Waring’s problem, the declining exchange rate between small powers, and the story of 13,792

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1. Introduction to Waring’s problem

Conjecture (E. Waring, 1770)

“Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, &c. usque ad novemdecim compositus, & sic deinceps”
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“All integer is a cube or the sum of two, three, … nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth.”
“... every natural number is a sum of at most 9 positive integral cubes, also a sum of at most 19 biquadrates, and so on.”
“... every natural number is a sum of at most 9 positive integral cubes, also a sum of at most 19 biquadrates, and so on.”

**Notation**

Let $g(k)$ denote the least number $s$ such that every natural number is the sum of at most $s$ $k$th powers of natural numbers.

$$n = x_1^k + \cdots + x_s^k$$

Waring’s Conjecture claims that

$$g(3) \leq 9, \quad g(4) \leq 19, \quad \ldots, \quad g(k) < \infty.$$
Note

Lagrange’s four square theorem (1770) shows that \( g(2) = 4 \).

Example

\[ 2007 = 43^2 + 11^2 + 6^2 + 1^2. \]
Note

Lagrange’s four square theorem (1770) shows that $g(2) = 4$.

Example

$2007 = 43^2 + 11^2 + 6^2 + 1^2$.

Notation

When $\alpha \in \mathbb{R}$, write

$$[\alpha] := \max\{n \in \mathbb{Z} : n \leq \alpha\},$$

$$\{\alpha\} := \alpha - [\alpha].$$
Consider

\[ n = 2^k \left\lceil \left( \frac{3}{2} \right)^k \right\rceil - 1. \]

Since \( n < 3^k \), then whenever \( n = x_1^k + \cdots + x_s^k \), one has \( x_i \leq 2 \) for every \( i \).

Most “efficient” to use as many 2’s as possible, so “most efficient” representation is

\[ n = (2^k + \cdots + 2^k) + (1^k + \cdots + 1^k) \]

\( \left\lceil \left( \frac{3}{2} \right)^k \right\rceil - 1 \) copies \( 2^k - 1 \) copies
Observation

Consider

\[ n = 2^k \left\lfloor (3/2)^k \right\rfloor - 1. \]

Since \( n < 3^k \), then whenever \( n = x_1^k + \cdots + x_s^k \), one has \( x_i \leq 2 \) for every \( i \).

Most “efficient” to use as many 2’s as possible, so “most efficient” representation is

\[
 n = (2^k + \cdots + 2^k) + (1^k + \cdots + 1^k) \\
 \left\lfloor (3/2)^k \right\rfloor - 1 \text{ copies} \quad 2^k - 1 \text{ copies}
\]

Conclusion

For every natural number \( k \), one has

\[ g(k) \geq 2^k + \left\lfloor (3/2)^k \right\rfloor - 2. \]
Conjecture

When \( k \geq 2 \), one has

\[
g(k) = 2^k + [(3/2)^k] - 2.
\]

Fact

This conjecture is “essentially” known from work spanning the 20th Century by Dickson, Pillai, . . . , Chen, Balasubramanian, Deshouillers, Dress.
Theorem

One has

\[ g(k) = 2^k + [(3/2)^k] - 2 \]

provided that

\[ 2^k \{ (3/2)^k \} + [(3/2)^k] \leq 2^k. \]

If

\[ 2^k \{ (3/2)^k \} + [(3/2)^k] > 2^k, \]

then

\[ g(k) = 2^k + [(3/2)^k] + [(4/3)^k] - \theta \]

where

\[ \theta = 2 \quad \text{when} \quad [(4/3)^k][(3/2)^k] + [(4/3)^k] + [(3/2)^k] = 2^k \]
\[ \theta = 3 \quad \text{when} \quad [(4/3)^k][(3/2)^k] + [(4/3)^k] + [(3/2)^k] > 2^k. \]
It is known that the condition

\[ 2^k \left\{ \left( \frac{3}{2} \right)^k \right\} + \left[ \left( \frac{3}{2} \right)^k \right] \leq 2^k \]

holds for \( k \leq 471,600,000 \) (Kubina and Wunderlich, 1990), and that

\[ 2^k \left\{ \left( \frac{3}{2} \right)^k \right\} + \left[ \left( \frac{3}{2} \right)^k \right] > 2^k \]

for at most finitely many natural numbers \( k \) (Mahler, 1957).

**Observation**

*The above lower bounds for \( g(k) \) are determined by the difficulty of representing small integers \( n \) (the integers \( 1^k, 2^k, 3^k \) are, in a relative sense, widely spaced apart compared to \( 1000^k, 1001^k, 1002^k \)).*
Definition

Let $G(k)$ denote the least number $s$ with the property that every large natural number is the sum of at most $s$ $k$th powers of natural numbers.

We have

\[
G(2) = 4 \quad \text{(Lagrange, 1770)}
\]
\[
G(3) \leq 7 \quad \text{(Linnik, 1943)}
\]
\[
G(4) = 16 \quad \text{(Davenport, 1939)}
\]
\[
G(5) \leq 17 \quad \text{(Vaughan and Wooley, 1995)}
\]
\[
G(6) \leq 24 \quad \text{(Vaughan and Wooley, 1994)}
\]

and

\[
G(k) \leq k(\log k + \log \log k + 2 + o(1)) \quad \text{(Wooley, 1992, 1995).}
\]
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Table 2. Upper bounds for $G(k)$ when $5 \leq k \leq 20$
Fact

There are infinitely many integers $n$ that are not the sum of 15 fourth powers

Consider $n = 16t \cdot 31$ ($t \in \mathbb{N}$).

