Waring's problem, the declining exchange rate between small powers, and the story of 13,792

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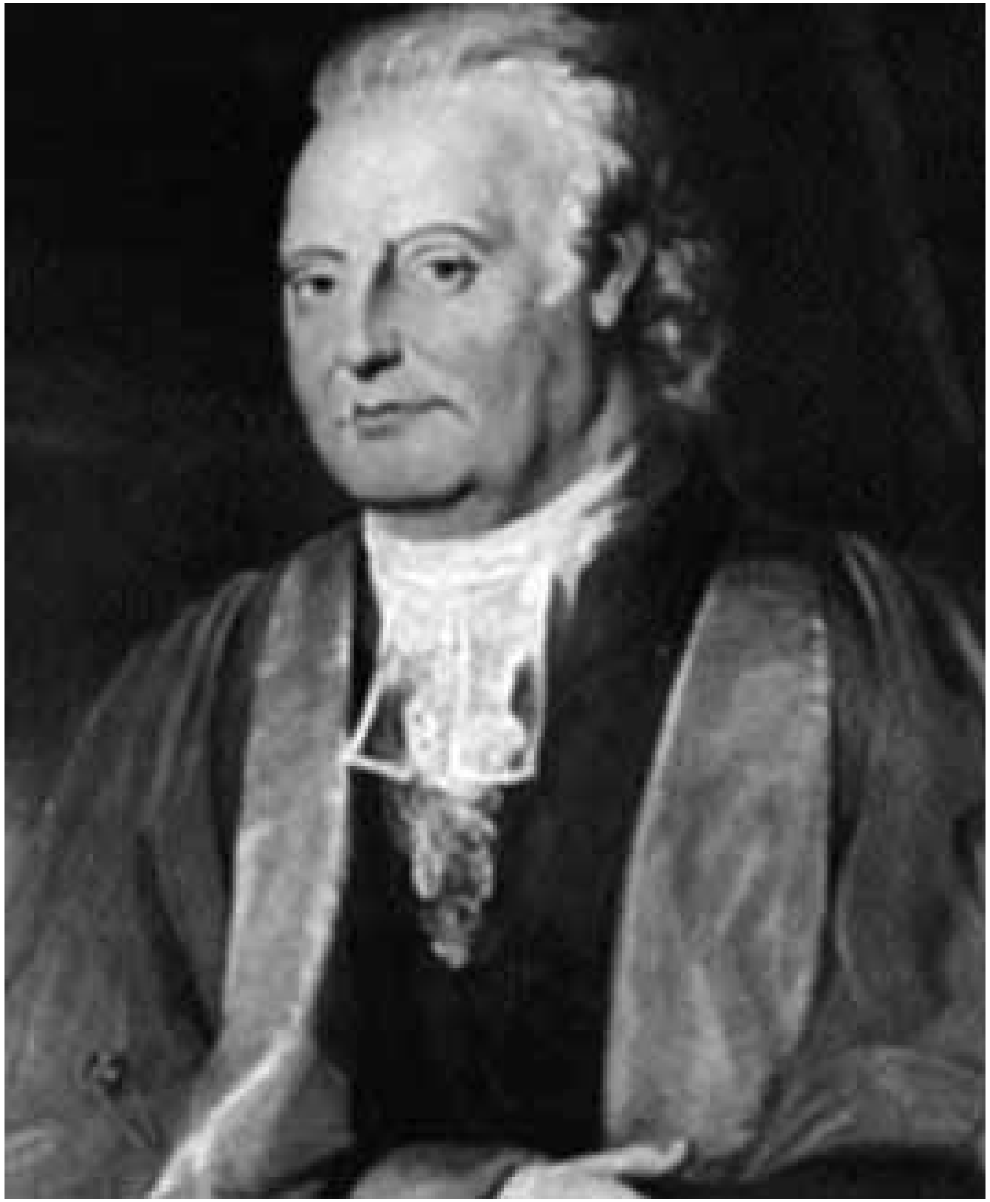

1. Introduction to Waring's problem

Conjecture (E. Waring, 1770)

"Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &c. usque ad novemdecim compositus, & sic deinceps"

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"Every integer is a cube or the sum of two, three, ... nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth."

"... every natural number is a sum of at most 9 positive integral cubes, also a sum of at most 19 biquadrates, and so on."

