THE NUMBER OF REPRESENTATIONS OF RATIONALS AS A SUM OF UNIT FRACTIONS

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Abstract. For given positive integers m and n, we consider the frequency of representations of $\frac{m}{n}$ as a sum of unit fractions.

1. Introduction

This paper centres on the question of representing fractions as sums of unit fractions. Specifically, for a positive integer $k \geqslant 2$ and given $m, n \in \mathbb{N}$, we would like a better understanding of the counting function

$$f_k(m,n) = \#\left\{(t_1,\dots,t_k) \in \mathbb{N}^k : t_1 \leqslant \cdots \leqslant t_k \text{ and } \frac{m}{n} = \frac{1}{t_1} + \cdots + \frac{1}{t_k}\right\}.$$

We will be mainly concerned with upper bounds for $f_k(m,n)$ which are uniform in k, m and n. On observing the trivial upper bound $f_k(m,n) \leqslant f_k(1,n)$, we will generally be interested in bounds for $f_k(m,n)$ that get sharper as the size of m increases.

The easiest case to deal with is the case $k = 2$, for which we have the following essentially complete description.

Theorem 1. *We have*

$$f_2(m,n) \leqslant \exp\Big((\log 3 + o(1))\frac{\log n}{\log\log n}\Big).$$

Furthermore, for fixed $m \in \mathbb{N}$, there are infinitely many values of n for which

$$f_2(m,n) \gg_m \exp\Big((\log 3 + o(1))\frac{\log n}{\log\log n}\Big).$$

When $k = 3$ the equation appearing in the definition of $f_3(m,n)$ has received much attention in the context of the conjecture[1] of Erdős and Straus [5]. This predicts that $f_3(4,n) > 0$ for any $n \geqslant 2$. The conjecture has since been generalised to arbitrary numerators by Schinzel [10]. Thus for any $m \geqslant 4$ one expects the existence of $N_m \in \mathbb{N}$ such that $f_3(m,n) > 0$ for $n \geqslant N_m$. Both of these conjectures are still wide open and have generated a lot of attention in the literature. An overview of the domain can be found in work of the second author [4]. The following result provides an upper bound for $f_3(m,n)$ which is uniform in m and n.

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[1]The earliest reference in the literature to this conjecture appears to be a paper by Obláth [7], submitted in 1948.

Theorem 2. *For any $\varepsilon > 0$ we have*

$$f_3(m,n) \ll_\varepsilon \left(\frac{n}{m}\right)^{\frac{2}{3}} n^\varepsilon.$$

It follows from the theorem that $f_3(m,n) \ll_\varepsilon n^{\frac{2}{3}+\varepsilon}$. Numerical experimentation reveals that $f_3(m,n)$ varies considerably as n varies but nonetheless ought to correspond to a superposition of divisor functions. Indeed we would conjecture that $f_3(m,n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$. Moreover our numerical investigations lead us to expect that $f_3(m,n) \to \infty$ as $n \to \infty$, for fixed m.

Once the denominators are cleared the equation appearing in $f_3(m,n)$ takes the shape

$$mxyz = n(xy + xz + yz).$$

This is one of several affine cubic equations for which the number of solutions in positive integers is expected to grow like the divisor function. In private communication with the authors Brian Conrey has asked whether the number of solutions in positive integers to the equation

$$n = xyz + x + y$$

can be bounded by $O_\varepsilon(n^\varepsilon)$ for any $\varepsilon > 0$. Kevin Ford[2] has posed a generalisation of this problem, in which one would like to show that there are $O_\varepsilon(|AB|)^\varepsilon)$ non-trivial positive integer solutions to the equation $xyz = A(x+y) + B$, for given non-zero $A, B \in \mathbb{Z}$. A further problem of this type has been posed by Pelling[3], in which it is asked whether there are $O_\varepsilon(n^\varepsilon)$ solutions to the cubic equation

$$xyz = n(x + y + z),$$

with $x, y, z \in \mathbb{N}$. For this equation it is known that the relevant counting function grows at most like $O_\varepsilon(n^{\frac{1}{2}+\varepsilon})$ but the original question is open. We shall not say anything more about these equations here.

