COUNTING RATIONAL POINTS ON CUBIC HYPERSURFACES: CORRIGENDUM

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There is an error in [1] which invalidates the proof of the main theorem from [1] and also the proof of Lemma 11 from [2]. In attempting to apply Proposition 3 in [1, §5], it is claimed that

\[
\sum_{R_0 < b_1 \leq 2R_0} M_1 \ll R_0^{-1/2} \sum_{R_0 < b_1 \leq 2R_0} \max_{0 < N \ll (HP)^{\theta}} \gcd(b_1, N)^{1/2} \\
\ll R_0^{-1/2} \sum_{R_0 < b_1 \leq 2R_0} \max_{0 < N \ll (HP)^{\theta}} \gcd(b_1, N)^{1/2} \\
\ll R_0^{1/2} (HP)^{\varepsilon}.
\]

The second line is false and in fact one has \( M_1 = 1 \) in Proposition 3. The author is very grateful to Professor Hongze Li for drawing his attention to this flaw.

The error can be fixed by introducing an average over \( b_1 \) into the statement of Proposition 3. This allows us to recover the main theorem in [1], and also [2, Lemma 11], via the following modification.

**Proposition 3.** Let \( w \in W_n \), let \( \varepsilon > 0 \) and let \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) be a cubic polynomial such that \( g_0 \) is non-singular and \( \|g\|_P \leq H \), for some \( H \leq P \). Let \( \tilde{q} = b_2^2 c^2 d \), where

\[
b_2 := \prod_{p \nmid \tilde{q}^2} p, \quad d := \prod_{\rho \mid \tilde{q}, e \geq 3, 2 \nmid e} p,
\]

and let \( R_0 \geq 1/2 \). Define

\[
V := R_0 \tilde{q} P^{-1} \max \{1, \sqrt{|z| P^3}\}, \quad (4.2)
\]

and

\[
W := V + (e^2 d)^{1/3}, \quad (4.3)
\]

Then there exists a positive number \( \theta \) such that

\[
\sum_{R_0 < b_1 \leq 2R_0} |S_u(b_1 \tilde{q}; z)| \ll H^\theta (R_0 \tilde{q}^{-n/2 + 1} P^{n+\varepsilon} \\
\times (W^n \tilde{M}_1 + R_0 \min\{M_2, M_3\})),
\]

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where
\[ M_1 := \min \left\{ R_0, \frac{P^{3/4}}{q^{1/2}} \right\} \]
and
\[ M_2 := c^n \left( 1 + \frac{V}{c} \right)^{n-3/2}, \quad M_3 := V^n \left( 1 + \frac{c^2d}{V^3} \right)^{n/2}. \]

In order to prove this result we will need a new technical lemma, which allows us to separate variables at a crucial point in our argument.

**Lemma A.** Let \( h \in \mathbb{R}^n \), let \( M, N > 0 \) and let \( f(m; n) \geq 0 \) for every \( m \in \mathbb{N} \) and \( n \in \mathbb{Z}^n \). Then we have
\[
\sum_{M < m \leq 2M} \sum_{n \in \mathbb{Z}^n \atop |n - mh| \leq N} f(m; n) \leq \sum_{1 \leq I \leq L} \sum_{n \in \mathbb{Z}^n \atop |n - M_i h| \leq 2N} \sum_{M < m \leq 2M} f(m; n),
\]
for appropriate \( M_i \in (M, 2M) \), where \( L = M \min \{1, |h|/N\} + 1 \).

**Proof.** We break the outer sum into smaller intervals of length \( U \geq 1 \), writing
\[
(M, 2M) = \bigcup_{1 \leq i \leq M/U+1} (M_i, M_{i+1}],
\]
with \( M_i = M + (i - 1)U \). We will take \( U \) to be maximal so that \( U \geq 1 \) and \( |h|U \leq N \). Let \( m \in (M_i, M_{i+1}] \) and note that
\[
N \geq |n - mh| = |n - M_i h + M_i h - mh| \geq ||n - M_i h| - (m - M_i)|h||.
\]
Since \( m - M_i \leq M_{i+1} - M_i = U \), we see that the overall contribution to the left-hand side from such \( m \) is at most
\[
\sum_{M_i < m \leq M_{i+1}} \sum_{n \in \mathbb{Z}^n \atop |n - M_i h| \leq 2N} f(m; n).
\]
We conclude the proof on enlarging the outer sum to all \( m \in (M, 2M) \) and interchanging it with the sum over \( n \). \( \square \)

