Mechanics 1: Week 20 Problems

Marked problems 1, 4 & 6 below.

1. A **central force** is a force having the following form:

\[ \mathbf{F} = f(r) \hat{r} = f(r) \frac{\mathbf{r}}{r}, \]

where \(f(r)\) is an arbitrary function of the magnitude of the position vector. Show that angular momentum about the origin is conserved for a particle of constant mass \(m\) moving under the influence of a central force field.

2. Show that if a particle moves under the influence of a central force field, then its path must always lie in a plane. (Hint. A plane is defined by a constant vector. It then suffices to show that the position vector is perpendicular to an appropriately chosen constant vector. Use the previous problem to choose this constant vector.)

3. Prove that in cartesian coordinates the magnitude of the areal velocity is \( \frac{1}{2} (x \dot{y} - y \dot{x}) \).

4. Derive the equation:

\[ \frac{d^2 r}{d\theta^2} - \frac{2}{r} \left( \frac{dr}{d\theta} \right)^2 - r = \frac{r^4 f(r)}{mh^2}. \]

5. Show that the position of a particle as a function of time moving in a central force field can be determined from the equations:

\[ t = \int \frac{1}{\sqrt{G(r)}} dr, \quad t = \frac{1}{\hbar} \int r^2 d\theta, \]

where

\[ G(r) = \frac{2E}{m} + \frac{2}{m} \int f(r)dr - \frac{2h^2}{m^2 r^2}. \]

6. (a) Find the potential energy for a particle which moves in the force field:

\[ \mathbf{F} = -\frac{K}{r^2} \hat{r}, \]

where \(K\) is some positive constant.

(b) How much work is done by the force field in moving a particle from a point on the circle of radius \(r = a > 0\) to a point on the circle of radius \(r = b > 0\)? Does the work depend on the path?

7. Consider the relation:

\[ r^2 \dot{\theta} = \hbar = \text{constant}, \]

that we derived in class. Explain how it enables us to determine the \(\theta\) component of motion if we know the \(r\) component of motion.

8. How is the expression \(r^2 \dot{\theta}\) related to the angular momentum of the particle about \(O\)?