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Notation

Let $g(k)$ denote the least number s such that every natural number is the sum of at most s kth powers of natural numbers.

$$n = x_1^k + \cdots + x_s^k$$

Waring's Conjecture claims that

$$g(3) \le 9, \quad g(4) \le 19, \quad \ldots, \quad g(k) < \infty.$$

Note

Lagrange's four square theorem (1770) shows that $g(2) = 4$*.*

Example

$$2007 = 43^2 + 11^2 + 6^2 + 1^2.$$

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Notation

When $\alpha \in \mathbb{R}$, *write*

$$\begin{aligned} [\alpha] &:= \max\{n \in \mathbb{Z} : n \leq \alpha\}, \\ \{\alpha\} &:= \alpha - [\alpha]. \end{aligned}$$

Observation

Consider

$$n = 2^k \left[(3/2)^k \right] - 1.$$

Since $n < 3^k$*, then whenever* $n = x_1^k + \cdots + x_s^k$*, one has* $x_i \leq 2$ *for every* i*. Most "efficient" to use as many* 2*'s as possible, so "most efficient" representation is*

$$n = \underbrace{(2^k + \cdots + 2^k)}_{[(3/2)^k]-1 \text{ copies}} + \underbrace{(1^k + \cdots + 1^k)}_{2^k - 1 \text{ copies}}$$

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Conclusion

For every natural number k*, one has*

$$g(k) \geq 2^k + \left[(3/2)^k\right] - 2.$$

Conjecture

When $k \geq 2$*, one has*

$$g(k) = 2^k + [(3/2)^k] - 2.$$

Fact

This conjecture is "essentially" known from work spanning the 20th Century by Dickson, Pillai, . . . , Chen, Balasubramanian, Deshouillers, Dress.

Theorem

One has

$$g(k) = 2^k + [(3/2)^k] - 2$$

provided that

$$2^k\{(3/2)^k\} + [(3/2)^k] \le 2^k.$$

If

$$2^k\{(3/2)^k\} + [(3/2)^k] > 2^k,$$

then

$$g(k) = 2^k + [(3/2)^k] + [(4/3)^k] - \theta$$

where

$$\theta = 2 \quad \textit{when} \quad [(4/3)^k][(3/2)^k] + [(4/3)^k] + [(3/2)^k] = 2^k$$

$$\theta = 3 \quad \textit{when} \quad [(4/3)^k][(3/2)^k] + [(4/3)^k] + [(3/2)^k] > 2^k.$$

It is known that the condition

$$2^k\{(3/2)^k\} + [(3/2)^k] \le 2^k$$

holds for $k \le 471,600,000$ (Kubina and Wunderlich, 1990), and that

$$2^k\{(3/2)^k\} + [(3/2)^k] > 2^k$$

for at most finitely many natural numbers k (Mahler, 1957).

Observation

The above lower bounds for $g(k)$ are determined by the difficulty of representing small integers n (the integers 1^k, 2^k, 3^k are, in a relative sense, widely spaced apart compared to 1000^k, 1001^k, 1002^k).

Definition

Let $G(k)$ denote the least number s with the property that every large natural number is the sum of at most s kth powers of natural numbers.

We have

$$\begin{aligned} G(2) &= 4 && \text{(Lagrange, 1770)}\\ G(3) &\leq 7 && \text{(Linnik, 1943)}\\ G(4) &= 16 && \text{(Davenport, 1939)}\\ G(5) &\leq 17 && \text{(Vaughan and Wooley, 1995)}\\ G(6) &\leq 24 && \text{(Vaughan and Wooley, 1994)} \end{aligned}$$

and

$$G(k) \leq k(\log k + \log\log k + 2 + o(1)) \quad \text{(Wooley, 1992, 1995).}$$

k	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Hardy & Littlewood (1925)	41	87	192	425	949	2113										
James (1934)	35		164		824											
Heilbronn (1936)				164	190	217	244	272	300	329	359	388	418	449	480	511
Estermann (1937)	29	42	59	78	101	125	153	184	217	253	292	333	377	424	474	
Hua (1938a)	28															
Davenport (1939b, 1942b)	23	36	(53)													
Narasimhamurti (1941)				73	99	122										
Chen (1958)						121										
Cook (1973)					96											
Vaughan (1977)					91	107	122	137	153	168	184	200	216			
Thanigasalam (1980)					90	106	121	136	152	167	183	199	215	231	248	264
Thanigasalam (1982)					88	104	119	134	150	165	181	197	213	229	245	262
Thanigasalam (1985)			50	68	87	103										
Vaughan (1986a,c)	21	31	45	62	82											
Vaughan (1989a,c)	19	29	41	57	75	93	109	125	141	156	171	187	202	217	232	248
Brüdern (1990)	18															
Vaughan & Wooley (1991)	18	28				92	108	124	139	153	168	183	198	213	228	243
Wooley (1992a)		27	36	47	55	63	70	79	87	95	103	112	120	129	138	146
Wooley (1995a)						62		78	86	94	102	110	118	127	135	144
Vaughan & Wooley (1995a)	17	25	33	43	51	(59)	(67)	(76)	(84)	(92)	(100)					
Vaughan & Wooley (1993)				42												
Vaughan & Wooley (1994)		24														
Meng (1997)																143
Vaughan & Wooley (2000)					50	59	67	76	84	92	100	109	117	125	134	142
Conjectured	6	9	8	32	13	12	12	16	14	15	16	64	18	27	20	25

Table 2. Upper bounds for $G(k)$ when $5 \leq k \leq 20$

Fact

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Why? Consider $n = 16^t \cdot 31$ $(t \in \mathbb{N})$.