**Proof of Proposition 3.** We adopt the equation numbering from [1] and write \( B \) for the set of square-free integers \( b_1 \in (R_0, 2R_0] \). For given \( b_1 \in B \) we write \( q = b_1 \hat{q} \) and \( b = b_1 b_2^2 \). Our chief difficulty in introducing averaging over \( b_1 \) will be that we can no longer merely take a maximum over \( v_0 \ll H P \) in (4.5) in every case. We begin, using (4.5) and (4.11), by noting that
\[
S_u(q; z) \ll P^{-N} + q^{-n} \int_{x \ll P} \left| \sum_{|v - qz \nabla g(x)| \ll P^\epsilon} S_u(q; v) \right| \, dx,
\]
where
\[
S_u(q; v) \ll H^{\theta} b^{(n+1)/2+\epsilon} b_2 \gcd(b_1, u, g^*(v))^{1/2}
\]
\[ \times \max_{\tilde{b} \in (\mathbb{Z}/c^2d\mathbb{Z})^*} |S_{\tilde{u} b^2}(c^2d; \tilde{b}v)|. \] \( (*) \)
Let $S_2(v_0)$ be the overall contribution obtained by taking $u = 0$ and summing the right-hand side of (*) over $|v - v_0| \leq P^\varepsilon V$ for which $g^*(v) = 0$. Then

$$
\sum_{b_1 \in B} q^{-n} \int_{x \ll P} S_2(qz \nabla g(x)) \, dx \ll \sum_{b_1 \in B} q^{-n} P^n \max_{v_0 \ll HP} S_2(v_0).
$$

But the treatment of $S_2(v_0)$, which is uniform in $v_0$, is correct and leads via (4.15)–(4.16) to

$$
\sum_{b_1 \in B} q^{-n} \int_{x \ll P} S_2(qz \nabla g(x)) \, dx \ll H^\theta R_0(R_0q)^{-n/2+1} P^{n+\varepsilon} \min\{M_2, M_3\},
$$

the effect of the sum over $b_1$ being merely to multiply the bound by $R_0$.

Interchanging the sum over $b_1$ and the integral over $x$, we are now led to examine

$$
J = \sum_{b_1 \in B} S_1(b_1 \tilde{q}_z \nabla g(x)),
$$

for given $x \ll P$, where for given $v_0 \in \mathbb{R}^n$, we denote by $S_1(v_0)$ the overall contribution from summing (*) over $|v - v_0| \leq P^\varepsilon V$ for which

$$(u, g^*(v)) \neq (0, 0).$$

We will produce two bounds for $J$. The first arises from taking

$$
gcd(b_1, u, g^*(v)) \ll b_1
$$

in the existing argument and summing trivially over $b_1$. This leads to the estimate

$$
J \ll H^\theta R_0(R_0q)^{n/2+1+\varepsilon} W^n. \tag{**}
$$

To deduce an alternative estimate we first analyze

$$
J(v_0) = \sum_{b_1 \in B} S_1(v_0)
$$

$$
\ll H^\theta R_0^{(n+1)/2+\varepsilon} b_2^{n+2+\varepsilon} \sum_{|v - v_0| \leq P^\varepsilon V \atop (u, g^*(v)) \neq (0, 0)} \max_{b \in (\mathbb{Z}/c^2d\mathbb{Z})^*} |S_{ub^2}(c^2d; \tilde{b}v)|
$$

$$
\times \sum_{b_1 \in B} \gcd(b_1, u, g^*(v))^{1/2},
$$

for fixed $v_0 \in \mathbb{R}^n$. The inner sum over $b_1$ is $O(R_0P^\varepsilon)$, by the third displayed equation on page 107 of [1], whence