Recording anything meaningful for $f_k(m,n)$ when $k \geqslant 4$ seems to be a harder problem. Nonetheless we are able to build Theorem 2 into an induction argument which leads us to the following result.

Theorem 3. *Let $k \geqslant 4$. For any $\varepsilon > 0$ we have*

$$f_4(m,n) \ll_\varepsilon n^\varepsilon \Big\{ \left(\frac{n}{m}\right)^{\frac{5}{3}} + \frac{n^{\frac{4}{3}}}{m^{\frac{2}{3}}} \Big\},$$

and for $k \geqslant 5$

$$f_k(m,n) \ll_\varepsilon (kn)^\varepsilon \left(\frac{k^{\frac{4}{3}} n^2}{m}\right)^{\frac{5}{3}\times 2^{k-5}}.$$

The special case $f_k(1,1)$ has received special attention in the literature. In one direction Croot [2] has solved a difficult problem of Erdős by showing that any finite colouring of the positive integers allows a monochromatic solution of the equation

$$1 = \sum_{i=1}^{k} \frac{1}{t_i} \tag{1}$$

for unspecified k. In a different direction, for given $k \in \mathbb{N}$, let $K(k) = f_k(1,1)$ denote the number of vectors $(t_1, \ldots, t_k) \in \mathbb{N}^k$ with $t_1 \leqslant \cdots \leqslant t_k$, for which (1)

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[2]First presented at the DIMACS Meeting in Rutgers in 1996.

[3]*Problem 10745, Solution in: American Math. Monthly*, **108** (2001), 668–669.

holds. Define the sequence u_n via $u_1 = 1$ and $u_{n+1} = u_n(u_n + 1)$. This sequence grows doubly exponentially and one has $c_0 = \lim_{n\to\infty} u_n^{2^{-n}} = 1.264\ldots$. Building on earlier work of Erdős, Graham and Straus [6], Sándor [8] has established the upper bound

$$K(k) < c_0^{(1+\varepsilon)2^{k-1}}$$

for any $\varepsilon > 0$ and any $k \geqslant k(\varepsilon)$. Taking $m = n = 1$ in Theorem 3 we deduce the following estimate.

Corollary. *For any $\varepsilon > 0$ we have*

$$K(k) \ll_\varepsilon k^{\frac{5}{9}\times 2^{k-3}+\varepsilon}.$$

For intermediate k this improves upon Sándor's result. By revisiting Sándor's argument we achieve the following sharpening for large k.

Theorem 4. *Let $\varepsilon > 0$ and assume that $k \geqslant k(\varepsilon)$. Then we have*

$$K(k) < c_0^{(\frac{5}{24}+\varepsilon)2^{k-1}}.$$

While interesting in its own right it transpires that the study of Egyptian fractions has applications to various problems in topology. For example, Brenton and Hall [1] have established a bijection between solutions $(t_1, \ldots, t_k) \in \mathbb{N}^k$ to the equation

$$1 = \sum_{i=1}^{k} \frac{1}{t_i} + \prod_{i=1}^{k} \frac{1}{t_i}$$

and homeomorphism equivalence classes of homologically trivial complex surface singularities whose dual intersection graph is a star with central weight 1 and weights t_i on the arms. In [1, §4] the authors ask for a better understanding of the counting function $S(k)$ for large k, which is defined to be the number of solutions $(t_1, \ldots, t_k) \in \mathbb{N}^k$ to the above equation with $t_1 \leqslant \cdots \leqslant t_k$. On observing that $S(k) \leqslant K(k+1)$ we observe the following trivial consequence of Theorem 4.