Suppose that n is the sum of 15 fourth powers.

Note

One has

$$x^4 \equiv \begin{cases} 0 \pmod{16}, & \text{when } x \text{ is even,} \\ 1 \pmod{16}, & \text{when } x \text{ is odd.} \end{cases}$$

So whenever

$$x_1^4 + \cdots + x_{15}^4 \equiv 0 \pmod{16},$$

one has $2|x_i$ $(1 \leq i \leq 15)$, and hence

$$16|(x_1^4 + \cdots + x_{15}^4).$$

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But the "most efficient" representation of 31 as the sum of fourth powers is

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Conclusion

One has $G(4) \geq 16$.

2. Polynomial identities

Example

Observe that

$$\begin{aligned}6(a^2+b^2+c^2+d^2)^2 = {} & (a+b)^4+(a-b)^4+(c+d)^4+(c-d)^4 \\ & +(a+c)^4+(a-c)^4+(b+d)^4+(b-d)^4 \\ & +(a+d)^4+(a-d)^4+(b+c)^4+(b-c)^4.\end{aligned}$$

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Then $6x^2$ is always a sum of 12 fourth powers.
Given a natural number n, write

$$n = 6m + r \quad (0 \le r \le 5).$$

Then r is the sum of at most 5 fourth powers, and by Lagrange,

$$6m = 6x^2 + 6y^2 + 6z^2 + 6w^2$$

(for some $x, y, z, w \in \mathbb{Z}$) is the sum of at most $4 \cdot 12 = 48$ fourth powers.

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Clever variants of such polynomial identity arguments provide bounds on $g(k)$ for $k = 3, 4, 5, 6, 7, 8, 10$. In particular,

$$g(6) \leq 2\,451 \quad \text{and} \quad g(8) \leq 42\,273.$$

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A more complicated argument gives:

Theorem (Wieferich, 1909 and Kempner, 1912)

One has $g(3) = 9$.

Theorem (Dickson, 1939)

All integers except 23 *and* 239 *are sums of at most* 8 *cubes of natural numbers.*

Theorem (Hilbert, 1909)

For each natural number k, one has

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What about sharper upper bounds?

3. The Hardy-Littlewood (circle) method

Developed by Hardy and Littlewood (1920's) and Vinogradov (1930's).

Given a large integer n, we seek to write n as the sum of kth powers of positive integers in the form

$$n = x_1^k + \cdots + x_s^k.$$

Notation

Define

$$P = [n^{1/k}]$$

$$f(\alpha) = \sum_{1 \le x \le P} e(\alpha x^k).$$

Here, as usual, we write $e(z) = e^{2\pi i z}$.

We apply Fourier analysis. If we write

$$R_s(n) = \operatorname{card}\{\mathbf{x} \in \mathbb{N}^s \,:\, x_1^k + \cdots + x_s^k = n\},$$

then one has

$$\begin{aligned} R_s(n) &= \sum_{1 \le x_1, \ldots, x_s \le P} \int_0^1 e(\alpha(x_1^k + \cdots + x_s^k - n))\, d\alpha \\ &= \int_0^1 f(\alpha)^s e(-n\alpha)\, d\alpha. \end{aligned}$$

Strategy

Use this relation to obtain an asymptotic formula for $R_s(n)$*, and show that when* s *is large enough, one has* $R_s(n) \to \infty$ *as* $n \to \infty$*.*

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Use this relation to obtain an asymptotic formula for $R_s(n)$*, and show that when* s *is large enough, one has* $R_s(n) \to \infty$ *as* $n \to \infty$*.*
From this one obtains $G(k) \le s$*.*

Observation

One finds that $f(\alpha)$ tends to be "large" when α is "close" to a rational number a/q with q "small", and otherwise $f(\alpha)$ is "small".