$$
J(v_0) \ll H^\theta R_0^{(n+3)/2} b_2^{n+2} P^\varepsilon \sum_{|v - v_0| \leq P^\varepsilon V} \max_{\tilde{b} \in (\mathbb{Z}/c^2d\mathbb{Z})^*} \sum_{a \mod c^2d \atop \gcd(a, c^2d) = 1} |T(a, c^2d; \tilde{b}v)|,
$$

where $T(a, c^2d; \tilde{b}v)$ is given by (4.6). The path is now clear for the final bound

$$
J(v_0) \ll H^\theta R_0^{1/2} (R_0q)^{n/2+1+\varepsilon} W^n,
$$

where
which is obtained by combining [3, Lemmas 11, 15 and 16] in the manner indicated at the close of [3, §5]. In particular, this bound is uniform in \( v_0 \).

Returning to the estimation of \( J \) we apply Lemma A with

\[
M = R_0, \quad N = P^\varepsilon V, \quad h = \tilde{q}z \nabla g(x),
\]

which leads to the bound

\[
J \ll \min\left\{ \frac{R_0 \tilde{q}|z|HP^2}{V} \right\} \max_{v_0 \ll HP} J(v_0)
\]

\[
\ll H \sqrt{\frac{P^{3/2}}{R_0 \tilde{q}}} \max_{v_0 \ll HP} J(v_0),
\]

since

\[
\frac{R_0 \tilde{q}|z|P^2}{V} = \frac{|z|P^3}{\max\{1, \sqrt{|z|P^5}\}} \leq \sqrt{|z|P^3} \leq \sqrt{\frac{P^{3/2}}{R_0 \tilde{q}}},
\]

by (3.2). Drawing our argument together with (**), this therefore shows that

\[
(R_0 \tilde{q})^{-n} \int_{x \ll P} \sum_{b_1 \in B} S_1(q \tilde{z} \nabla g(x)) \, dx \ll H^\theta (R_0 \tilde{q})^{-n/2+1} P^{n+\varepsilon} W^n \tilde{M}_1,
\]

which concludes our proof of the proposition. \( \Box \)

It remains to show that our modified Proposition 3 suffices to prove [1, Proposition 1] and [2, Lemma 11].

**Proof of Proposition 1.** Let us adopt the equation and page numbering from [1]. We begin as in §5, with the aim of showing (5.2) for \( i = 1, 2 \), under the assumption that \( n \geq 5 \) and \( s(g_0) = -1 \). We supplant Lemma 3 with the modified bound

\[
\#(\tilde{q} = b_2^2 c^2 d : (5.1) holds) \ll R_1 R_2^{1/2} R_3^{1/2}.
\]

The estimation of \( \Sigma_2(R, R; t) = \Sigma_2(R, R) \) in §5.1 begins with (5.5), the estimation of \( \Sigma_{2,b} \) running through unchanged. On the other hand, we now have

\[
\Sigma_{2,a} \ll H^\theta P^{n-3+\varepsilon} M \sum_{\tilde{q}} R_0^{1/2} R^{1-n/2} \max_{|z| \gg (RQ)^{-1}} W^n,
\]

where

\[
M = \min\left\{ R_0^{1/2}, \frac{P^{3/4}}{R^{1/2}} \right\}
\]

and the summation over \( \tilde{q} \) is over all \( \tilde{q} = b_2^2 c^2 d \) such that \( b_2, c, d \) are constrained to lie in the dyadic ranges (5.1). Hence

\[
\Sigma_{2,a} \ll H^\theta \frac{P^{n-3+\varepsilon}}{R^{n/2-3/2}} M (R^{1/2} P^{-1/4} + (R_2 R_3)^{1/3})^n. \tag{***}
\]
This is the same bound for $\Sigma_{2,a}$ that features in the middle of page 107, except that we have an additional factor $\mathcal{M}$. The term involving $R^{1/2}P^{-1/4}$ is now found to contribute

$$\ll H^{0}P^{3n/4-3/4+\varepsilon} R \ll H^{0}P^{3n/4-3/4+\varepsilon},$$

since $R \leq P^{3/2}$, whereas the term involving $(R_2^2R_3)^{1/3}$ contributes

$$\ll H^{0}P^{n-3+3/4+\varepsilon} (R_2^2R_3)^{n/3} R^{n/2-1} \ll H^{0}P^{n-3+3/4+\varepsilon} R^{1-n/6},$$

since $R_2^2R_3 \ll R$. Both of these are satisfactory, concluding the proof of (5.7).