Corollary. *Let $\varepsilon > 0$ and assume that $k \geqslant k(\varepsilon)$. Then we have*

$$S(k) < c_0^{(\frac{5}{24}+\varepsilon)2^{k}}.$$

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2. Sums of two unit fractions

In this section we establish Theorem 1. Beginning with the upper bound, Sándor [8, Lemma 4] has shown that

$$f_2(m, n) \leqslant f_2(1, n) = \frac{1}{2}(d(n^2) + 1),$$

where d denotes the divisor function. To see this we note that if $\frac{1}{n} = \frac{1}{t_1} + \frac{1}{t_2}$ then $t_2 = \frac{nt_1}{t_1-n} = n + \frac{n^2}{t_1-n}$, which is an integer if and only if $t_1 - n \mid n^2$. The condition $t_1 \leqslant t_2$ ensures that $t_1 \leqslant 2n$, so that $0 < t_1 - n \leqslant n$ and indeed $f_2(1, n) = \frac{1}{2}(d(n^2) + 1)$. Applying work of Shiu [9] on the maximum order of multiplicative functions we easily deduce the upper bound in Theorem 1.

We now turn to the lower bound for $f_2(m,n)$ for fixed $m \in \mathbb{N}$. It will suffice to examine $g_2(m,n)$, which is defined as for $f_2(m,n)$, but without the restriction that $t_1 \leqslant t_2$ in each solution. Indeed we plainly have

$$g_2(m,n) \leqslant 2f_2(m,n).$$

Let $n = \prod_{i=1}^s q_i$, where s is odd and q_i denotes the ith prime which is congruent to $-1 \bmod m$. Then we claim that

$$g_2(m,n) \geqslant \frac{3^s}{2}.$$

To see this, let x_1 be the product of any subset of an odd number i of the s prime factors. Let x_2 be a product of an even number j of the remaining $s-i$ prime factors. Then $x_{12} = \frac{n}{x_1x_2}\frac{x_1+x_2}{m}$ is an integer and we have

$$\frac{m}{n} = \frac{1}{x_1x_{12}} + \frac{1}{x_2x_{12}}.$$

Counting up the number of available x_1, x_2 gives the contribution

$$S_1 = \sum_{i \text{ odd}}^{s} \sum_{j \text{ even}}^{s-i} \binom{s}{i}\binom{s-i}{j} = \sum_{i \text{ odd}}^{s} \binom{s}{i} 2^{s-i-1}.$$

Likewise we can instead choose x_1 to consist of an even number i of the s primes, and x_2 an odd number j of the remaining $s-i$ primes. This gives the contribution

$$S_2 = \sum_{i \text{ even}}^{s} \sum_{j \text{ odd}}^{s-i} \binom{s}{i}\binom{s-i}{j} = \sum_{i \text{ even}}^{s} \binom{s}{i} 2^{s-i-1}.$$

Thus we deduce that

$$g_2(m,n) \geqslant S_1 + S_2 = \sum_{i=0}^{s} \binom{s}{i} 2^{s-i-1} = \frac{3^s}{2},$$

as required. To complete the proof of the theorem we note that

$$n = \prod_{i=1}^{s} q_i = \exp\left(\sum_{i=1}^{s} \log q_i\right).$$

By the prime number theorem for arithmetic progressions

$$\sum_{i=1}^{s} \log q_i \sim \sum_{i=1}^{s} \log(i(\log i)\varphi(m)) \sim s\log s + s\log\log s + s\log\varphi(m).$$

It follows that $s \sim \frac{\log n}{\log\log n + \log\varphi(m)} \sim \frac{\log n}{\log\log n}$, for fixed m. Therefore, there are at least $\frac{1}{4}3^s = \exp\big((\log 3 + o(1))\frac{\log n}{\log\log n}\big)$ solutions counted by $f_2(m,n)$, which thereby completes the proof of Theorem 1.

3. Sums of three unit fractions

In this section we establish the upper bound in Theorem 2 for $f_3(m,n)$. It will clearly suffice to assume that $\gcd(m,n)=1$. Since $t_1 \leqslant t_2 \leqslant t_3$ in the definition of the counting function it is clear that

$$\frac{n}{m} < t_1 \leqslant \frac{3n}{m}. \qquad (2)$$

In particular we must have $m \leqslant 3n$. We can get an upper bound for t_2 via the expression

$$\frac{m}{n} - \frac{1}{t_1} = \frac{1}{t_2} + \frac{1}{t_3} \leqslant \frac{2}{t_2}.$$