Suppose that $n$ is the sum of 15 fourth powers. Note one has $x^4 \equiv \begin{cases} 0 \pmod{16}, & \text{when } x \text{ is even}, \\ 1 \pmod{16}, & \text{when } x \text{ is odd}. \end{cases}$ So whenever $x^4 + \cdots + x^4 \equiv 0 \pmod{16}$, one has $2 | x_i$ ($1 \leq i \leq 15$), and hence $16 | (x^4 + \cdots + x^4)$.

Thus $n := 16t - 1 \cdot 31$ is also the sum of 15 fourth powers.
Fact

*There are infinitely many integers n that are not the sum of 15 fourth powers*

Why?
**Fact**

*There are infinitely many integers \( n \) that are **not** the sum of 15 fourth powers*

Why? Consider \( n = 16^t \cdot 31 \) \((t \in \mathbb{N})\).

Suppose that \( n \) is the sum of 15 fourth powers.

**Note**

*One has*

\[
x^4 \equiv \begin{cases} 
0 \pmod{16}, & \text{when } x \text{ is even}, \\
1 \pmod{16}, & \text{when } x \text{ is odd}.
\end{cases}
\]

So whenever

\[x_1^4 + \cdots + x_{15}^4 \equiv 0 \pmod{16},\]

one has \(2|x_i\) \((1 \leq i \leq 15)\), and hence

\[16|(x_1^4 + \cdots + x_{15}^4).\]
There are infinitely many integers $n$ that are not the sum of 15 fourth powers

Why? Consider $n = 16^t \cdot 31$ ($t \in \mathbb{N}$).

Suppose that $n$ is the sum of 15 fourth powers.

One has
\[
x^4 \equiv \begin{cases} 
0 \pmod{16}, & \text{when } x \text{ is even}, \\
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\end{cases}
\]

So whenever
\[
x_1^4 + \cdots + x_{15}^4 \equiv 0 \pmod{16},
\]
one has $2|x_i$ ($1 \leq i \leq 15$), and hence
\[
16|(x_1^4 + \cdots + x_{15}^4).
\]

Thus $n_1 := 16^{t-1} \cdot 31$ is also the sum of 15 fourth powers.
If $n = 16^t \cdot 31$ is the sum of 15 fourth powers, then $n_1 := 16^{t-1} \cdot 31$ is also the sum of 15 fourth powers.
If \( n = 16^t \cdot 31 \) is the sum of 15 fourth powers, then \( n_1 := 16^{t-1} \cdot 31 \) is also the sum of 15 fourth powers.

Now repeat until we run out of powers of 16.
If $n = 16^t \cdot 31$ is the sum of 15 fourth powers, then $n_1 := 16^{t-1} \cdot 31$ is also the sum of 15 fourth powers.

Now repeat until we run out of powers of 16.

In this way we find that 31 is the sum of 15 fourth powers.
If \( n = 16^t \cdot 31 \) is the sum of 15 fourth powers, then \( n_1 := 16^{t-1} \cdot 31 \) is also the sum of 15 fourth powers.

Now repeat until we run out of powers of 16.

In this way we find that 31 is the sum of 15 fourth powers.

But the “most efficient” representation of 31 as the sum of fourth powers is

\[
31 = 2^4 + 1^4 + \cdots + 1^4 = 2^4 + 15 \cdot 1^4.
\]
If $n = 16^t \cdot 31$ is the sum of 15 fourth powers, then $n_1 := 16^{t-1} \cdot 31$ is also the sum of 15 fourth powers.

Now repeat until we run out of powers of 16.

In this way we find that 31 is the sum of 15 fourth powers.

But the “most efficient” representation of 31 as the sum of fourth powers is

$$31 = 2^4 + 1^4 + \cdots + 1^4 = 2^4 + 15 \cdot 1^4.$$  

**Conclusion**

One has $G(4) \geq 16$. 
2. Polynomial identities

Example

Observe that

\[
6(a^2 + b^2 + c^2 + d^2)^2 = (a + b)^4 + (a - b)^4 + (c + d)^4 + (c - d)^4 \\
+ (a + c)^4 + (a - c)^4 + (b + d)^4 + (b - d)^4 \\
+ (a + d)^4 + (a - d)^4 + (b + c)^4 + (b - c)^4.
\]

Then \(6x^2\) is always a sum of 12 fourth powers.
2. Polynomial identities

Example

Observe that
\[ 6(a^2 + b^2 + c^2 + d^2)^2 = (a + b)^4 + (a - b)^4 + (c + d)^4 + (c - d)^4 \]
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Then \(6x^2\) is always a sum of 12 fourth powers.

Given a natural number \(n\), write
\[ n = 6m + r \quad (0 \leq r \leq 5). \]
2. Polynomial identities

Example

Observe that

\[ 6(a^2 + b^2 + c^2 + d^2)^2 = (a + b)^4 + (a - b)^4 + (c + d)^4 + (c - d)^4 \]
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\[ + (a + d)^4 + (a - d)^4 + (b + c)^4 + (b - c)^4. \]

Then \( 6x^2 \) is always a sum of 12 fourth powers.