We now turn to the treatment of $\Sigma_1(R; R; t)$ in §5.2, with the estimation of $\Sigma_{1,b}$ running through unchanged. On the other hand, we now have

$$\Sigma_{1,a} \ll H^{0}P^{n+\varepsilon} t \mathcal{M} \left( \frac{R^{3/2-n/2}(V + (R_2^2R_3)^{1/3})^n}{R_2^{1/2}} \right),$$

where $V$ has order (5.10) and the difference between this and the existing bound for $\Sigma_{1,a}$ is the additional factor $\mathcal{M}$. Following the argument in §5.2, we need to check that this does not alter the truth of (5.9). Thus, when $t \geq P^{-3}$, we take $\mathcal{M} \leq R^{-1/2}P^{3/4}$ and find that the term involving $V$ makes the contribution

$$\ll H^{0}P^{3n/2+3/4+\varepsilon} t^{1+n/2} R^{1+n/2} \ll H^{0}P^{3n/4-3/4+\varepsilon},$$

since $t \leq (RP^{3/2})^{-1}$. This is satisfactory for $n \geq 5$. Likewise, when $t < P^{-3}$, one obtains a satisfactory contribution. Turning to the contribution from the term involving $(R_2^2R_3)^{1/3}$, we suppose first that $t < P^{-3}$. Taking $R_2 \geq (R_2^2R_3)^{1/3}$, the contribution from this case is found to be

$$\ll H^{0}P^{n+\varepsilon} \mathcal{M} \frac{R^{3/2-n/2} t (R_2^2R_3)^{n/3}}{R_2^{1/2}} \ll H^{0}P^{n-3+\varepsilon} \mathcal{M} R^{3/2-n/2} (R_2^2R_3)^{n/3-1-6}.$$

Taking $\mathcal{M} \leq R^{-1/2}P^{3/4}$ gives $O(H^{0}P^{n-3+3/4+\varepsilon} R^{(5-n)/6})$, which is satisfactory since $n \geq 5$. Next, assuming that $t \geq P^{-3}$ and adjoining Proposition 2, it remains to analyze the contribution

$$\ll H^{0}P^{n+\varepsilon} \min \left\{ \mathcal{M} R^{3/2-n/2} t (R_2^2R_3)^{n/3-1-6}, \frac{R_2^{2-n/8} t^{1-n/8}}{(R_2^2R_3)^{2/3} P^{3n/8}} \right\}. \tag{****}$$

For $n \geq 6$ we apply the inequality $\min \{A, B\} \leq A^{1/3}B^{2/3}$, to get the overall contribution $O(H^{0}P^{n-2+\varepsilon} \mathcal{M}^{1/3}E_n)$, with $E_n$ given at the bottom of page 109. When $n \geq 13$ we take $t \geq P^{-3}$, getting

$$\mathcal{M}^{1/3}E_n \ll P^{-3/4} R^{7/6-5n/36} \ll 1.$$
When $6 \leq n \leq 12$ we take $t \leq (RP^{3/2})^{-1}$ to deduce that

$$\mathcal{M}^{1/3} E_n \ll P^{3/4-n/8} R^{1/6-n/18} \ll 1.$$ 

Finally we dispatch the case $n = 5$, for which we return to (****) and take $t \leq (RP^{3/2})^{-1}$. This leads to the contribution

$$\ll H^\theta P^{3+\varepsilon} \times \min \left\{ P^{1/2} R^{-2} R_0^{1/2} (R_2^2 R_3)^{3/2}, P^{5/4} R^{-5/2} (R_2^2 R_3)^{3/2}, \frac{P^{-7/16} R}{(R_2^2 R_3)^{2/3}} \right\}$$

$$\ll H^\theta P^{3+\varepsilon} \min \left\{ P^{1/2} R^{-3/2} R_2^2 R_3, P^{5/4} R^{-5/2} (R_2^2 R_3)^{3/2}, \frac{P^{-7/16} R}{(R_2^2 R_3)^{2/3}} \right\}.$$ 

Taking $\min \{A, B, C\} \leq A^{17/75} B^{2/15} C^{16/25}$ leads to the contribution $O(H^\theta p^{3+\varepsilon} R^{-1/30})$. This is satisfactory and so concludes the proof of Proposition 1 in [1].

