Expanding measures and random walks on homogeneous spaces

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What is in this talk?

1. Introduction: Ratner and Benoist–Quint Theorems
2. H-expanding measures
3. Measure classification
4. Equidistribution and orbit closures
5. Proof roadmap: height functions and countability
6. Diagonal flows

The abstract problem

Let X be a locally compact second countable space and suppose $\Gamma \curvearrowright X$ by homeomorphisms.

Problems:

1. *(Orbit closure description)* Describe the orbit closures $\overline{\Gamma x} \subseteq X$ <u>for every</u> $x \in X$.
2. *(Equidistribution)* Describe how does $\Gamma.x$ densify in $\overline{\Gamma.x}$.
3. *(Measure classification)* Describe the Γ-invariant (probability) measures on X.

Some examples

0) $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ by shift. In this case, all three questions are intractable.

Odd cases

1) $X = \mathbb{T}^d$ and $\mathbb{Z} \curvearrowright X$ by irrational translation. In this case: question 1. is elementary and 2. 3. can be deduced by Weyl equidistribution criterion and ergodic theorem.

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3) Let X be the unit tangent bundle of a compact hyperbolic surface $S \simeq \mathrm{PSL}_2(\mathbb{R})/\Lambda$. Consider the horocycle flow on X. This corresponds to the action of $\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ by left-multiplication on $\mathrm{PSL}_2(\mathbb{R})/\Lambda$.

The answers are similar to 1). (Hedlund, Furstenberg, Margulis, Dani, Smillie).

Even cases

2) $X = \mathbb{T}^d$ and $g \in \mathrm{SL}_d(\mathbb{Z})$ with no eigenvalue of modulus one. e.g. $d = 2$ and $g = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

There is a copy of 0) in this system.

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4) Consider geodesic flow g_t on X. It corresponds to the action of $\begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$ by left-multiplication on $\mathrm{PSL}_2(\mathbb{R})/\Lambda$. (Anosov)

Ratner's theorems

In all cases above, the space $X = G/\Lambda$ for some Lie group G and lattice $\Lambda < G$ and $G > \Gamma \curvearrowright X$.

The odd cases where we can answer questions 1. 2. and 3. correspond to the situation where the element u defining the $\mathbb{Z}$ action has the property that $\mathrm{Ad}(u) \in \mathrm{GL}(\mathfrak{g})$ is unipotent.

In fact, they all correspond to particular cases of Ratner's theorems.

Ratner's theorems

Theorem (Ratner)

Let G be a Lie group, $\Lambda < G$ a lattice and $\Gamma < G$ a subgroup generated by 1-parameter Ad*-unipotent elements. Then,*

1. *(Orbit closures) For every $x \in G/\Lambda$, $\overline{\Gamma x}$ is a finite volume homogeneous space, i.e. there exists a closed group $G > L > \Gamma$ such that $\overline{\Gamma x} = Lx$ and Lx admit a L-invariant probability measure ν_x.*

2. *(Equidistribution) Let u_s be a 1-parameter* Ad*-unipotent subgroup. Then, for every $x \in G/\Lambda$, for every $\phi \in C_b(G/\Lambda)$,*
$\frac{1}{T}\int_0^T \phi(u_s x)ds \to \int \phi(y)d\nu_x(y)$ *as $T \to \infty$.*

3. *(Measure classification) Every Γ-invariant and ergodic probability probability measure is homogeneous: there exists a closed group $L > \Gamma$ and $x \in G/\Lambda$ such that ν is L-invariant and $\nu(Lx) = 1$.*

Ratner's theorems: extensions, applications

There is a lot to comment on this theorem, its extensions, versions (by Shah, Mozes, Margulis, Tomanov, Weiss and many others) and applications (and motivations) in number theory and geometry.

A conjecture

To move on with the results of Benoist–Quint, here is a conjecture I learned from Jean-François Quint:

Conjecture

Ratner's theorem 1. and 3. hold when the assumption on Γ *("generated by 1-parameter* Ad*-unipotents") is replaced by "the Zariski closure of* $\mathrm{Ad}(\Gamma) < \mathrm{GL}(\mathfrak{g})$ *is generated by 1-parameter unipotent subgroups".*

Benoist–Quint (and slightly earlier Bourgain–Furman–Lindenstrauss–Mozes for actions on tori $\mathbb{T}^d$) is a step towards the above conjecture where the assumption on Γ is: the Zariski-closure of $\mathrm{Ad}(\Gamma)$ is semisimple without compact factors.