Given a natural number \( n \), write

\[ n = 6m + r \quad (0 \leq r \leq 5). \]

Then \( r \) is the sum of at most 5 fourth powers, and by Lagrange,

\[ 6m = 6x^2 + 6y^2 + 6z^2 + 6w^2 \]

(for some \( x, y, z, w \in \mathbb{Z} \)) is the sum of at most \( 4 \cdot 12 = 48 \) fourth powers.
Theorem (Liouville, 1859)

One has $g(4) \leq 53$.

(Compare with the lower bound $g(4) \geq 19$.)
Theorem (Liouville, 1859)

One has \( g(4) \leq 53 \).

(Compare with the lower bound \( g(4) \geq 19 \).)

Clever variants of such polynomial identity arguments provide bounds on \( g(k) \) for \( k = 3, 4, 5, 6, 7, 8, 10 \). In particular,

\[
g(6) \leq 2451 \quad \text{and} \quad g(8) \leq 42273.
\]
Theorem (Liouville, 1859)

One has $g(4) \leq 53$.

(Compare with the lower bound $g(4) \geq 19$.)

Clever variants of such polynomial identity arguments provide bounds on $g(k)$ for $k = 3, 4, 5, 6, 7, 8, 10$. In particular,

$$g(6) \leq 2451 \quad \text{and} \quad g(8) \leq 42273.$$ 

A more complicated argument gives:

Theorem (Wieferich, 1909 and Kempner, 1912)

One has $g(3) = 9$.

Theorem (Dickson, 1939)

All integers except 23 and 239 are sums of at most 8 cubes of natural numbers.
Theorem (Hilbert, 1909)

For each natural number $k$, one has

$$g(k) < \infty.$$ 

This provides a “weak” affirmative answer to Waring’s Problem (conjecture).
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For each natural number $k$, one has

$$g(k) < \infty.$$ 

This provides a “weak” affirmative answer to Waring’s Problem (conjecture).

What about sharper upper bounds?
3. The Hardy-Littlewood (circle) method

Developed by Hardy and Littlewood (1920’s) and Vinogradov (1930’s).

Given a large integer \( n \), we seek to write \( n \) as the sum of \( k \)th powers of positive integers in the form

\[
  n = x_1^k + \cdots + x_s^k.
\]

**Notation**

**Define**

\[
P = \left\lfloor n^{1/k} \right\rfloor
\]

\[
f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k).
\]

*Here, as usual, we write \( e(z) = e^{2\pi iz} \).*
We apply Fourier analysis. If we write

$$R_s(n) = \text{card} \{ \mathbf{x} \in \mathbb{N}^s : x_1^k + \cdots + x_s^k = n \},$$

then one has

$$R_s(n) = \sum_{1 \leq x_1, \ldots, x_s \leq P} \int_0^1 e(\alpha(x_1^k + \cdots + x_s^k - n)) \, d\alpha$$

$$= \int_0^1 f(\alpha)^s e(-n\alpha) \, d\alpha.$$ 

**Strategy**

*Use this relation to obtain an asymptotic formula for $R_s(n)$, and show that when $s$ is large enough, one has $R_s(n) \to \infty$ as $n \to \infty$.***
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\]

**Strategy**

*Use this relation to obtain an asymptotic formula for \( R_s(n) \), and show that when \( s \) is large enough, one has \( R_s(n) \to \infty \) as \( n \to \infty \). From this one obtains \( G(k) \leq s \).*
Observation

One finds that $f(\alpha)$ tends to be “large” when $\alpha$ is “close” to a rational number $a/q$ with $q$ “small”, and otherwise $f(\alpha)$ is “small”.

Definition (Hardy-Littlewood dissection)

$$\mathcal{M}(q, a) := \{\alpha \in [0, 1) : |\alpha - a/q| \leq q^{-1}P^{1-k}\}$$

$$\mathcal{M} := \bigcup_{0 \leq a \leq q \leq P, (a, q) = 1} \mathcal{M}(q, a)$$ (“Major arcs”).

$$m := [0, 1) \setminus \mathcal{M}.$$
When \( \alpha \in \mathcal{M}(q, a) \subseteq \mathcal{M} \), one can obtain the asymptotic relation

\[
f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k) \]

\[
= q^{-1}S(q, a)\nu(\alpha - a/q) + O(P^{1/2+\varepsilon}).
\]

where

\[
S(q, a) = \sum_{r=1}^{q} e(ar^k/q) \quad \text{and} \quad \nu(\beta) = \int_{0}^{P} e(\beta \gamma^k) \, d\gamma.
\]

From this, by integrating over each major arc \( \mathcal{M}(q, a) \) comprising \( \mathcal{M} \), one obtains

\[
\int_{\mathcal{M}} f(\alpha)^s e(-n\alpha) \, d\alpha = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} \mathcal{G}_s(n)n^{s/k-1} + o(n^{s/k-1}),
\]

valid for \( s \geq k + 1 \), where
From this, by integrating over each major arc $\mathcal{M}(q, a)$ comprising $\mathcal{M}$, one obtains

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valid for $s \geq k + 1$, where

$$\mathcal{G}_s(n) = \prod_p \varpi_p(n),$$

in which

$$\varpi_p(n) = \lim_{h \to \infty} p^{h(1-s)} \text{card} \{ x \in (\mathbb{Z}/p^h\mathbb{Z})^s : x_1^k + \cdots + x_s^k \equiv n \pmod{p^h} \}. $$

Note here that $\varpi_p(n) = 0$ whenever there is a $p$-adic obstruction to solubility.
Major arcs:

\[
\int_{\mathcal{M}} f(\alpha)^s e(-n\alpha) \, d\alpha \gg n^{s/k-1} \quad (s \geq 4k).
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Now consider the **minor arcs** \( m = [0, 1) \setminus \mathcal{M} \).