**Proof of Lemma 11.** We now adopt the equation and page numbering from [2]. The treatments of $\Sigma_{1,b}$ and $\Sigma_{2,b}$ go through unchanged, leaving us the task of showing that

$$\Sigma_{i,a} \ll H^\theta P^{n-5/2+\varepsilon},$$

for $i = 1, 2$ and $n \geq 8$. Beginning with $i = 2$, it follows from (***), that our estimate at the top of page 866 gets replaced by

$$\Sigma_{2,a} \ll H^\theta P^{3n/4-3/4+\varepsilon} + H^\theta \frac{P^{n-3+\varepsilon} (R_2^2 R_3)^{n/3}}{R^{n/2-3/2}} \min \left\{ R_0^{1/2}, \frac{P^{3/4}}{R^{1/2}} \right\}.$$ 

The first term is satisfactory. We take $\min \{\cdot, \cdot\} \leq R_0^{1/2}$ in the second term and note that $R_0^{1/2} (R_2^2 R_3)^{n/3} \ll R^{n/3}$. Thus the second term is

$$\ll H^\theta P^{n-3+\varepsilon} R^{-n/6+3/2},$$

which is satisfactory for $n \geq 8$, since $R \leq P^{3/2}$.

Turning to $i = 1$, our analogue of the third displayed equation on page 866 is

$$\Sigma_{1,a} \ll H^\theta P^{\varepsilon} (P^{3n/4-3/4} + P^{n-3} \mathcal{M} R^{3/2-n/2} (R_2^2 R_3)^{n/3-1/6} + \mathcal{E}),$$

where, in view of (****),

$$\mathcal{E} = P^n \min \left\{ \mathcal{M} R^{3/2-n/2} t (R_2^2 R_3)^{n/3-1/6}, \frac{R^{2-n/8} t^{1-n/8}}{(R_2^2 R_3)^{2/3} P^{3n/8}} \right\}.$$ 

In our bound for $\Sigma_{1,a}$ the second and third terms correspond to the contribution from the term involving $(R_2^2 R_3)^{1/3}$, with the second dealing with the case $t < P^{-3}$ and the third dealing with the case $t \geq P^{-3}$. The first term
is satisfactory. Taking $\mathcal{M} \leq R_0^{1/2}$ shows that the second term makes the satisfactory contribution
\[
\ll H^\theta P^{n-3+\varepsilon} R^{3/2-n/2} R_0^{1/2} (R_2^2 R_3) n/3-1/6 \ll H^\theta P^{n-3+\varepsilon} R^{4/3-n/6}.
\]

We handle $\mathcal{E}$ as in [1, §5.2] by applying the inequality $\min\{A, B\} \leq A^{1/3} B^{2/3}$, to get the overall contribution $O(H^\theta P^{n-3+\varepsilon} \mathcal{M}^{1/3} E_n)$, with
\[
E_n = P^{2-n/4} t^{1-n/12} R^{11/6-n/4} (R_2^2 R_3) n/9-1/2.
\]

We need to check that $P^{1/2} \mathcal{M}^{1/3} E_n \ll 1$ for $n \geq 8$. When $n \geq 13$ we take $t \geq P^{-3}$, getting
\[
P^{1/2} \mathcal{M}^{1/3} E_n \ll P^{-1/4} R^{7/6-5n/36} \ll 1.
\]

When $8 \leq n \leq 12$ we take $t \leq (RP^{3/2})^{-1}$ to deduce that
\[
P^{1/2} \mathcal{M}^{1/3} E_n \ll R_0^{1/6} P^{1-n/8} R^{5/6-n/6} (R_2^2 R_3) n/9-1/2 \ll P^{1-n/8} R^{1/3-n/18} \ll 1.
\]

This is satisfactory and so concludes the proof of Lemma 11 in [2].

References


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