Mechanics 1: Week 20 Problem Solutions

1. The time rate of change of the angular momentum about the origin is given by the torque about the origin. Therefore we only need to show that the torque about the origin is zero. This is a trivial computation:

\[ A = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times f(r) \frac{\mathbf{r}}{r} = \frac{f(r)}{r} (\mathbf{r} \times \mathbf{r}) = 0. \]

2. It follows from the previous problem that:

\[ \mathbf{r} \times \mathbf{F} = 0, \]

and therefore

\[ \mathbf{r} \times m \frac{d\mathbf{v}}{dt} = 0, \]

or

\[ \mathbf{r} \times \frac{d\mathbf{v}}{dt} = 0, \]

which is the same as (why?)

\[ \frac{d}{dt} (\mathbf{r} \times \mathbf{v}) = 0. \]

Integrating this equation with respect to time gives:

\[ \mathbf{r} \times \mathbf{v} = \mathbf{h}, \quad (1) \]

where \( \mathbf{h} \) is a constant vector. Now \( \mathbf{r} \times \mathbf{v} \) is perpendicular to \( \mathbf{r} \) (why?). Therefore taking the dot product of both sides of (1) with \( \mathbf{r} \) gives:

\[ \mathbf{r} \cdot (\mathbf{r} \times \mathbf{v}) = 0 = \mathbf{r} \cdot \mathbf{h}. \]

Therefore the position vector is always perpendicular to the constant vector \( \mathbf{h} \), so that the motion is always in a plane.

3. The magnitude of the areal velocity is given by \( \frac{1}{2} |\mathbf{r} \times \mathbf{v}|. \) Hence, we need to compute \( \mathbf{r} \times \mathbf{v} \) in cartesian coordinates, and then compute the magnitude of the resulting vector.

\[ \mathbf{r} \times \mathbf{v} = (x \mathbf{i} + y \mathbf{j}) \times (\dot{x} \mathbf{i} + \dot{y} \mathbf{j}) = x \dot{y} \mathbf{k} - y \dot{x} \mathbf{k}. \]

Then

\[ |\mathbf{r} \times \mathbf{v}| = x \dot{y} - y \dot{x}. \]

4. We start with the equation derived in class:

\[ \ddot{\mathbf{r}} - \frac{h^2}{r^3} = \frac{f(r)}{m}. \quad (2) \]

We need two preliminary relations. From \( r^2 \dot{\theta} = h \) we have:
\[ \dot{\theta} = \frac{h}{r^2}. \]  

Differentiating \( r^2 \dot{\theta} = h \) with respect to \( t \) gives:

\[ 2r \ddot{r} + r^2 \ddot{\theta} = 0, \]

or

\[ \ddot{\theta} = -\frac{2 \dot{\theta}}{r} \frac{\dot{r}}{r} = -\frac{2h \dot{r}}{r^3}. \]  

Now we use the chain rule:

\[ \frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \dot{\theta} \frac{dr}{d\theta} = \frac{h r}{r^2} \frac{dr}{d\theta}. \]  

\[ \frac{d^2r}{dt^2} = \left( \frac{d}{dt} \left( \frac{dr}{d\theta} \right) \right) \dot{\theta} + \frac{dr}{d\theta} \ddot{\theta}, \]

\[ = \frac{d^2r}{d\theta^2} \ddot{\theta} + \frac{dr}{d\theta} \ddot{\theta}, \]

\[ = \frac{h^2 d^2r}{r^4 d\theta^2} + \frac{2h^2}{r^5} \left( \frac{dr}{d\theta} \right)^2, \]  

where we have used (3), (4) and (5).

Now substituting (6) into (2) gives the result.

5. This result uses conservation of energy. From class we derived the following equation that expresses conservation of energy for a particle moving in a central force field:

\[ \frac{1}{2} m \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) - \int f(r) dr = E. \]

Substituting \( \dot{\theta} = \frac{h}{r^2} \) into this equation gives:

\[ \frac{1}{2} m \left( \dot{r}^2 + \frac{h^2}{r^2} \right) - \int f(r) dr = E, \]

or

\[ \dot{r}^2 = \frac{2E}{m} + \frac{2}{m} \int f(r) dr - \frac{2h^2}{mr^2} \equiv G(r). \]

From this expression we obtain:

\[ \frac{dr}{dt} = \sqrt{G(r)}, \]

or

\[ t = \int \frac{1}{\sqrt{G(r)}} dr. \]

The second equation follows by writing \( \dot{\theta} = \frac{h}{r^2} \) as:

\[ dt = \frac{1}{h} \dot{r}^2 d\theta. \]
6. (a) The potential is given by:

\[ V(r) = \int \frac{K}{r^2} dr = -\frac{K}{r}. \]

(b) The work done is given by:

\[ V(r = a) - V(r = b) = \frac{K}{b} - \frac{K}{a} = \frac{K(a - b)}{ab}. \]

7. From

\[ r^2 \dot{\theta} = h = \text{constant}, \]

we derive the quadrature:

\[ \int d\theta = h \int \frac{dt}{r(t)^{\frac{1}{2}}}. \]

8. In the lectures we showed that:

\[ \mathbf{r} \times \mathbf{v} = r^2 \dot{\theta} \mathbf{k}. \]

Hence, \( mr^2 \dot{\theta} \mathbf{k} \) is the angular momentum of the particle about \( O \).