**Theorem (Weyl, 1916)**

*Suppose that \( \alpha \in \mathbb{R} \) and that \( a \in \mathbb{Z} \), \( q \in \mathbb{N} \) satisfy \( (a, q) = 1 \) and \( |\alpha - a/q| \leq q^{-2} \). Then for each \( \varepsilon > 0 \), one has*

\[
f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k) \ll P^{1+\varepsilon} (q^{-1} + P^{-1} + qP^{-k})^{2^{1-k}}.
\]
Major arcs:

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$$f(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k) \ll P^{1+\varepsilon} (q^{-1} + P^{-1} + qP^{-k})^{2^{1-k}}.$$

Suppose now that $\alpha \in m$, and apply Dirichlet’s theorem on diophantine approximation to find $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $1 \leq q \leq P^{k-1}$ and $|\alpha - a/q| \leq q^{-1}P^{1-k}$.
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If \( q \leq P \), then it follows from the definition of \( \mathcal{M} \) that \( \alpha \in \mathcal{M} \[\gg \ll\]. So by hypothesis we have \( q > P \), whence from Weyl’s inequality

\[
f(\alpha) \ll P^{1-2^{1-k}+\varepsilon}.
\]
\[
\sup_{\alpha \in m} |f(\alpha)| \ll P^{1-2^{1-k}+\varepsilon}.
\]
\[ \sup_{\alpha \in m} |f(\alpha)| \ll P^{1-2^{-k}+\varepsilon}. \]

This is appreciably better than the trivial estimate

\[ |f(\alpha)| \leq \sum_{1 \leq x \leq P} 1 = P. \]
\[
\sup_{\alpha \in \mathbb{R}} |f(\alpha)| \ll P^{1-2^{1-k}+\varepsilon}.
\]

This is appreciably better than the trivial estimate
\[
|f(\alpha)| \leq \sum_{1 \leq x \leq P} 1 = P.
\]

We may infer that for \( s \geq k2^{k-1} + 1 \), one has
\[
\int_{m} f(\alpha)^s e(-n\alpha) \, d\alpha \ll (P^{1-2^{1-k}+\varepsilon})^s = o(P^{s-k}).
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\[ \int_{m} f(\alpha)^s e(-n\alpha) \, d\alpha \ll (P^{1-2^{1-k}+\varepsilon})^s = o(P^{s-k}). \]

But one has a much sharper mean value estimate:

**Theorem (Hua, 1938)**

*For each \( \varepsilon > 0 \), one has*

\[ \int_{0}^{1} |f(\alpha)|^{2^k} \, d\alpha \ll P^{2^k-k+\varepsilon}. \]
\[
\int_0^1 |f(\alpha)|^{2^k} \, d\alpha \ll P^{2^k-k+\varepsilon}.
\]

So for \( s \geq 2^k + 1 \) one has

\[
\int_m f(\alpha)^s e(-n\alpha) \, d\alpha \ll (\sup_{\alpha \in m} |f(\alpha)|)^{s-2^k} \int_0^1 |f(\alpha)|^{2^k} \, d\alpha
\]
\[
\ll P^{s-k-k\eta} \quad \text{(some } \eta > 0).\]
\[
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\ll P^{s-k-k\eta} \quad \text{(some } \eta > 0).\]

So combining this estimate with our major arc lower bound, we find that

\[
\int_0^1 f(\alpha)^s e(-n\alpha) \, d\alpha \gg n^{s/k-1} + o(n^{s/k-1})
\]

whenever \( s \geq \max\{4k, 2^k + 1\} \).
Idea

One has

\[ |f(\alpha)|^2 = \sum_{1 \leq x \leq P} \sum_{1 \leq y \leq P} e(\alpha(x^k - y^k)) \]
\[ = \sum_{|h| < P} \sum_{1 \leq x \leq P} e(\alpha hp(x; h)), \]

in which \( p(x; h) := h^{-1}((x + h)^k - x^k) \) is a polynomial of degree \( k - 1 \).

Now repeat this process in combination with Cauchy’s inequality:

\[ f(\alpha)^{2^{k-1}} \ll P^{2^{k-1} - k} \sum_{|h_1| < P} \ldots \sum_{|h_{k-1}| < P} \sum_{1 \leq x \leq P} e(\alpha h_1 \ldots h_{k-1} q(x; h)), \]

in which \( q \) is a linear polynomial in \( x \) and \( h \).
Recall:

$$f(\alpha)^{2^{k-1}} \ll P^{2^{k-1}-k} \sum |h_1| < P \sum |h_{k-1}| < P \sum_{1 \leq x \leq P} e(\alpha h_1 \ldots h_{k-1} q(x; \mathbf{h})),$$

Now observe that

$$\int_0^1 |f(\alpha)|^{2^k} d\alpha = \int_0^1 |f(\alpha)|^{2^{k-1}} \cdot |f(\alpha)|^{2^{k-1}} d\alpha \ll P^{2^{k-1}-k} \text{card} \left\{ h_1 \ldots h_{k-1} q(x; \mathbf{h}) = \sum_{j=1}^{2^{k-2}} (x_j^k - y_j^k) \right\}$$

Now isolate diagonal and non-diagonal solutions to obtain

$$\int_0^1 |f(\alpha)|^{2^k} d\alpha \ll P^{2^k-k+\varepsilon}.$$
Theorem

For $k \geq 3$ one has $G(k) \leq 2^k + 1$, and hence $g(k) < \infty$.

