Parameter dependent invariant measures for IFS - dimension and absolute continuity

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arXiv:2107.03692

UK One Day Ergodic Theory Meeting

15th September 2021

Invariant measures for Iterated Function Systems (IFS)

IFS: collection $f_1, ..., f_m : X \to X$ of contractions on a compact set $X \subset \mathbb{R}^d$

(soon: X - compact interval and f_j - $C^{2+\delta}$ hyperbolic)

Attractor: a non-empty compact set $A \subset X$ with

$$A = f_1(A) \cup \ldots \cup f_m(A)$$

(it exists and is unique)

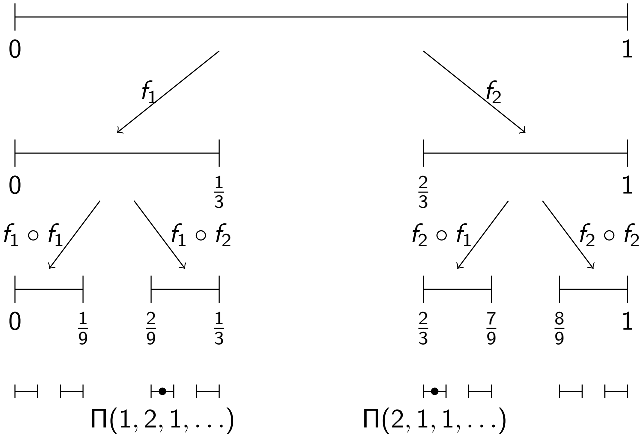
Symbolic space: $\Omega := \mathcal{A}^{\mathbb{N}}$, where $\mathcal{A} = \{1, \ldots, m\}$

Natural projection: $\Pi : \Omega \to X$,

$$\Pi(\omega) := \bigcap_{n=1}^{\infty} f_{\omega_1} \circ \cdots \circ f_{\omega_n}(X)$$

where $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$. **Then** $A = \Pi(\Omega)$.

The middle-thirds Cantor set: $X = [0,1],\ f_1(x) = \frac{1}{3}x,\ f_2(x) = \frac{1}{3}x + \frac{2}{3}.$



Invariant measures: let μ be (a shift-invariant, ergodic) finite measure on Ω. We are interested in properties of the projected measure $\Pi_*\mu$ on A.

E.g. if μ is the Bernoulli measure $(p_1, \ldots, p_m)^{\otimes \mathbb{N}}$, then $\nu = \Pi_*\mu$ is the unique probability measure satisfying

$$\nu = \sum_{j=1}^{m} p_j (f_j)_* \nu,$$

i.e. it is the stationary measure for the Markov process on X generated by applying f_j with probability p_j.

Questions

$$\dim_H(A) = ?,\ \dim_H(\Pi_*\mu) = ?,\ \text{is } \Pi_*\mu \text{ absolutely continuous ?}$$

Recall:

$$\dim_H(\nu) = \operatorname*{ess\,sup}_{x\sim\nu} \underline{d}(\nu, x) := \operatorname*{ess\,sup}_{x\sim\nu} \liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r}.$$

Natural upper bound for ergodic μ on Ω and $C^{1+\delta}$ IFS on interval:

$$\dim_H(\Pi_*\mu) \leq \min\left\{1, \frac{h_\mu}{\chi_\mu}\right\}$$

where h_μ - entropy of μ (w.r.t to shift on Ω) and χ_μ - Lyapunov exponent

$$\chi_\mu = -\int_\Omega \log|f'_{\omega_1}(\Pi(\sigma\omega))| d\mu(\omega)$$

Heurestics: Assume first $f_i(X) \cap f_j(X) = \emptyset$ for $i \neq j$. Let $\nu = \Pi_* \mu$ and pick a μ-typical $\omega \in \Omega$

$$\underline{d}(\nu, \Pi(\omega)) \approx \frac{\log \nu(|f_{\omega_1 \ldots \omega_n}(X)|)}{\log |f_{\omega_1 \ldots \omega_n}(X)|} = \frac{\log \mu([\omega_1, \ldots, \omega_n])}{\log |f_{\omega_1 \ldots \omega_n}(X)|} \approx \frac{\log e^{-nh_\mu}}{\log e^{-n\chi_\mu}} = \frac{h_\mu}{\chi_\mu}$$

If there are **overlaps** between cylinders, then

$$\nu(|f_{\omega_1 \ldots \omega_n}(X)|) \geq \mu([\omega_1, \ldots, \omega_n]),$$

hence we only get an upper bound on the dimension and the dimension can drop

However in many *families* of iterated function systems with overlaps, the dimension formula holds *generically*

Transversality

$U \subset \mathbb{R}^d$ - open and bounded *parameter space*.

