m\omega^2(a^2 - b^2) \sin^2 \omega t \bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}m\omega^2(a^2 - b^2). \]

(e) Using the previous two results:

Work done \( = \frac{1}{2}m\omega^2(a^2 - b^2) = \frac{1}{2}m\omega^2a^2 - \frac{1}{2}m\omega^2b^2 \),

\[ = \text{kinetic energy at B} - \text{kinetic energy at A}. \]

(f) Using the result above from (d), in making a complete circuit around the ellipse we go from \( t = 0 \) to \( t = \frac{2\pi}{\omega} \). Therefore:

Work done \( = \int_0^{\frac{2\pi}{\omega}} m\omega^3(a^2 - b^2) \sin \omega t \cos \omega t \, dt, \)

\[ = \frac{1}{2}m\omega^2(a^2 - b^2) \sin^2 \omega t \bigg|_0^{\frac{2\pi}{\omega}} = 0. \]

(g) The force was obtained in b). A direct calculation shows that \( \nabla \times \mathbf{F} = 0 \).

(h) Since the force is conservative there exists a potential \( V \) such that:

\[ \mathbf{F} = -m\omega^2x \mathbf{i} - m\omega^2y \mathbf{j} = -\nabla V = -\frac{\partial V}{\partial x} \mathbf{i} - \frac{\partial V}{\partial y} \mathbf{j} - \frac{\partial V}{\partial z} \mathbf{k}. \]

Then we have:

\[ m\omega^2x = \frac{\partial V}{\partial x}, \quad m\omega^2y = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial z} = 0. \]

Solving these equations (and setting the integration constant to zero) gives the potential:

\[ V = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m\omega^2y^2 = \frac{1}{2}m\omega^2(x^2 + y^2) = \frac{1}{2}m\omega^2r^2. \]
(i)

Potential at point A (where \( r=a \)) = \( \frac{1}{2} m \omega^2 a^2 \),
Potential at point B (where \( r=b \)) = \( \frac{1}{2} m \omega^2 b^2 \).

Then we have

\[
\text{Work done from A to B} = \text{Potential at A} - \text{Potential at B},
\]

\[
= \frac{1}{2} m \omega^2 a^2 - \frac{1}{2} m \omega^2 b^2,
\]

which agrees with the result obtained in d).

5. First we collect together the relevant results from the example that have already been computed in the notes.

\[
F = -mgk,
\]
\[
v = v_0 \cos \alpha j + (v_0 \sin \alpha - gt)k,
\]
\[
r = (v_0 \cos \alpha) t j + \left( (v_0 \sin \alpha) t - \frac{1}{2} gt^2 \right) k,
\]

and, if the particle is launched at \( t = 0 \), the time required for the particle to reach its highest point is:

\[
t_h = \frac{v_0 \sin \alpha}{g}.
\]

(a) We have:

\[
\int_0^{t_h} F \cdot dr = \int_0^{t_h} F \cdot \frac{dr}{dt} dt = \int_0^{t_h} -mg(v_0 \sin \alpha - gt)dt,
\]

\[
= -\left( mg \sin \alpha \right) t + \left. \left( \frac{m g^2}{2} t^2 \right) \right|_0^{v_0 \sin \alpha},
\]

\[
= -\frac{m}{2} v_0^2 \sin^2 \alpha.
\]

(b) Let \( P_1 \) denote the point where the projectile is launched (the origin) and \( P_2 \) denote the highest point of the projectile. Then we have:

\[
T_{P_2} - T_{P_1} = \frac{1}{2} m v_0^2 \cos^2 \alpha - \frac{1}{2} m v_0^2 = -\frac{1}{2} m v_0^2 \sin^2 \alpha.
\]

6. First we collect together the relevant results from the Week 17 Problem Solutions (Problem 1):

\[
F = (mg \sin \alpha - \mu mg \cos \alpha) e_1,
\]
\[
v = (g \sin \alpha - \mu g \cos \alpha) t e_1,
\]
\[
s = \frac{g}{2} (\sin \alpha - \mu \cos \alpha) t^2,
\]
The particle starts at rest from the top of the incline. If the incline is of length $L$, then the time to reach the bottom is obtained by solving:

$$L = \frac{g}{2} (\sin \alpha - \mu \cos \alpha) t^2,$$

or

$$t_b = \sqrt{\frac{2L}{g(\sin \alpha - \mu \cos \alpha)}}.$$

(a) We have:

$$\int_0^{t_b} \mathbf{F} \cdot d\mathbf{r} = \int_0^{t_b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt,$$

$$= \int_0^{t_b} mg^2 (\sin \alpha - \mu \cos \alpha)^2 t dt,$$

$$= \frac{mg^2}{2} (\sin \alpha - \mu \cos \alpha)^2 \left( \sqrt{\frac{2L}{g(\sin \alpha - \mu \cos \alpha)}} \right)^2,$$

$$= mg (\sin \alpha - \mu \cos \alpha) L.$$

(b) The particle starts from rest at the top of the incline, so at $t = 0$ we have $T_{top} = 0$. At the bottom of the incline the velocity is given by:

$$\mathbf{v}_b = (g \sin \alpha - \mu g \cos \alpha) \sqrt{\frac{2L}{g(\sin \alpha - \mu \cos \alpha)}} \mathbf{e}_1.$$

Then the kinetic energy at the bottom of the incline, $T_b$, is given by:

$$T_b = \frac{1}{2} m \left( (g \sin \alpha - \mu g \cos \alpha) \sqrt{\frac{2L}{g(\sin \alpha - \mu \cos \alpha)}} \right)^2 = mg (\sin \alpha - \mu \cos \alpha) L.$$