This argument has “good” explicit control of error terms with $2^k + 1$ $k$th powers.

Note: for fourth powers this entails using $\geq 17$ variables.

Note (Davenport, 1939)

Using a refinement of diminishing ranges, the analytic argument works for 14 fourth powers. This gives $G(4) = 16$.

Note (Vaughan, 1989)

Using a method based on exponential sums over smooth numbers (integers having only small prime divisors), the analytic argument works for 12 fourth powers.

Last two arguments — no “good” control of error terms available.
Calculating $g(k)$:

By the mid-1960’s the above methods had established that

$$g(k) = 2^k + [(3/2)^k] - 2$$

(with modifications in exceptional circumstances as described earlier)

**EXCEPT** when $k = 4$. 
Calculating $g(4)$

Advances in computers, divisor sum techniques cleared the way for:

**Theorem (Balasubramanian, Deshouillers and Dress, 1985)**

*One has* $g(4) = 19$.

Roughly speaking, analytic machinery (circle method) shows that each

$$n \geq 10^{367}$$

is the sum of 19 fourth powers.

Then use computers to check that the integers $n$ with $1 \leq n < 10^{367}$ are each represented in the shape $n = x_1^4 + \cdots + x_{19}^4$. 

Trevor D. Wooley* (University of Bristol)
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Use greedy ascent. Suppose that $1 \leq n < 10^{367}$. Choose

$$x_{19} = \left\lfloor n^{1/4} \right\rfloor$$

and put $n_1 = n - x_{19}^4$.

Then

$$n_1 \approx n^{3/4} < 10^{3/4(367)},$$

and since nearly every integer is (expected to be) the sum of 18 biquadrates, we can hope to cover exceptional cases.
Now iterate this process, taking care to keep track of the ambient (mod 16) conditions, until we are left with 5 fourth powers. The number of cases that we must check is reduced from $10^{367}$ to approximately

$$2^{14} \cdot 10^{367} \cdot (3/4)^{14} \approx 10^{11}.$$
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Problem

$g(4) = 19 \Rightarrow \textbf{all integers are the sum of 19 fourth powers.}$

$G(4) = 16 \Rightarrow \text{there is an integer } n_0 \text{ with the property that each } n > n_0$

$\text{is the sum of 16 fourth powers.}$

Enquiring minds must know ... what is $n_0$?
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Enquiring minds must know ... what is $n_0$?

Note

Balasubramanian, Deshouillers and Dress need at least 17 fourth powers to get the analytic part of their argument started.
The polynomial identity strikes back
(Joint work with Koichi Kawada)
Consider the identity
\[ x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2 \]
due to Ramanujan, Proth, ... ancient?
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\[
\text{card} \{ n \leq X : n = a^2 + ab + b^2 \} \sim \frac{X}{\sqrt{\log X}},
\]
one might optimistically suppose that
\[ x^4 + y^4 + (x + y)^4 = 2m^2 \]
(with \( m = x^2 + xy + y^2 \)) really behaves like twice a square.
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one might optimistically suppose that
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(with \(m = x^2 + xy + y^2\)) really behaves like twice a square.

Note

3 fourth powers \(\leftrightarrow\) 1 square

is a “good” rate of exchange.
Analytically, squares are much easier to handle than fourth powers — easiest to see in an application.  

Define $N(X) := \text{card} \{ n \leq X : n = x_1^4 + \cdots + x_5^4, \ x_i \in \mathbb{N} \}$. 

**Theorem (Vaughan, 1989)**  
One has $N(X) \gg X^{0.9417\ldots}$. 

**Theorem (Kawada and Wooley, 1999)**  
For each $\varepsilon > 0$, one has $N(X) \gg X(\log X)^{-1-\varepsilon}$. 
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**Theorem (Kawada and Wooley, 1999)**

*For each* \( \varepsilon > 0 \), *one has* \( N(X) \gg X(\log X)^{1-\varepsilon} \).

**Sketch of proof:** Consider the set \( C \) of integers \( n \) of the shape

\[
 n = 2m^2 + u^4 + v^4,
\]

with

\[
 1 \leq u, v \leq \frac{1}{4}X^{1/4} \quad \text{and} \quad m \in \mathcal{B},
\]

in which

\[
 \mathcal{B} = \{ m \in [1, \frac{1}{2}X^{1/2}] : m = x^2 + xy + y^2, x, y \in \mathbb{N} \}.
\]
\( \mathcal{B} = \{ m \in [1, \frac{1}{2}X^{1/2}] : m = x^2 + xy + y^2, \ x, y \in \mathbb{N} \}. \)

**Note**

If \( m \in \mathcal{B} \), then there exist integers \( x \) and \( y \) with

\[
2m^2 = 2(x^2 + xy + y^2)^2 = x^4 + y^4 + (x + y)^4.
\]