Definition (Hardy-Littlewood dissection)

$$\mathfrak{M}(q,a) := \{\alpha \in [0,1) \,:\, |\alpha - a/q| \le q^{-1}P^{1-k}\}$$

$$\mathfrak{M} := \bigcup_{\substack{0\le a\le q\le P\\ (a,q)=1}} \mathfrak{M}(q,a) \quad (\text{"Major arcs"}).$$

$$\mathfrak{m} := [0,1) \setminus \mathfrak{M}.$$

When $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, one can obtain the asymptotic relation

$$\begin{aligned} f(\alpha) &= \sum_{1 \le x \le P} e(\alpha x^k) \\ &= q^{-1} S(q,a) v(\alpha - a/q) + O(P^{1/2+\varepsilon}). \end{aligned}$$

where

$$S(q,a) = \sum_{r=1}^{q} e(ar^k/q) \quad \text{and} \quad v(\beta) = \int_0^P e(\beta \gamma^k)\, d\gamma.$$

From this, by integrating over each major arc $\mathfrak{M}(q,a)$ comprising $\mathfrak{M}$, one obtains

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)\, d\alpha = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_s(n) n^{s/k-1} + o(n^{s/k-1}),$$

valid for $s \ge k+1$, where

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valid for $s \geq k+1$, where

$$\mathfrak{S}_s(n) = \prod_p \varpi_p(n),$$

in which

$$\varpi_p(n) = \lim_{h\to\infty} p^{h(1-s)}\text{card}\,\{\mathbf{x} \in (\mathbb{Z}/p^h\mathbb{Z})^s \,:\, x_1^k + \cdots + x_s^k \equiv n \pmod{p^h}\}.$$

Note here that $\varpi_p(n) = 0$ whenever there is a p-adic obstruction to solubility.

Major arcs:

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha)\, d\alpha \gg n^{s/k-1} \quad (s \geq 4k).$$

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Now consider the **minor arcs** $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$.

Theorem (Weyl, 1916)

Suppose that $\alpha \in \mathbb{R}$ *and that* $a \in \mathbb{Z}$, $q \in \mathbb{N}$ *satisfy* $(a,q) = 1$ *and* $|\alpha - a/q| \leq q^{-2}$. *Then for each* $\varepsilon > 0$, *one has*

$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k) \ll P^{1+\varepsilon}(q^{-1} + P^{-1} + qP^{-k})^{2^{1-k}}.$$

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Suppose now that $\alpha \in \mathfrak{m}$, and apply Dirichlet's theorem on diophantine approximation to find $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$, $1 \leq q \leq P^{k-1}$ and

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$$|\alpha - a/q| \leq q^{-1}P^{1-k}.$$

If $q \leq P$, then it follows from the definition of $\mathfrak{M}$ that $\alpha \in \mathfrak{M}$ $[\gg\ll]$. So by hypothesis we have $q > P$, whence from Weyl's inequality

$$f(\alpha) \ll P^{1-2^{1-k}+\varepsilon}.$$

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$$|f(\alpha)| \le \sum_{1 \le x \le P} 1 = P.$$

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We may infer that for $s \ge k2^{k-1}+1$, one has

$$\int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\, d\alpha \ll (P^{1-2^{1-k}+\varepsilon})^s = o(P^{s-k}).$$

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But one has a much sharper mean value estimate:

Theorem (Hua, 1938)

For each $\varepsilon > 0$, one has

$$\int_0^1 |f(\alpha)|^{2^k}\,d\alpha \ll P^{2^k-k+\varepsilon}.$$

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So for $s \geq 2^k + 1$ one has

$$\begin{aligned}\int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\, d\alpha &\ll \big(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\big)^{s-2^k} \int_0^1 |f(\alpha)|^{2^k}\, d\alpha \\ &\ll P^{s-k-k\eta} \quad \text{(some } \eta > 0\text{)}.\end{aligned}$$

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So combining this estimate with our major arc lower bound, we find that

$$\int_0^1 f(\alpha)^s e(-n\alpha)\, d\alpha \gg n^{s/k-1} + o(n^{s/k-1})$$

whenever $s \geq \max\{4k, 2^k + 1\}$.

Idea

One has

$$\begin{aligned} |f(\alpha)|^2 &= \sum_{1 \le x \le P} \sum_{1 \le y \le P} e(\alpha(x^k - y^k)) \\ &= \sum_{|h| < P} \sum_{\substack{1 \le x \le P \\ 1 \le x+h \le P}} e(\alpha h p(x; h)), \end{aligned}$$

in which $p(x; h) := h^{-1}((x+h)^k - x^k)$ *is a polynomial of degree* $k-1$.