For each $\lambda \in U$ consider an IFS $\{f_j^\lambda\}_{j\in\mathcal{A}}$ on a compact interval X and the corresponding natural projection $\Pi^\lambda : \Omega \to X$

Definition

The family $\{f_j^\lambda\}_{j\in\mathcal{A}}$ satisfies the transversality condition on U if there exists $C > 0$ s.t. for every $r > 0$ and $\omega, \tau \in \Omega$ with $\omega_1 \neq \tau_1$

$$\mathcal{L}^d\left(\left\{\lambda \in U : |\Pi^\lambda(\omega) - \Pi^\lambda(\tau)| \leq r\right\}\right) \leq Cr$$

Theorem (Simon, Solomyak, Urbański)

Let f_j^λ be $C^{1+\delta}$ with $0 < \gamma_1 \leq |\frac{d}{dx} f_j^\lambda(x)| \leq \gamma_2 < 1$ and $\lambda \mapsto f_j^\lambda \in C^{1+\delta}$ continuous. Let μ be an ergodic shift-invariant prob. measure on Ω. If the tranversality holds on U, then for Lebesgue a.e. $\lambda \in U$

- $\dim_H(\Pi_*^\lambda \mu) = \min\{1, \frac{h_\mu}{\chi_\mu}\}$,
- $\Pi_*^\lambda \mu$ is absolutely continuous if $\frac{h_\mu}{\chi_\mu} > 1$.

Our setting

We allow the measure on the symbolic space to *also depend on the parameter*, i.e. we study a family of projected measures

$$\Pi^{\lambda}_{*}\mu_{\lambda}, \ \lambda \in U,$$

where μ_λ are probability measures on Ω.

Under suitable regularity assumptions on the IFS and the family μ_λ, we obtain an analog of the previous theorem.

Main difficulties are in the absolute continuity part.

General examples / motivations

1. Stationary measures for place-dependent probabilities

Let $\{f_j^\lambda\}_{j\in\mathcal{A}}$ be an IFS on interval X and let $p_j : X \to (0,1)$ be the probability functions satisfying $\sum_{j=1}^{m} p_j(x) \equiv 1$. A *place-dependent stationary measure* ν_λ on X is one satisfying

$$\int \varphi(x) d\nu_\lambda(x) = \int \sum_{j\in\mathcal{A}} p_j(x)\varphi(f_i^\lambda(x)) d\nu_\lambda(x)$$

for any continuous test function φ.

If p_j are Hölder continuous, then ν_λ is unique and $\nu_\lambda = \Pi^\lambda_* \mu_\lambda$, where μ_λ is a Gibbs measure of the potential

$$\phi^\lambda(\omega) = \log |p_{\omega_1}(\Pi^\lambda(\sigma\omega))|,$$

i.e. there exists $P_\lambda \in \mathbb{R}$ and $C_\lambda \geq 1$ such that for every $\omega \in \Omega$ and $n \in \mathbb{N}$

$$C_\lambda^{-1} \leq \frac{\mu([\omega|_n])}{\exp(-P_\lambda n + \sum\limits_{k=0}^{n-1} \phi^\lambda(\sigma^k \omega))} \leq C_\lambda.$$

Note that ϕ^λ (and hence μ_λ) depends on the parameter, even if p_j's do not!

Typical dimension formula and absolute continuity of such place-dependent invariant measures were obtained by Balázs Bárány[1]. Unfortunately, the proof contains an error (see Corrigendum on Balázs' webpage).

[1] On Iterated Function Systems with place-dependent probabilities, *Proc. Amer. Math. Soc.* 143 no. 1 (2015), 419-432.

2. Equilibrium measures. Consider the pressure function

$$P_\lambda(t) = \lim_{n\to\infty} n^{-1} \log \sum_{\omega\in\mathcal{A}^n} \left\| \frac{d}{dx} f^\lambda_\omega \right\|^t$$

and the corresponding roots $s_\lambda > 0$, i.e. solutions of the Bowen's equation

$$P_\lambda(s_\lambda) = 0.$$

s_λ is the "natural guess" for the dimension of the attractor (and in general an upper bound for it).