Hence when \( n \in \mathcal{C} \), there exist integers \( x, y, u, v \) for which

\[
n = x^4 + y^4 + (x + y)^4 + u^4 + v^4
\]

is a sum of 5 fourth powers.
\[ \mathcal{B} = \{ m \in [1, \frac{1}{2} X^{1/2}] : m = x^2 + xy + y^2, \ x, y \in \mathbb{N} \}. \]

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n = x^4 + y^4 + (x + y)^4 + u^4 + v^4
\]

*is a sum of 5 fourth powers.*

Let \( r(n) \) denote the number of representations of \( n \) in the above form.
Then by Cauchy’s inequality,

\[ N(X) \geq \sum_{1 \leq n \leq X} 1 \geq \frac{\left( \sum_{1 \leq n \leq X} r(n) \right)^2}{\left( \sum_{1 \leq n \leq X} r(n)^2 \right)}. \]
Then by Cauchy’s inequality,

\[ N(X) \geq \sum_{1 \leq n \leq X} 1 \geq \frac{\left( \sum_{1 \leq n \leq X} r(n) \right)^2}{\left( \sum_{1 \leq n \leq X} r(n)^2 \right)}. \]

On the one hand,

\[ \sum_{1 \leq n \leq X} r(n) \geq \sum_{1 \leq u, v \leq \frac{1}{4} X^{1/4}} \sum_{m \in B} 1 \gg (X^{1/4})^2 (X^{1/2}/\sqrt{\log X}) \gg X/\sqrt{\log X}. \]

On the other hand, as we’ll shortly show (essentially), one has

\[ \sum_{1 \leq n \leq X} r(n)^2 \ll X (\log X)^\varepsilon. \]

Thus we find that

\[ N(X) \gg \frac{(X/\sqrt{\log X})^2}{X (\log X)^\varepsilon} \gg X (\log X)^{-1-\varepsilon}. \]
One has

\[ \sum_{1 \leq n \leq X} r(n)^2 = \text{card} \{ 2m_1^2 + u_1^4 + v_1^4 = 2m_2^2 + u_2^4 + v_2^4 : \]

\[ 1 \leq u_i, v_i \leq \frac{1}{4} X^{1/4}, m_1, m_2 \in \mathcal{B} \subseteq [1, \frac{1}{2} X^{1/2}] \} . \]

Two types of solutions:
One has

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Two types of solutions:

(a) those with \( u_1^4 + v_1^4 = u_2^4 + v_2^4 \), which forces \( m_1 = m_2 \), of which there are

\[ O(X^{1/2})O((X^{1/4})^{2+\varepsilon}) \ll X^{1+\varepsilon} . \]
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\[O(X^{1/2})O((X^{1/4})^{2+\varepsilon}) \ll X^{1+\varepsilon}.\]

(b) those with \(u_1^4 + v_1^4 \neq u_2^4 + v_2^4 = h \neq 0\), say. For these one has

\[2(m_1 - m_2)(m_1 + m_2) = h \neq 0,\]

and so the number of solutions here is

\[O(X^\varepsilon) \cdot O(X) \ll X^{1+\varepsilon}.\]

So the total number of solutions is \(O(X^{1+\varepsilon})\), an estimate that may be refined with additional work to \(O(X(\log X)^\varepsilon)\).
Further consequences:

Theorem (Vaughan, 1989)

Suppose that \( s \geq 12 \) and that \( n \) is a large integer with \( n \equiv r \) (mod 16) for some integer \( r \) with \( 1 \leq r \leq s \). Then \( n \) is the sum of \( s \) fourth powers.

Theorem (Kawada and Wooley, 1999)

Suppose that \( s \geq 11 \) and that \( n \) is a large integer with \( n \equiv r \) (mod 16) for some integer \( r \) with \( 1 \leq r \leq s - 1 \). Then \( n \) is the sum of \( s \) fourth powers.

Note

The value of \( x^4 + y^4 + (x+y)^4 \) is always even, whereas \( x^4 + y^4 + z^4 \) can be either even or odd. This interferes with solubility modulo 16.
Further consequences:

Theorem (Vaughan, 1989)
Suppose that $s \geq 12$ and that $n$ is a large integer with $n \equiv r \pmod{16}$ for some integer $r$ with $1 \leq r \leq s$. Then $n$ is the sum of $s$ fourth powers.

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can be either even or odd. This interferes with solubility modulo 16.
But one can remove these difficulties with variants of the basic problem:

**Theorem (Kawada and Wooley, 1999)**

*All large integers* \( n \) *are represented in the form*

\[
n = x_1^4 + x_2^4 + \cdots + x_{10}^4 + y^{2007}.
\]
But one can remove these difficulties with variants of the basic problem:

**Theorem (Kawada and Wooley, 1999)**

All large integers $n$ are represented in the form

$$n = x_1^4 + x_2^4 + \cdots + x_{10}^4 + y^{2007}.$$

... but back to sums of 16 fourth powers ...
Strategy

Write

\[ g(\alpha) = \sum_{1 \leq x, y \leq n^{1/4}} e(2\alpha(x^2 + xy + y^2)^2) \quad \text{and} \quad f(\alpha) = \sum_{1 \leq z \leq n^{1/4}} e(\alpha z^4). \]

From Hua’s lemma we have

\[ \int_{0}^{1} |f(\alpha)|^{16} \, d\alpha \ll n^{3+\varepsilon}, \]

which saves \( n^{1-\varepsilon} \) over the trivial estimate, and uses 16 variables. But

\[ \int_{0}^{1} |g(\alpha)^2 f(\alpha)^4| \, d\alpha \sim \sum_{m} r(m)^2 \ll n^{1+\varepsilon}, \]

which saves \( n^{1-\varepsilon} \) over the trivial estimate, and uses only 10 variables. So we have saved 6 variables in the critical mean value estimates.
5. Back to $g(4)$ and $n_0$

(Joint work with J.-M. Deshouillers and K. Kawada)

We want to study sums of 16 fourth powers, but cannot afford to miss any congruence classes (mod 16) in the represented integers!