Suppose that $m < n$. Let $n = mq + r$ for $0 < r \leqslant m - 1$. We have $t_1 \geqslant \lceil \frac{n}{m} \rceil = q+1$ and it follows that the left hand side is at least

$$\frac{m}{mq+r} - \frac{1}{q+1} \geqslant \frac{1}{(q+1)(mq+r)} \geqslant \frac{m}{2(mq+r)(mq+r)} = \frac{m}{2n^2} \geqslant \frac{m}{3n^2},$$

giving

$$t_2 \leqslant \frac{6n^2}{m}. \tag{3}$$

Suppose now that $m > n$, with $m \leqslant 3n$. Then we have $t_1 \geqslant \lceil \frac{n}{m} \rceil = 1$, whence

$$\frac{m}{n} - 1 \geqslant \frac{1}{n} \geqslant \frac{m}{3n^2},$$

whence (3) holds in this case also. Once combined with the underlying equation in $f_3(m,n)$ the inequalities (2) and (3) are enough to show that

$$f_3(m,n) \leqslant \frac{3n}{m}\frac{6n^2}{m} = \frac{18n^3}{m^2}.$$

Proceeding to the proof of the sharper bound in Theorem 2, we may henceforth assume that t_1, t_2 satisfy $t_1 \leqslant t_2$ and lie in the ranges given by (2) and (3), in any given solution $(t_1, t_2, t_3) \in \mathbb{N}^3$ counted by $f_3(m,n)$.

In what follows let i, j, k denote distinct elements from the set $\{1, 2, 3\}$. Let

$$x_{123} = \gcd(t_1, t_2, t_3), \quad x_{ij} = \frac{\gcd(t_i, t_j)}{x_{123}} \quad x_i = \frac{t_i}{x_{ij}x_{ik}x_{123}},$$

with $x_{ij} = x_{ji}$. Then

$$t_1 = x_1x_{12}x_{13}x_{123}, \quad t_2 = x_2x_{12}x_{23}x_{123}, \quad t_3 = x_3x_{13}x_{23}x_{123}, \tag{4}$$

with

$$\gcd(x_ix_{ik}, x_jx_{jk}) = 1. \tag{5}$$

Substituting these values for t_1, t_2, t_3 into the equation in the definition of $f_3(m,n)$, we obtain

$$mx_1x_2x_3x_{12}x_{13}x_{23}x_{123} = n(x_1x_2x_{12} + x_1x_3x_{13} + x_2x_3x_{23}).$$

It follows from (5) that $x_1x_2x_3 \mid n$. Since $\gcd(m,n) = 1$, we may conclude that

$$n = x_1x_2x_3h_{12}h_{13}h_{23}h_{123}, \tag{6}$$

where

$$h_{ij} = \gcd\left(\frac{n}{x_1x_2x_3}, x_{ij}\right), \qquad h_{123} = \gcd\left(\frac{n}{x_1x_2x_3}, x_{123}\right).$$

If we write $x_{ij} = h_{ij}y_{ij}$ and $x_{123} = dh_{123}$, then we obtain the simplification

$$mdy_{12}y_{13}y_{23} = x_1x_2h_{12}y_{12} + x_1x_3h_{13}y_{13} + x_2x_3h_{23}y_{23}. \tag{7}$$

Furthermore, we have the additional coprimality relations

$$\gcd(y_{ij}, h_{ik}h_{jk}h_{123}) = \gcd(d, h_{ij}) = 1.$$

Thus (5) and (7) imply that any two elements of the set $\{x_1, x_2, x_3, d\}$ must be coprime.

Let $D>0$. It will be convenient to consider the overall contribution to $f_3(m,n)$ from $x_1,x_2,x_3,d,h_{ij},h_{123},y_{ij}$ such that that d is constrained to lie in an interval

$$D \leqslant d < 2D.$$

We will write $F(m,n;D)$ for this quantity. It follows from (2), (3) and (4) that

$$y_{12}y_{13} = \frac{x_1x_{123}x_{12}x_{13}}{x_1x_{123}h_{12}h_{13}} = \frac{t_1}{x_1h_{123}dh_{12}h_{13}} \leqslant \frac{3n}{x_1mh_{12}h_{13}h_{123}D}, \tag{8}$$

and similarly

$$y_{12}y_{23} \leqslant \frac{6n^2}{x_2mh_{12}h_{23}h_{123}D}.$$

We proceed to estimate $F(m,n;D)$ in two different ways.