Now repeat this process in combination with Cauchy's inequality:

$$f(\alpha)^{2^{k-1}} \ll P^{2^{k-1}-k} \sum_{|h_1|<P} \cdots \sum_{|h_{k-1}|<P} \sum_{\substack{1 \le x \le P \\ x \in \mathcal{B}(\mathbf{h})}} e(\alpha h_1 \dots h_{k-1} q(x; \mathbf{h})),$$

in which q *is a linear polynomial in* x *and* $\mathbf{h}$.

Recall:

$$f(\alpha)^{2^{k-1}} \ll P^{2^{k-1}-k} \sum_{|h_1|<P} \cdots \sum_{|h_{k-1}|<P} \sum_{\substack{1\le x\le P\\ x\in\mathcal{B}(\mathbf{h})}} e(\alpha h_1 \dots h_{k-1} q(x;\mathbf{h})),$$

Now observe that

$$\begin{aligned}\int_0^1 |f(\alpha)|^{2^k}\, d\alpha &= \int_0^1 |f(\alpha)|^{2^{k-1}} \cdot |f(\alpha)|^{2^{k-1}}\, d\alpha\\ &\ll P^{2^{k-1}-k} \mathrm{card}\left\{ h_1 \dots h_{k-1} q(x;\mathbf{h}) = \sum_{j=1}^{2^{k-2}} (x_j^k - y_j^k) \right\}\end{aligned}$$

Now isolate diagonal and non-diagonal solutions to obtain

$$\int_0^1 |f(\alpha)|^{2^k}\, d\alpha \ll P^{2^k-k+\varepsilon}.$$

Theorem

For $k \geq 3$ one has $G(k) \leq 2^k + 1$, and hence $g(k) < \infty$.

This argument has "good" explicit control of error terms with $2^k + 1$ kth powers.
Note: for fourth powers this entails using ≥ 17 variables.

Note (Davenport, 1939)

Using a refinement of diminishing ranges, the analytic argument works for 14 *fourth powers. This gives $G(4) = 16$.*

Note (Vaughan, 1989)

Using a method based on exponential sums over smooth numbers (integers having only small prime divisors), the analytic argument works for 12 *fourth powers.*

Last two arguments — no "good" control of error terms available.

Calculating $g(k)$:

By the mid-1960's the above methods had established that

$$g(k) = 2^k + [(3/2)^k] - 2$$

(with modifications in exceptional circumstances as described earlier)
EXCEPT when $k = 4$.

Calculating $g(4)$

Advances in computers, divisor sum techniques cleared the way for:

Theorem (Balasubramanian, Deshouillers and Dress, 1985)

One has $g(4) = 19$.

Roughly speaking, analytic machinery (circle method) shows that each

$$n \geq 10^{367}$$

is the sum of 19 fourth powers.

Then use computers to check that the integers n with $1 \leq n < 10^{367}$ are each represented in the shape $n = x_1^4 + \cdots + x_{19}^4$.

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Use *greedy ascent*. Suppose that $1 \leq n < 10^{367}$. Choose

$$x_{19} = [n^{1/4}] \quad \text{and put} \quad n_1 = n - x_{19}^4.$$

Then

$$n_1 \approx n^{3/4} < 10^{\frac{3}{4}(367)},$$

and since nearly every integer is (expected to be) the sum of 18 biquadrates, we can hope to cover exceptional cases.

Now iterate this process, taking care to keep track of the ambient (mod 16) conditions, until we are left with 5 fourth powers. The number of cases that we must check is reduced from 10^{367} to approximately

$$2^{14} \cdot 10^{367 \cdot (3/4)^{14}} \approx 10^{11}.$$

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Problem

$g(4) = 19 \Rightarrow$ **all** *integers are the sum of* 19 *fourth powers.*

$G(4) = 16 \Rightarrow$ *there is an integer* n_0 *with the property that each* $n > n_0$ *is the sum of* 16 *fourth powers.*

Enquiring minds must know ... what is n_0*?*

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Note

Balasubramanian, Deshouillers and Dress need at least 17 *fourth powers to get the analytic part of their argument started.*

The polnomial identity strikes back

(Joint work with Koichi Kawada)

Consider the identity

$$x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2$$

due to Ramanujan, Proth, ... ancient?

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$$\text{card}\,\{n \le X \,:\, n = a^2 + ab + b^2\} \sim X/\sqrt{\log X},$$

one might optimistically suppose that

$$x^4 + y^4 + (x + y)^4 = 2m^2$$

(with $m = x^2 + xy + y^2$) really behaves like twice a square.

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Note

3 fourth powers $\leftrightarrow$ *1 square*

is a "good" rate of exchange.

Analytically, squares are much easier to handle than fourth powers — easiest to see in an application.

Define $N(X) := \text{card}\,\{n \leq X \,:\, n = x_1^4 + \cdots + x_5^4,\ x_i \in \mathbb{N}\}$.