**Observation**

*Every integer $n$ with*

\[ 13,793 \leq n \leq 220,688 = 16 \times 13,793 \]

*is the sum of 16 fourth powers.*
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*Suppose now that $n > 220688$ and $16|n$.*
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**Observation**

*Every integer $n$ with*

$$13\,793 \leq n \leq 220\,688 = 16 \times 13\,793$$

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*Suppose now that $n > 220\,688$ and $16|n$. Then there exists a natural number $i$ with $16^i|n$ satisfying the condition that either*

$$13\,793 \leq n/16^i \leq 220\,688,$$

*or else*

$$n/16^i > 220\,688 \quad \text{and} \quad 16 \text{ does not divide } n/16^i.$$
Observation

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In the former case, since $n/16^i$ is a sum of 16 fourth powers, so too is

$$n = (2^i)^4(n/16^i).$$
Observation

Every integer \( n \) with

\[
13\,793 \leq n \leq 220\,688 = 16 \times 13\,793
\]

is the sum of 16 fourth powers. Suppose now that \( n > 220\,688 \) and \( 16 | n \). Then there exists a natural number \( i \) with \( 16^i | n \) satisfying the condition that either

\[
13\,793 \leq n/16^i \leq 220\,688,
\]

or else

\[
n/16^i > 220\,688 \quad \text{and} \quad 16 \text{ does not divide } n/16^i.
\]

In the former case, since \( n/16^i \) is a sum of 16 fourth powers, so too is

\[
n = (2^i)^4(n/16^i).
\]

In the latter case, it suffices to represent \( n/16^i \) as the sum of 16 fourth powers, in which

\[
n/16^i \not\equiv 0 \pmod{16}.
\]
Conclusion

It suffices to represent only those integers $n$ with $n \equiv r \pmod{16}$ for $1 \leq r \leq 15$.

Strategy

Try using one copy of the identity (which loses one congruence class modulo 16). So seek representations in the shape

$$n = x^4 + y^4 + (x + y)^4 + \sum_{j=1}^{13} z_j^4.$$ 

The corresponding integral formulation is

$$\int_0^1 g(\alpha) f(\alpha)^{13} e(-n\alpha) \, d\alpha.$$
**Major arcs** (easy part)

\[ \int_{\mathcal{M}} g(\alpha) f(\alpha)^{13} e(-n\alpha) \, d\alpha \gg n^{11/4}. \]

**Minor arcs** (hard part)

\[ \int_{\mathcal{M}} g(\alpha) f(\alpha)^{13} e(-n\alpha) \, d\alpha \leq \left( \sup_{\alpha \in \mathcal{M}} |f(\alpha)| \right)^3 \left( \int_{0}^{1} |g(\alpha)^2 f(\alpha)^4| \, d\alpha \right)^{1/2} \]
\[ \quad \times \left( \int_{0}^{1} |f(\alpha)|^{16} \, d\alpha \right)^{1/2} \]
\[ \ll n^{11/4 - 3/8 + \varepsilon}. \]

A careful analysis based on this discussion shows that whenever \( n \equiv r \pmod{16} \) and \( 1 \leq r \leq 15 \), and

\[ n > 10^{356}, \]

then \( n \) is the sum of 16 biquadrates.
Major arcs (easy part)

\[ \int_{\mathcal{M}} g(\alpha)f(\alpha)^{13} e(-n\alpha) \, d\alpha \gg n^{11/4}. \]

Minor arcs (hard part)

\[ \int_{\mathcal{M}} g(\alpha)f(\alpha)^{13} e(-n\alpha) \, d\alpha \leq \left( \sup_{\alpha \in \mathcal{M}} |f(\alpha)| \right)^3 \left( \int_0^1 |g(\alpha)^2 f(\alpha)^4| \, d\alpha \right)^{1/2} \times \left( \int_0^1 |f(\alpha)|^{16} \, d\alpha \right)^{1/2} \ll n^{11/4-3/8+\varepsilon}. \]

A careful analysis based on this discussion shows that whenever \( n \equiv r \) (mod 16) and \( 1 \leq r \leq 15, \) and

\[ n > 10^{356}, \]

then \( n \) is the sum of 16 biquadrates. But ...

\[ 10^{356} \rightarrow \text{greedy algorithm} \rightarrow 2^{11} \cdot 10^{(3/4)^{11\cdot356}} \approx 2 \times 10^{18} \ (\text{too big!}). \]
6. The polynomial identity strikes back — again!

Observation

One has

\[(w + x)^4 + (w - x)^4 + (w + y)^4 + (w - y)^4 \]
\[+ (w + x + y)^4 + (w - x - y)^4 = 4(x^2 + xy + y^2 + 3w^2)^2 - 30w^4.\]

So

6 fourth powers \(\sim\) 1 square and 1 fourth power
6. The polynomial identity strikes back — again!