The "natural" measure ν_λ is the projection $\nu_\lambda = \Pi^\lambda_* \mu_\lambda$ of the equilibrium measure μ_λ, i.e. the Gibbs measure of the potential

$$\phi^\lambda(\omega) = s_\lambda \log \left| \frac{d}{dx} f^\lambda_{\omega_1}(\Pi^\lambda(\sigma\omega)) \right|.$$

It satisfies

$$s_\lambda = \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}}$$

Assumptions on the IFS

IFS $\{f_j^\lambda\}_{j\in\mathcal{A}}$ on a **compact interval** $X \subset \mathbb{R}$ with $\lambda \in \overline{U} \subset \mathbb{R}$, where U is an **open and bounded interval**. We assume that there exists $\delta \in (0,1]$ such that

(A1) the maps f_j^λ are $C^{2+\delta}$-smooth on X (uniformly w.r.t. λ)

(A2) the maps $\lambda \mapsto f_j^\lambda(x)$ are $C^{1+\delta}$-smooth on U (uniformly w.r.t. x)

(A3) the second partial derivatives $\frac{d^2}{dx d\lambda} f_j^\lambda(x)$, $\frac{d^2}{d\lambda dx} f_j^\lambda(x)$ are δ-Hölder (uniformly, both in λ and x)

(A4) the system $\{f_j^\lambda\}_{j\in\mathcal{A}}$ is *uniformly hyperbolic and contractive*: there exists $\gamma_1,\ \gamma_2 > 0$ such that

$$0 < \gamma_1 \leq |(\tfrac{d}{dx} f_j^\lambda)(x)| \leq \gamma_2 < 1$$

Assumptions on the measures

Let $\{\mu_\lambda\}_{\lambda \in \overline{U}}$ be a collection of finite Borel measures on Ω. We will consider two continuity assumptions on μ_λ:

(M0) for every λ_0 and every $\varepsilon > 0$ there exist $C, \xi > 0$ such that

$$C^{-1} e^{-\varepsilon|\omega|} \mu_{\lambda_0}([\omega]) \leq \mu_\lambda([\omega]) \leq C e^{\varepsilon|\omega|} \mu_{\lambda_0}([\omega])$$

holds for every $\omega \in \Omega^*,\ |\omega| \geq 1$ and $\lambda \in \overline{U}$ with $|\lambda - \lambda_0| < \xi$;

(M) there exists $c > 0$ and $\theta \in (0, 1]$ such that for all $\omega \in \Omega^*,\ |\omega| \geq 1$, and all $\lambda, \lambda' \in \overline{U}$,

$$e^{-c|\lambda - \lambda'|^\theta |\omega|} \mu_{\lambda'}([\omega]) \leq \mu_\lambda([\omega]) \leq e^{c|\lambda - \lambda'|^\theta |\omega|} \mu_{\lambda'}([\omega]).$$

Example: μ_λ - Gibbs measures of Hölder potentials ϕ^λ

- if $\lambda \mapsto \phi^\lambda$ is continuous, then (M0) holds
- if $\lambda \mapsto \phi^\lambda$ is Hölder, then (M) holds

Main results

From now on, we always assume that $\{f_j^\lambda\}_{j\in\mathcal{A}}$ is a parametrized IFS satisfying smoothness assumptions (A1) - (A4) and the transversality condition (T) on U.

Theorem 1.

Let $\{\mu_\lambda\}_{\lambda\in\overline{U}}$ be a collection of finite ergodic shift-invariant Borel measures on Ω satisfying (M0), such that h_{μ_λ} and χ_{μ_λ} are continuous in λ. Then equality

$$\dim_H(\Pi_*^\lambda \mu_\lambda) = \min\left\{1, \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}}\right\}$$

holds for Lebesgue almost every $\lambda \in U$.

This was essentially proved by Balázs Bárány and Michał Rams[2] in a more specific context

[2]Dimension maximizing measures for self-affine systems, *Trans. Amer. Math. Soc.* 370 (2018), 553-576.

Theorem 3.

Let $\{\mu_\lambda\}_{\lambda\in\overline{U}}$ be a family of Gibbs measures on Ω corresponding to a family of potentials $\phi^\lambda\colon \Omega \mapsto \mathbb{R}$ which are uniformly Hölder. Moreover, suppose that there exist constants $c_0 > 0$ and $\theta > 0$ such that

$$|\phi^\lambda(\omega) - \phi^{\lambda'}(\omega)| \leq c_0|\lambda - \lambda'|^\theta \text{ for every } \omega \in \Omega \text{ and } \lambda, \lambda' \in \overline{U}. \qquad (1)$$

Then $\{\mu_\lambda\}_{\lambda\in\overline{U}}$ satisfies (M) and $\Pi^\lambda_*\mu_\lambda$ is absolutely continuous for Lebesgue almost every λ in the set $\{\lambda \in U : \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}} > 1\}$.