Theorem (Vaughan, 1989)

One has $N(X) \gg X^{0.9417\ldots}$*.*

Theorem (Kawada and Wooley, 1999)

For each $\varepsilon > 0$*, one has* $N(X) \gg X(\log X)^{-1-\varepsilon}$*.*

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Theorem (Kawada and Wooley, 1999)

For each $\varepsilon > 0$, *one has* $N(X) \gg X(\log X)^{-1-\varepsilon}$.

Sketch of proof: Consider the set $\mathcal{C}$ of integers n of the shape

$$n = 2m^2 + u^4 + v^4,$$

with

$$1 \le u, v \le \tfrac{1}{4}X^{1/4} \quad \text{and} \quad m \in \mathcal{B},$$

in which

$$\mathcal{B} = \{m \in [1, \tfrac{1}{2}X^{1/2}] \,:\, m = x^2 + xy + y^2,\ x, y \in \mathbb{N}\}.$$

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Note

If $m \in \mathcal{B}$*, then there exist integers* x *and* y *with*

$$2m^2 = 2(x^2 + xy + y^2)^2 = x^4 + y^4 + (x + y)^4.$$

Hence when $n \in \mathcal{C}$*, there exist integers* x*,* y*,* u*,* v *for which*

$$n = x^4 + y^4 + (x + y)^4 + u^4 + v^4$$

is a sum of 5 *fourth powers.*

$$\mathcal{B} = \{m \in [1, \tfrac{1}{2}X^{1/2}] \,:\, m = x^2 + xy + y^2,\ x, y \in \mathbb{N}\}.$$

Note

If $m \in \mathcal{B}$, then there exist integers x and y with

$$2m^2 = 2(x^2 + xy + y^2)^2 = x^4 + y^4 + (x + y)^4.$$

Hence when $n \in \mathcal{C}$, there exist integers x, y, u, v for which

$$n = x^4 + y^4 + (x + y)^4 + u^4 + v^4$$

is a sum of 5 *fourth powers.*

Let $r(n)$ denote the number of representations of n in the above form.

Then by Cauchy's inequality,

$$N(X) \geq \sum_{\substack{1 \leq n \leq X \\ r(n) > 0}} 1 \geq \frac{\left(\sum_{1 \leq n \leq X} r(n) \right)^2}{\left(\sum_{1 \leq n \leq X} r(n)^2 \right)}.$$

Then by Cauchy's inequality,

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On the one hand,

$$\sum_{1 \leq n \leq X} r(n) \geq \sum_{1 \leq u, v \leq \frac{1}{4} X^{1/4}} \sum_{m \in \mathcal{B}} 1 \gg (X^{1/4})^2 (X^{1/2}/\sqrt{\log X}) \gg X/\sqrt{\log X}.$$

On the other hand, as we'll shortly show (essentially), one has

$$\sum_{1 \leq n \leq X} r(n)^2 \ll X(\log X)^{\varepsilon}.$$

Thus we find that

$$N(X) \gg \frac{(X/\sqrt{\log X})^2}{X(\log X)^{\varepsilon}} \gg X(\log X)^{-1-\varepsilon}.$$

One has

$$\sum_{1\le n\le X} r(n)^2 = \mathrm{card}\,\{2m_1^2+u_1^4+v_1^4=2m_2^2+u_2^4+v_2^4 \ : \\ 1\le u_i,v_i\le \tfrac{1}{4}X^{1/4}, m_1,m_2\in \mathcal{B}\subseteq [1,\tfrac{1}{2}X^{1/2}]\}.$$

Two types of solutions:

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Two types of solutions:

(a) those with $u_1^4+v_1^4=u_2^4+v_2^4$, which forces $m_1=m_2$, of which there are

$$O(X^{1/2})O((X^{1/4})^{2+\varepsilon}) \ll X^{1+\varepsilon}.$$

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(b) those with $u_1^4+v_1^4 \ne u_2^4+v_2^4 = h \ne 0$, say. For these one has

$$2(m_1-m_2)(m_1+m_2) = h \ne 0,$$

and so the number of solutions here is

$$O(X^{\varepsilon}) \cdot O(X) \ll X^{1+\varepsilon}.$$

So the total number of solutions is $O(X^{1+\varepsilon})$, an estimate that may be refined with additional work to $O(X(\log X)^{\varepsilon})$.