Lemma 1. *For any $\varepsilon>0$ we have*

$$F(m,n;D) \ll_\varepsilon \frac{n^{1+\varepsilon}}{mD}.$$

Proof. It follows from (7) that there exists an integer r such that

$$y_{23}r = x_2h_{12}y_{12} + x_3h_{13}y_{13}.$$

For fixed $x_2,x_3,h_{12},h_{13},y_{12},y_{13}$ the trivial estimate for the divisor function implies that there are $O_\varepsilon(n^\varepsilon)$ choices for y_{23},r. Summing over y_{12},y_{13} we conclude from (8) that there are $O_\varepsilon(m^{-1}D^{-1}n^{1+\varepsilon})$ choices for the y_{ij} and r. A choice of d is fixed by (7). Since there are $O_\varepsilon(n^\varepsilon)$ possible choices for $x_1,x_2,x_3,h_{ij},h_{123}$, by (6), so it follows that

$$F(m,n;D) \ll_\varepsilon \sum_{x_1,x_2,x_3,h_{ij},h_{123}} \frac{n^{1+\varepsilon}}{mD} \ll_\varepsilon \frac{n^{1+2\varepsilon}}{mD}.$$

The statement of the lemma follows on redefining the choice of $\varepsilon>0$. □

Lemma 2. *For any $\varepsilon>0$ we have*

$$F(m,n;D) \ll_\varepsilon \frac{D^{\frac12}n^{\frac12+\varepsilon}}{m^{\frac12}}.$$

Proof. Assume without loss of generality that $y_{12}\leqslant y_{13}$. Fixing y_{12}, we then estimate the number of integers $A,B \ll n^2$ for which

$$mdy_{12}AB = x_1x_2h_{12}y_{12} + x_1x_3h_{13}A + x_2x_3h_{23}B.$$

But we may rewrite this equation as

$$(mdy_{12}A - x_2x_3h_{23})(mdy_{12}B - x_1x_3h_{13}) = mx_1x_2dh_{12}y_{12}^2 + x_1x_2x_3^2h_{13}h_{23}.$$

For each $x_1,x_2,x_3,d,h_{ij},h_{123},y_{12}$, there are clearly $O_\varepsilon(n^\varepsilon)$ possible values of A,B, by elementary estimates for the divisor function. Moreover, (8) and the assumption $y_{12}\leqslant y_{13}$ together imply that $y_{12}\ll\sqrt{\frac{n}{mD}}$. Thus we obtain the bound

$$F(m,n;D) \ll_\varepsilon \sum_{x_1,x_2,x_3,d,h_{ij}} \frac{n^{\frac12+\varepsilon}}{(mD)^{\frac12}} \ll_\varepsilon \frac{D^{\frac12}n^{\frac12+2\varepsilon}}{m^{\frac12}},$$

on summing over values of d in the range $D\leqslant d<2D$, and the $O_\varepsilon(n^\varepsilon)$ possible values of $x_1,x_2,x_3,h_{ij},h_{123}$ for which (6) holds. The lemma follows on redefining the choice of $\varepsilon>0$. □

We are now ready to complete the proof of the theorem. There are $O(\log n)$ possible dyadic ranges for d, such that $d \leqslant n$. Theorem 2 therefore follows on applying Lemma 1 to deal with the contribution from $d \geqslant (\frac{n}{m})^{\frac{1}{3}}$, and Lemma 2 to handle $d < (\frac{n}{m})^{\frac{1}{3}}$.