Sketch of the proof of Theorem 1.

Tools

Correlation dimension: for a finite Borel measure μ on a metric space X and $\alpha > 0$, let

$$\mathcal{E}_\alpha(\mu, d) = \int\limits_X \int\limits_X \frac{1}{d(x,y)^\alpha} d\mu(x) d\mu(y)$$

be the α-**energy** of μ. The **correlation dimension** of μ is

$$\dim_{cor}(\mu, d) = \sup\{\alpha > 0 : \mathcal{E}_\alpha(\mu, d) < \infty\}.$$

Fact

$\dim_{cor}(\mu, d) \leq \dim_H(\mu, d)$

Metric on Ω: for $\lambda \in U$ and $\omega, \tau \in \Omega$ define

$$d_\lambda(\omega, \tau) = |f^\lambda_{\omega \wedge \tau}(X)|.$$

In this metric, $\Pi^\lambda : \Omega \to \mathbb{R}$ is Lipschitz and for an ergodic shift-invariant measure μ one has

$$\dim_H(\mu, d_\lambda) = \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}}$$

We are proving:

$$\dim_H(\Pi^\lambda_* \mu_\lambda) = \min\{1, \dim_H(\mu_\lambda, d_\lambda)\}$$

Sketch of the proof

Goal 1.: $\dim_{cor}(\Pi^{\lambda}_{*}\mu_{\lambda}) \geq \min\{1, \dim_{cor}(\mu_{\lambda}, d_{\lambda})\}$ for a.e. $\lambda \in U$

Classical transversality argument:

$$\dim_{cor}(\Pi^{\lambda}_{*}\mu_{\lambda_0}) \geq \min\{1, \dim_{cor}(\mu_{\lambda_0}, d_{\lambda})\} \text{ for a.e. } \lambda \in U$$

Enough: map $\lambda \mapsto \dim_{cor}(\Pi_{*}\mu_{\lambda})$ is continuous, uniformly for Π in our family
(and $\lambda \mapsto \dim_{cor}(\mu_{\lambda}, d)$ as well)

Then: for every $\lambda_0 \in U$ and $\varepsilon > 0$, there exists an open neighbourhood V of λ_0 such that for a.e. $\lambda \in V$

$$\dim_{cor}(\Pi^{\lambda}_{*}\mu_{\lambda}) \geq \dim_{cor}(\Pi^{\lambda}_{*}\mu_{\lambda_0}) - \varepsilon \geq \min\{1, \dim_{cor}(\mu_{\lambda_0}, d_{\lambda})\} - \varepsilon$$

$$\geq \min\{1, \dim_{cor}(\mu_{\lambda}, d_{\lambda})\} - 2\varepsilon$$

Continuity of $\lambda \mapsto \dim_{cor}(\Pi_* \mu_\lambda)$

Enough: for λ_0 and $\varepsilon > 0$ there exists a neighbourhood V of λ_0 such that $\mathcal{E}_\alpha(\Pi_* \mu_\lambda) \lesssim \mathcal{E}_{\alpha+\varepsilon}(\Pi_* \mu_{\lambda_0})$ for $\lambda \in V$

$$\mathcal{E}_\alpha(\Pi_* \mu_\lambda) = \int\limits_\Omega \int\limits_\Omega |\Pi(\omega) - \Pi(\tau)|^{-\alpha} d\mu_\lambda(\omega) d\mu_\lambda(\tau)$$

$$\approx \sum_{n=0}^{\infty} 2^{\alpha n} \mu_\lambda \otimes \mu_\lambda(\{|\Pi(\omega) - \Pi(\tau)| \leq 2^{-n}\})$$

$$\approx \sum_{n=0}^{\infty} 2^{\alpha n} \mu_\lambda \otimes \mu_\lambda(\{|\Pi(\omega|_{qn} 1^\infty) - \Pi(\tau|_{qn} 1^\infty)| \leq 2^{-n}\})$$

$$\lesssim \sum_{n=0}^{\infty} 2^{(\alpha+\varepsilon)n} \mu_{\lambda_o} \otimes \mu_{\lambda_0}(\{|\Pi(\omega|_{qn} 1^\infty) - \Pi(\tau|_{qn} 1^\infty)| \leq 2^{-n}\})$$

$$\approx \mathcal{E}_{\alpha+\varepsilon}(\Pi_* \mu_{\lambda_0})$$

We have proved:

$$\dim_H(\Pi_*^\lambda \mu_\lambda) \geq \dim_{cor}(\Pi_*^\lambda \mu_\lambda) \geq \min\{1, \dim_{cor}(\mu_\lambda, d_\lambda)\}$$