Further consequences:

Theorem (Vaughan, 1989)

Suppose that $s \geq 12$ *and that* n *is a large integer with* $n \equiv r \pmod{16}$ *for some integer* r *with* $1 \leq r \leq s$. *Then* n *is the sum of* s *fourth powers.*

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Theorem (Kawada and Wooley, 1999)

Suppose that $s \geq 11$ *and that* n *is a large integer with* $n \equiv r \pmod{16}$ *for some integer* r *with* $1 \leq r \leq s - 1$. *Then* n *is the sum of* s *fourth powers.*

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Note

The value of

$$x^4 + y^4 + (x+y)^4 = 2(x^2 + xy + y^2)^2$$

is always even, whereas

$$x^4 + y^4 + z^4$$

can be either even or odd. This interferes with solubility modulo 16.

But one can remove these difficulties with variants of the basic problem:

Theorem (Kawada and Wooley, 1999)

All large integers n are represented in the form

$$n = x_1^4 + x_2^4 + \cdots + x_{10}^4 + y^{2007}.$$

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All large integers n are represented in the form

$$n = x_1^4 + x_2^4 + \cdots + x_{10}^4 + y^{2007}.$$

... but back to sums of 16 fourth powers ...

Strategy

Write

$$g(\alpha) = \sum_{1 \le x, y \le n^{1/4}} e(2\alpha(x^2 + xy + y^2)^2) \quad \textit{and} \quad f(\alpha) = \sum_{1 \le z \le n^{1/4}} e(\alpha z^4).$$

From Hua's lemma we have

$$\int_0^1 |f(\alpha)|^{16} \, d\alpha \ll n^{3+\varepsilon},$$

which saves $n^{1-\varepsilon}$ *over the trivial estimate, and uses* 16 *variables. But*

$$\int_0^1 |g(\alpha)^2 f(\alpha)^4| \, d\alpha \sim \sum_m r(m)^2 \ll n^{1+\varepsilon},$$

which saves $n^{1-\varepsilon}$ *over the trivial estimate, and uses only* 10 *variables. So we have saved* 6 *variables in the critical mean value estimates.*

5. Back to $g(4)$ and n_0

(Joint work with J.-M. Deshouillers and K. Kawada)

We want to study sums of 16 fourth powers, but cannot afford to miss any congruence classes (mod 16) in the represented integers!

Observation

Every integer n with

$$13\,793 \leq n \leq 220\,688 = 16 \times 13\,793$$

is the sum of 16 *fourth powers.*

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Suppose now that $n > 220\,688$ and $16|n$.

Then there exists a natural number i with $16^i|n$ satisfying the condition that either

$$13\,793 \leq n/16^i \leq 220\,688,$$

or else

$$n/16^i > 220\,688 \quad \textit{and} \quad 16 \textit{ does not divide } n/16^i.$$

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In the former case, since $n/16^i$ *is a sum of* 16 *fourth powers, so too is*

$$n = (2^i)^4(n/16^i).$$

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In the latter case, it suffices to represent $n/16^i$ *as the sum of* 16 *fourth powers, in which*

$$n/16^i \not\equiv 0 \pmod{16}.$$

Conclusion

It suffices to represent only those integers n *with* $n \equiv r \pmod{16}$ *for* $1 \leq r \leq 15$.

Strategy

Try using one copy of the identity (which loses one congruence class modulo 16*). So seek representations in the shape*

$$n = x^4 + y^4 + (x+y)^4 + \sum_{j=1}^{13} z_j^4.$$

The corresponding integral formulation is

$$\int_0^1 g(\alpha) f(\alpha)^{13} e(-n\alpha)\, d\alpha.$$

Major arcs (easy part)

$$\int_{\mathfrak{M}} g(\alpha)f(\alpha)^{13}e(-n\alpha)\,d\alpha \gg n^{11/4}.$$

Minor arcs (hard part)

$$\begin{aligned}\int_{\mathfrak{m}} g(\alpha)f(\alpha)^{13}e(-n\alpha)\,d\alpha &\le \left(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\right)^3 \left(\int_0^1 |g(\alpha)^2f(\alpha)^4|\,d\alpha\right)^{1/2}\\ &\qquad\times\left(\int_0^1 |f(\alpha)|^{16}\,d\alpha\right)^{1/2}\\ &\ll n^{11/4-3/8+\varepsilon}.\end{aligned}$$

A careful analysis based on this discussion shows that whenever $n \equiv r \pmod{16}$ and $1 \le r \le 15$, and

$$n > 10^{356},$$

then n is the sum of 16 biquadrates.