4. Sums of k unit fractions

In this section we establish Theorems 3 and 4. Beginning with the former, let $(t_1, \ldots, t_k) \in \mathbb{N}^k$ be a point with $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_k$ counted by $f_k(m,n)$. Then

$$\frac{mt_1 - n}{nt_1} = \frac{m}{n} - \frac{1}{t_1} = \frac{1}{t_2} + \cdots + \frac{1}{t_k}.$$

It is easy to see that $f_k(m,n) = 0$ unless $m \leqslant kn$ which we now assume. Furthermore the analogue of (2) in the preceding section is clearly

$$\frac{n}{m} < t_1 \leqslant \frac{kn}{m}. \tag{9}$$

Our induction is based on the observation that

$$f_k(m,n) \leqslant \sum_{t_1} f_{k-1}(mt_1 - n, nt_1),$$

where the summation is over $t_1 \in \mathbb{N}$ for which (9) holds. Making the change of variables $u = mt_1 - n$ we obtain

$$f_k(m,n) \leqslant \sum_{\substack{0<u\leqslant(k-1)n \\ m|u+n}} f_{k-1}\Big(u, \frac{n(u+n)}{m}\Big). \tag{10}$$

Note that $u + n \leqslant kn$ for each u under consideration.

Let $\varepsilon > 0$. We begin by establishing the theorem in the case $k = 4$. It follows from Theorem 2 that

$$f_4(m,n) \ll_\varepsilon n^\varepsilon \sum_{\substack{0<u\leqslant 3n \\ m|u+n}} \Big(\frac{\frac{n(u+n)}{m}}{u}\Big)^{\frac{2}{3}} \ll_\varepsilon \frac{n^{\frac{4}{3}+\varepsilon}}{m^{\frac{2}{3}}} \sum_{\substack{0<u\leqslant 3n \\ m|u+n}} u^{-\frac{2}{3}}.$$

Given $\theta \in [0,1)$ we now require the estimate

$$S_\theta(x) = \sum_{\substack{n\leqslant x \\ n\equiv a \bmod q}} n^{-\theta} = \frac{x^{1-\theta}}{(1-\theta)q} + O_\theta(1),$$

which is valid uniformly for $a \in \mathbb{Z}$ and $q \in \mathbb{N}$. This follows from combining partial summation with the familiar estimate $S_0(x) = q^{-1}x + O(1)$. If $\theta \geqslant 1 + \delta$ for some fixed $\delta > 0$, then $S_\theta(x) \ll_\delta 1$. We may now conclude that

$$f_4(m,n) \ll_\varepsilon \frac{n^{\frac{4}{3}+\varepsilon}}{m^{\frac{2}{3}}}\Big(\frac{n^{\frac{1}{3}}}{m} + 1\Big). \tag{11}$$

This establishes the theorem in the case $k = 4$. Turning to the case $k = 5$ we repeat the above analysis based on (10), but use the inequality in (11) as our bound for $f_4(m,n)$. It follows that

$$f_5(m,n) \ll_\varepsilon n^\varepsilon \sum_{\substack{0<u\leqslant 4n \\ m|u+n}} \Big\{\Big(\frac{m^{-1}n^2}{u}\Big)^{\frac{5}{3}} + \frac{(m^{-1}n^2)^{\frac{4}{3}}}{u^{\frac{2}{3}}}\Big\} \ll_\varepsilon n^\varepsilon\Big(\frac{n^2}{m}\Big)^{\frac{5}{3}},$$

which thereby establishes the theorem when $k=5$.

It remains to establish Theorem 3 for $k\geqslant 6$. We will begin by showing that

$$f_k(m,n)\ll_{\varepsilon,k} n^{\varepsilon}\Big(\frac{n^2}{m}\Big)^{\frac{5}{3}\times 2^{k-5}}, \tag{12}$$

for $k\geqslant 5$, where the implied constant is allowed to depend on k. This will be achieved by induction on k, the case $k=5$ already having been dealt with. When $k\geqslant 6$ we deduce from the induction hypothesis and (10) that

$$f_k(m,n)\ll_{\varepsilon,k} n^{\varepsilon}\sum_{0<u\leqslant(k-1)n}\Big(\frac{n^2(u+n)^2}{um^2}\Big)^{\frac{5}{3}\times 2^{k-6}}\ll_{\varepsilon,k} n^{\varepsilon}\Big(\frac{n^2}{m}\Big)^{\frac{5}{3}\times 2^{k-5}},$$

since $\frac{5}{3}\times 2^{k-6}\geqslant\frac{5}{3}$ for $k\geqslant 6$. This therefore establishes (12).