Remains: having $\min\{1, \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}}\}$ as the lower bound

Applying Egorov's theorem to the convergence in SMB and Birkhoff's theorems, one has

$$\dim_{cor}(\mu_\lambda|_A, d_\lambda) \geq \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}} - \varepsilon$$

on a set A of almost full μ_λ measure. To conclude the proof, one can repeat the reasoning for the restricted measures.

Absolute continuity?

For a **fixed** measure, almost sure absolute continuity is proved by showing

$$\int_U \int_{\mathbb{R}} \underline{D}(\Pi_*^\lambda \mu_{\lambda_0}, x) d\Pi_*^\lambda \mu_{\lambda_0}(x) d\lambda$$

$$\leq \liminf_{r \to 0} \frac{1}{2r} \int_U \int_{\mathbb{R}} \Pi_*^\lambda \mu_{\lambda_0}(B(x,r)) d\Pi_*^\lambda \mu_{\lambda_0}(x) d\lambda < \infty.$$

Transversality is used to obtain, roughly speaking,

$$\int_U \int_{\mathbb{R}} \Pi_*^\lambda \mu_{\lambda_0}(B(x,r)) d\Pi_*^\lambda \mu_{\lambda_0}(x) d\lambda \leq Cr$$

Problem in our case: using the previous approach, we only obtain

$$\int_V \int_{\mathbb{R}} \Pi_*^\lambda \mu_\lambda(B(x,r)) d\Pi_*^\lambda \mu_\lambda(x) d\lambda$$

$$\lesssim r^{-\varepsilon} \int_V \int_{\mathbb{R}} \Pi_*^\lambda \mu_{\lambda_0}(B(x,r)) d\Pi_*^\lambda \mu_{\lambda_0}(x) d\lambda \leq C r^{1-\varepsilon},$$

so we do not get the finiteness of the integral.

Sobolev dimension

Sobolev energy:

$$\mathcal{I}_\alpha(\nu) := \int_{\mathbb{R}} |\hat{\nu}(\xi)|^2 |\xi|^{\alpha-1} d\xi$$

Sobolev dimension:

$$\dim_S(\nu) := \sup\{\alpha > 0 : \mathcal{I}_\alpha(\nu) < \infty\}$$

- if $\dim_S(\nu) > 1$, then $\hat{\nu} \in L^2(\mathbb{R})$
- hence, **if** $\dim_S(\nu) > 1$ **then** ν **is absolutely continuous**
- for $\alpha \in (0,1)$ we have $\mathcal{I}_\alpha(\nu) = c_\alpha \mathcal{E}_\alpha(\nu)$
- (hence $\dim_{cor}(\nu) = \dim_S(\nu)$ provided $0 < \dim_S(\nu) < 1$)
- unfortunately, equality of energies does not extend to $\alpha \geq 1$

Recall:

(M) there exists $c > 0$ and $\theta \in (0,1]$ such that for all $\omega \in \Omega^*$, $|\omega| \geq 1$, and all $\lambda, \lambda' \in \overline{U}$,

$$e^{-c|\lambda-\lambda'|^\theta |\omega|}\mu_{\lambda'}([\omega]) \leq \mu_\lambda([\omega]) \leq e^{c|\lambda-\lambda'|^\theta |\omega|}\mu_{\lambda'}([\omega]).$$

Theorem 2.

Let $\{\mu_\lambda\}_{\lambda \in \overline{U}}$ be a collection of finite Borel measures on Ω satisfying (M). Then

$$\dim_S(\Pi^\lambda_* \mu_\lambda) \geq \min\left\{\dim_{cor}(\mu_\lambda, d_\lambda), 1 + \min\{\delta, \theta\}\right\}$$

holds for Lebesgue almost every $\lambda \in U$. Consequently, $\Pi^\lambda_* \mu_\lambda$ is absolutely continuous with a density in L^2 for Lebesgue almost every λ in the set $\{\lambda \in U : \dim_{cor}(\mu_\lambda, d_\lambda) > 1\}$.