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A careful analysis based on this discussion shows that whenever $n\equiv r \pmod{16}$ and $1\le r\le 15$, and

$$n>10^{356},$$

then n is the sum of 16 biquadrates. But ...

$$10^{356}\to \text{greedy algorithm}\to 2^{11}\cdot 10^{(3/4)^{11}\cdot 356}\approx 2\times 10^{18}\quad\text{(too big!)}.$$

6. The polynomial identity strikes back — again!

Observation

One has

$$\begin{aligned}&(w+x)^4+(w-x)^4+(w+y)^4+(w-y)^4\\&\quad+(w+x+y)^4+(w-x-y)^4=4(x^2+xy+y^2+3w^2)^2-30w^4.\end{aligned}$$

So

6 fourth powers $\sim$ 1 square and 1 fourth power

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So

6 fourth powers $\sim$ 1 square and 1 fourth power

5 fourth powers $\sim$ 1 square.

This is less efficient than the previous exchange rate of 3 fourth powers to 1 square, but in compensation there are no lost congruence classes.

Define

$$G(\alpha) := \sum_{x,y,w \sim n^{1/4}} e(\alpha(4(x^2+xy+y^2+3w^2)^2 - 30w^4))$$

Then

$$\int_0^1 |G(\alpha)^2 f(\alpha)^2|\, d\alpha \ll n^{1+\varepsilon} \quad \text{(14 fourth powers).}$$

Compare this with

$$\int_0^1 |g(\alpha)^2 f(\alpha)^4|\, d\alpha \ll n^{1+\varepsilon} \quad \text{(10 fourth powers)}$$

and

$$\int_0^1 |f(\alpha)|^{16}\, d\alpha \ll n^{3+\varepsilon} \quad \text{(16 fourth powers).}$$

Strategy

Consider representations of n in the form

$$\begin{aligned} n = {} & (w+x)^4 + (w-x)^4 + (w+y)^4 + (w-y)^4 \\ & + (w+x+y)^4 + (w-x-y)^4 + (u+v)^4 + u^4 + v^4 \\ & + z_1^4 + \cdots + z_7^4. \end{aligned}$$

This has integral representation

$$\int_0^1 G(\alpha) g(\alpha) f(\alpha)^7 e(-n\alpha)\, d\alpha.$$

Major arcs (easy):

$$\int_{\mathfrak{M}} G(\alpha)g(\alpha)f(\alpha)^7 e(-n\alpha)\,d\alpha \gg n^2.$$

Minor arcs (hard):

$$\begin{aligned}\int_{\mathfrak{m}} G(\alpha)g(\alpha)f(\alpha)^7 e(-n\alpha)\,d\alpha \le \left(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\right)^4 & \left(\int_0^1 |G(\alpha)^2 f(\alpha)^2|\,d\alpha\right)^{1/2} \\ & \times \left(\int_0^1 |g(\alpha)^2 f(\alpha)^4|\,d\alpha\right)^{1/2} \\ & \ll n^{2-4/8+\varepsilon}.\end{aligned}$$

Now make all the estimates explicit (non-trivial divisor sum estimates, estimates for complete exponential sums etc.)

Theorem (Deshouillers, Kawada and Wooley, 2005)

Suppose that $n \geq 10^{216}$. Then n is the sum of 16 fourth powers.

Note

Although 10^{216} may appear a formidable number of integers to check for representability, if we use the "greedy" algorithm, this scales down to something like

$$2^{11} \cdot 10^{(3/4)^{11} \cdot 216} \approx 3 \times 10^{12},$$

which is far more manageable.

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Theorem (Deshouillers, Hennecart and Landreau, 2000)

When $13\,793 \leq n < 10^{245}$, *then* n *is the sum of* 16 *fourth powers.*

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Combining these conclusions, we obtain:

Theorem (Deshouillers, Hennecart, Kawada, Landreau, Wooley, 2005)

All integers exceeding 13 792 *can be written as a sum of* 16 *fourth powers.*

The 96 exceptional integers

(1) The 7 integers that are not B_{18} are

$$79 + 80k \quad (k = 0, 1, \ldots, 6)$$

(2) The 24 integers that are B_{18} but not B_{17} are

$$\begin{aligned} 63 + 80k &\quad (k = 0, \ldots, 14) \\ 78 + 80k &\quad (k = 0, \ldots, 6) \\ 48 + 80k &\quad (k = 12, 15) \end{aligned}$$

(3) The 65 integers that are B_{17} but not B_{16} are

$$\begin{aligned} 47 + 80k &\quad (k = 0, \ldots, 22, 46) \\ 62 + 80k &\quad (k = 0, \ldots, 14) \\ 77 + 80k &\quad (k = 0, \ldots, 6) \\ 32 + 80k &\quad (k = 9, 12, 15, 25, 28, 44, 47, 57, 60, 79, 89, 108, 137, 172) \\ 48 + 80k &\quad (k = 44, 47, 76, 79) \\ 64 + 80k &\quad (k = 31) \end{aligned}$$