**Observation**

*One has*

\[(w + x)^4 + (w - x)^4 + (w + y)^4 + (w - y)^4 + (w + x + y)^4 + (w - x - y)^4 = 4(x^2 + xy + y^2 + 3w^2)^2 - 30w^4.\]

So

6 fourth powers \(\sim\) 1 square and 1 fourth power

5 fourth powers \(\sim\) 1 square.

This is less efficient than the previous exchange rate of 3 fourth powers to 1 square, but in compensation there are no lost congruence classes.
Define

\[ G(\alpha) := \sum_{x, y, w \sim n^{1/4}} e(\alpha(4(x^2 + xy + y^2 + 3w^2)^2 - 30w^4)) \]

Then

\[ \int_0^1 |G(\alpha)^2 f(\alpha)^2| \, d\alpha \ll n^{1+\varepsilon} \quad (14 \text{ fourth powers}). \]

Compare this with

\[ \int_0^1 |g(\alpha)^2 f(\alpha)^4| \, d\alpha \ll n^{1+\varepsilon} \quad (10 \text{ fourth powers}) \]

and

\[ \int_0^1 |f(\alpha)|^{16} \, d\alpha \ll n^{3+\varepsilon} \quad (16 \text{ fourth powers}). \]
Strategy

Consider representations of $n$ in the form

$$n = (w + x)^4 + (w - x)^4 + (w + y)^4 + (w - y)^4$$
$$+ (w + x + y)^4 + (w - x - y)^4 + (u + v)^4 + u^4 + v^4$$
$$+ z_1^4 + \cdots + z_7^4.$$ 

This has integral representation

$$\int_0^1 G(\alpha)g(\alpha)f(\alpha)^7 e(-n\alpha) d\alpha.$$
**Major arcs** (easy):

\[
\int_{\mathcal{M}} G(\alpha)g(\alpha)f(\alpha)^7 e(-n\alpha) \, d\alpha \gg n^2.
\]

**Minor arcs** (hard):

\[
\int_{\mathcal{m}} G(\alpha)g(\alpha)f(\alpha)^7 e(-n\alpha) \, d\alpha \leq \left( \sup_{\alpha \in \mathcal{m}} |f(\alpha)| \right)^4 \left( \int_0^1 |G(\alpha)^2 f(\alpha)^2| \, d\alpha \right)^{1/2}
\times \left( \int_0^1 |g(\alpha)^2 f(\alpha)^4| \, d\alpha \right)^{1/2}
\ll n^{2-4/8+\varepsilon}.
\]

Now make all the estimates explicit (non-trivial divisor sum estimates, estimates for complete exponential sums etc.)
Theorem (Deshouillers, Kawada and Wooley, 2005)

Suppose that $n \geq 10^{216}$. Then $n$ is the sum of 16 fourth powers.

Note

Although $10^{216}$ may appear a formidable number of integers to check for representability, if we use the “greedy” algorithm, this scales down to something like

$$2^{11} \cdot 10^{(3/4) \cdot 216} \approx 3 \times 10^{12},$$

which is far more manageable.
Theorem (Deshouillers, Kawada and Wooley, 2005)

Suppose that \( n \geq 10^{216} \). Then \( n \) is the sum of 16 fourth powers.

Note

Although \( 10^{216} \) may appear a formidable number of integers to check for representability, if we use the “greedy” algorithm, this scales down to something like

\[
2^{11} \cdot 10^{(3/4)^{11} \cdot 216} \approx 3 \times 10^{12},
\]

which is far more manageable.

Theorem (Deshouillers, Hennecart and Landreau, 2000)

When \( 13793 \leq n < 10^{245} \), then \( n \) is the sum of 16 fourth powers.
Theorem (Deshouillers, Kawada and Wooley, 2005)

Suppose that \( n \geq 10^{216} \). Then \( n \) is the sum of 16 fourth powers.

Note

Although \( 10^{216} \) may appear a formidable number of integers to check for representability, if we use the “greedy” algorithm, this scales down to something like

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which is far more manageable.

Theorem (Deshouillers, Hennecart and Landreau, 2000)

When \( 13793 \leq n < 10^{245} \), then \( n \) is the sum of 16 fourth powers.

Combining these conclusions, we obtain:

Theorem (Deshouillers, Hennecart, Kawada, Landreau, Wooley, 2005)

All integers exceeding 13,792 can be written as a sum of 16 fourth powers.
The 96 exceptional integers

(1) The 7 integers that are not $B_{18}$ are

$$79 + 80k \quad (k = 0, 1, \ldots, 6)$$

(2) The 24 integers that are $B_{18}$ but not $B_{17}$ are

$$63 + 80k \quad (k = 0, \ldots, 14)$$
$$78 + 80k \quad (k = 0, \ldots, 6)$$
$$48 + 80k \quad (k = 12, 15)$$

(3) The 65 integers that are $B_{17}$ but not $B_{16}$ are

$$47 + 80k \quad (k = 0, \ldots, 22, 46)$$
$$62 + 80k \quad (k = 0, \ldots, 14)$$
$$77 + 80k \quad (k = 0, \ldots, 6)$$
$$32 + 80k \quad (k = 9, 12, 15, 25, 28, 44, 47, 57, 60, 79, 89, 108, 137, 172)$$
$$48 + 80k \quad (k = 44, 47, 76, 79)$$
$$64 + 80k \quad (k = 31)$$