We now turn to a bound for $f_k(m,n)$ which is uniform in k, which we will again achieve via induction on k. Let $\varepsilon>0$. We will take for our induction hypothesis the estimate

$$f_k(m,n)\ll_{\varepsilon}(kn)^{\varepsilon}\Big(\frac{k^{\theta_k}n^2}{m}\Big)^{\frac{5}{3}\times 2^{k-5}}, \tag{13}$$

for an undetermined function θ_k. We may henceforth suppose that

$$k\geqslant\frac{\log 3-\log(5\varepsilon)}{\log 2}+5, \tag{14}$$

else (13) follows trivially from (12). Now for any $L\leqslant k$ it follows from (10) that

$$f_k(m,n)\leqslant\sum_{\substack{0<u\leqslant(L-1)n\\ m\mid u+n}}f_{k-1}\Big(u,\frac{n(u+n)}{m}\Big)+\sum_{\substack{(L-1)n<u\leqslant(k-1)n\\ m\mid u+n}}f_{k-1}\Big(u,\frac{n(u+n)}{m}\Big).$$

One notes that $u+n\leqslant Ln$ in the first sum and $u+n\leqslant kn$ in the second. The induction hypothesis therefore gives

$$f_k(m,n)\ll_{\varepsilon}(kn)^{\varepsilon}k^{\frac{5\theta_{k-1}}{3}\times 2^{k-6}}\Big\{\Big(\frac{Ln^2}{m}\Big)^{\frac{5}{3}\times 2^{k-5}}\Sigma_1+\Big(\frac{kn^2}{m}\Big)^{\frac{5}{3}\times 2^{k-5}}\Sigma_2\Big\},$$

where

$$\Sigma_1=\sum_{0<u\leqslant(L-1)n}\Big(\frac{1}{u}\Big)^{\frac{5}{3}\times 2^{k-6}}\ll 1,$$

$$\Sigma_2=\sum_{(L-1)n<u\leqslant(k-1)n}\Big(\frac{1}{u}\Big)^{\frac{5}{3}\times 2^{k-6}}\leqslant\sum_{u\geqslant L}\Big(\frac{1}{u}\Big)^{\frac{5}{3}\times 2^{k-6}}\ll L^{1-\frac{5}{3}\times 2^{k-6}}.$$

We deduce that

$$f_k(m,n)\ll_{\varepsilon}(kn)^{\varepsilon}\Big(\frac{k^{\frac{\theta_{k-1}}{2}}n^2}{m}\Big)^{\frac{5}{3}\times 2^{k-5}}\Big\{L+k\Big(\frac{1}{L}\Big)^{\frac{1}{2}-\frac{3}{5}2^{-(k-5)}}\Big\}^{\frac{5}{3}\times 2^{k-5}}.$$

Now (14) ensures that $\frac{1}{2}-\frac{3}{5}\times 2^{-(k-5)}\geqslant\frac{1}{2}-\varepsilon$. Hence, on taking $L=k^{\frac{2}{3}}$, we conclude that

$$f_k(m,n)\ll_{\varepsilon}k^{\varepsilon(1+\frac{5}{9}\times 2^{k-4})}n^{\varepsilon}\Big(\frac{k^{\frac{\theta_{k-1}}{2}+\frac{2}{3}}n^2}{m}\Big)^{\frac{5}{3}\times 2^{k-5}}.$$

Redefining the choice of ε therefore leads us to the induction hypothesis (13) with

$$\theta_k=\frac{\theta_{k-1}}{2}+\frac{2}{3}.$$

It is now easy to deduce that $\theta_k < \frac{4}{3}$, which completes the proof of Theorem 3.

We now turn to the proof of Theorem 4, for which we will modify the argument in [8]. Recall the definition of the sequence u_n from the introduction and let $c_0 = \lim_{n\to\infty} u_n^{2^{-n}}$. Since $u_n^{2^{-n}}$ is monotonically increasing we have $u_n < c_0^{2^n}$. Suppose that $1 = \sum_{i=1}^k \frac{1}{t_i}$, with $t_1 \leqslant \cdots \leqslant t_k$. Then Curtiss [3] has shown that

$$1 - \sum_{i=1}^{m} \frac{1}{t_i} \geqslant \frac{1}{u_{m+1}},$$

for $1 \leqslant m \leqslant k-1$. It follows that $t_j \leqslant (k-j+1)u_j$ for each j since otherwise

$$1 = \sum_{i=1}^{k} \frac{1}{t_i} = \sum_{i=1}^{j-1} \frac{1}{t_i} + \sum_{i=j}^{k} \frac{1}{t_i} < 1 - \frac{1}{u_j} + \frac{k-j+1}{(k-j+1)u_j} = 1,$$

which is a contradiction.