A fixed measure version of this theorem follows from results of Peres and Schlag

Instead of the usual energy integrals, one can rely on the decomposition obtained from the **Littlewood-Paley function**:

$\psi : \mathbb{R} \to \mathbb{R}$ of Schwarz class, with $\widehat{\psi} \geq 0$ and

$$\mathrm{supp}(\widehat{\psi}) \subset \{\xi : \ 1 \leq |\xi| \leq 4\}, \qquad \sum_{j \in \mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1 \ \text{ for all } \xi \neq 0.$$

$$\mathcal{I}_\alpha(\nu) \asymp \sum_{n=0}^{\infty} 2^{\alpha n} \int_\Omega \int_\Omega \psi\big(2^n(\Pi(\omega_1) - \Pi(\omega_2))\big)\, d\mu(\omega_1)\, d\mu(\omega_2)$$

Difficulties: ψ is not non-negative, so using bounds on the measures to change $\lambda \to \lambda_0$ requires extra care

$\lambda \mapsto \dim_S(\Pi_* \mu_\lambda)$ is no longer continuous

Our strategy:

$$\int_V \mathcal{I}_\alpha(\nu_\lambda) d\lambda$$

$$\approx \sum_{n=0}^{\infty} 2^{\alpha n} \int_{\mathbb{R}} \int_{\Omega} \int_{\Omega} \psi\big(2^n(\Pi^\lambda(\omega_1) - \Pi^\lambda(\omega_2))\big)\, d\mu_\lambda(\omega_1)\, d\mu_\lambda(\omega_2) d\lambda$$

$$\approx \sum_{n=0}^{\infty} 2^{\alpha n} \int_{\mathbb{R}} \int_{\Omega} \int_{\Omega} \psi\big(2^n(\Pi^\lambda(\omega_1) - \Pi^\lambda(\omega_2))\big)\, e_n(\omega_1, \omega_2, \lambda)\, d\mu_{\lambda_0}(\omega_1)\, d\mu_{\lambda_0}(\omega_2) d\lambda$$

We extended the proof of Peres and Schlag to apply transversality for the modified kernel $\psi(2^n \cdot) e_n(\cdot)$.

This requires certain regularity of $e_n(\cdot)$, coming from condition (M).

Theorem 3. (almost sure absolute continuity for projections of Gibbs measures in the region $\{\lambda \in U : \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}} > 1\}$)

follows from

Theorem 2. (almost sure absolute continuity in the region $\{\lambda \in U : \dim_{cor}(\mu_\lambda, d_\lambda) > 1\}$)

by finding $A \subset \Omega$ such that restrictions $\mu_\lambda|_A$ satisfy both $\dim_{cor}(\mu_\lambda|_A, d_\lambda) > \frac{h_{\mu_\lambda}}{\chi_{\mu_\lambda}} - \varepsilon$ and property (M)

We construct it using the Large Deviations Principle for Gibbs measures.

Applications

- absolute continuity for place-dependent Bernoulli convolutions ($\Rightarrow$ a.c. of SRB measures for certain modified fat baker's maps)
- a.c. of Blackwell measures for binary channels (transversality obtained by Bárány and Kolossváry)
- equilibrium measures for IFS satisfying (A1) - (A4) and transversality
- in particular: equilibrium measure for non-homogenous self-similar IFS

$$\{x \mapsto \lambda_1 x, x \mapsto \lambda_2 x + 1\}$$

  is absolutely continuous for a.e. $(\lambda_1, \lambda_2) \in (0,1)^2$ such that $\lambda_1 + \lambda_2 > 1$ and $\max\{\lambda_1, \lambda_2\} \leq 0.668$ (trasnversality by Ngai-Wang and Neunhäuserer)
- some hyperbolic random continued fractions:

$$\left\{f_1^{\alpha}, f_2^{\beta}\right\} = \left\{\frac{x+\alpha}{x+\alpha+1}, \frac{x+\beta}{x+\beta+1}\right\}$$

  for $\alpha \in (0, 10^{-4}]$ and $\beta = \sqrt{2} - 1$, the equilibrium measure $\nu_{\alpha,\beta+\lambda}$ is absolutely continuous for a.e. $\lambda \in (0, 0.077)$

Thank you for your attention!