Let $\varepsilon > 0$ and let L be chosen to be the least positive integer for which $2^{5-L} < \frac{\varepsilon}{2}$. The number of tuples $(t_1, \ldots, t_{k-L})$ with $t_j \leqslant (k-j+1)u_j$ is therefore

$$\prod_{j=1}^{k-L} (k-j+1)u_j \leqslant (k-L)! \prod_{j=1}^{k-L} c_0^{2^j} < k! c_0^{2^{k-L+1}}.$$

For a given $(t_1, \ldots, t_{k-L})$-tuple, it remains to estimate the number of vectors $(t_{k-L+1}, \ldots, t_k)$ that complete the sum $\sum_{i=1}^k \frac{1}{t_i} = 1$. We write

$$1 - \frac{1}{t_1} - \cdots - \frac{1}{t_{k-L}} = \frac{m}{n},$$

where $n \leqslant t_1 \cdots t_{k-L} < k! c_0^{2^{k-L+1}}$. Applying Theorem 3 we deduce that the number of available $(t_{k-L+1}, \ldots, t_k)$ is at most

$$f_L(m,n) \ll_\varepsilon \left(k! c_0^{2^{k-L+1}}\right)^{\frac{5}{3}\times 2^{L-5}+\varepsilon} \ll_\varepsilon (k!)^{\frac{5}{3}\times 2^{L-5}+\varepsilon} c_0^{\frac{5}{3}\times 2^{k-4}+\varepsilon},$$

for any $\varepsilon > 0$. Combining our two estimates we may now conclude that

$$K(k) \ll_\varepsilon (k!)^{e_L} \times c_0^{2^{k-L+1}} \times c_0^{\frac{5}{3}\times 2^{k-4}+\varepsilon} \ll_\varepsilon (k!)^{e_L} c_0^{(\frac{5}{3}+\varepsilon)\times 2^{k-4}},$$

where $e_L = 1 + \frac{5}{3} \times 2^{L-5} + \varepsilon$. This therefore concludes the proof of Theorem 4 on redefining the choice of ε.

References

[1] L. Brenton and R. Hill, On the Diophantine equation $1 = \sum 1/n_i + 1/\prod n_i$ and a class of homologically trivial complex surface singularities. *Pacific J. Math.* **133** (1988), 41–67.

[2] E.S. Croot III, On a coloring conjecture about unit fractions. *Annals of Math.* **157** (2003), 545–556.

[3] D.R. Curtiss, On Kellogg's Diophantine Problem, *American Math. Monthly* **29** (1922), 380–387

[4] C. Elsholtz, Sums of k unit fractions. *Trans. Amer. Math. Soc.* **353** (2001), 3209–3227.

[5] P. Erdős, Az $\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} = \frac{a}{b}$ egyenlet egész számú megoldásairól (On a Diophantine equation). *Mat. Lapok* **1** (1950), 192–210.

[6] P. Erdős and R.L. Graham, *Old and new problems and results in combinatorial number theory.* Monographies de L'Enseignement Mathématique **28**, Geneva, 1980.

[7] M.R. Obláth, Sur l' équation diophantienne $\frac{4}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}$. *Mathesis* **59** (1950), 308–316.

[8] C. Sándor, On the number of solutions of the Diophantine equation $\sum_{i=1}^n \frac{1}{x_i} = 1$. *Period. Math. Hung.* **47** (2003), 215–219.

[9] P. Shiu, The maximum order of multiplicative functions. *Quart. J. Math.* **31** (1980), 247–252.

[10] W. Sierpiński, Sur les décompositions de nombres rationnels en fractions primaires. *Mathesis* **65** (1956), 16–32.

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