- Before moving on: there is a transversal conjecture of Margulis about rigidity of higher rank abelian actions (paralleling x2x3 conjecture of Furstenberg): Einsiedler, Katok, Lindenstrauss. -

Towards random walks

Benoist–Quint's general strategy follows that of Ratner's. However, there is an important difference: such Γ are necessarily non-amenable. Why is this important?

Except obvious ones, there is no invariant measure to start working with (e.g. for a "drift argument", which is the core machinery behind Ratner's theorems excepting some other important results/ingredients such as Dani, Margulis recurrence).

To make up for this lack, Benoist–Quint work with random walks and stationary measures.

Stationary measures

Let $X = G/\Lambda$ and μ be a probability measure on G. Denote Γ_μ the semigroup generated by the support of μ.

- Example: $g_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mu = \frac{1}{2}\delta_{g_1} + \frac{1}{2}\delta_{g_2}$.
$\Gamma_\mu =< g_1, g_2 >$.

A probability measure ν on X is called μ-stationary if $\mu * \nu := \sum_g \mu(g) g_* \nu = \nu$. (invariant on μ-average. Clearly, Γ_μ-invariant $\implies$ μ-stationary)

Fact: There always exist μ-stationary probability measures ν (say if X is compact). Moreover they enjoy very useful decompositions such as $\nu = \int \nu_b d\mu^{\mathbb{N}}(b_1, \ldots)$, where $\nu_b = \lim_{n\to\infty} b_1 \ldots b_{n*}\nu$.

Benoist–Quint's theorems

Theorem (Benoist–Quint)

Let G be a Lie group and $\Lambda < G$ a lattice. Let $\Gamma < G$ be a compactly generated semigroup such that the Zariski closure of $\mathrm{Ad}(\Gamma)$ is semisimple without compact factors. Then,

1. *(Orbit closures) For every $x \in G/\Lambda$ there exist a closed group $G > L > \Gamma$ such that $\overline{\Gamma x} = Lx$ and Lx admit a L-invariant probability measure ν_x.*

Let μ be a compactly supported probability measure such that $\Gamma_\mu = \Gamma$.
2. *(Equidistribution) For every $x \in G/\Lambda$* **a)** $\frac{1}{N}\sum_{n=1}^{N} \mu^{*n} * \delta_x \to \nu_x$, *and*
b) *For $\mu^{\mathbb{N}}$ a.e. $(b_1, \dots)$, we have $\frac{1}{N}\sum_{n=1}^{N} \delta_{b_n \dots b_1 x} \to \nu_x$.*

3. *(Measure classification) Every μ-stationary and μ-ergodic ν on G/Λ is Γ-invariant and homogeneous, i.e. there exist closed $L > \Gamma$ and $x \in G/\Lambda$ such that ν is L-invariant and $\nu(Lx) = 1$.*

Further progress (0)

Before going further to mention the works of Simmons–Weiss and Eskin–Lindenstrauss, let's go back for a second:

BQ1: Benoist–Quint proved a first version of this result under the assumption that *G is a simple Lie group* and Γ *is Zariski dense in G.*

BQ2: In a second (and third) paper, they revisited their drift argument, and *using heavy random matrix products machinery* and *new non-escape of mass result*, they established the above version.

What is the difference? There are differences in each part 1., 2. and 3. Regarding 3. (i.e. exponential drift argument), perhaps most crucial, it is about controlling the expansion properties of random matrix products when there are several blocks.

Further progress (1)

SW: In the setting of first version of Benoist–Quint's result, Simmons–Weiss proved a similar statement replacing Zariski density by some precise dynamical assumptions (transversal to Zariski density).

EL: Finally, with a new drift argument "Factorization"*, Eskin–Lindenstrauss proved a *measure classification statement 3.*, which is based on *some expansion assumptions* and which envelops both Benoist–Quint and Simmons–Weiss directions (*building on some ideas from Eskin–Mirzakhani's work).

$-->$ The new argument significantly replaces the use of random matrix products machinery (most prominently, local limit theorem) by mere expansion assumptions.

Further progress (2)

In a nutshell, what we do is to

A) introduce and study a class of measures that we called *H-expanding measures*,

B) deduce *a measure classification result* using Eskin–Lindenstrauss,

C) prove *recurrence* and *countability results* after Benoist–Quint and Eskin–Mirzakhani–Mohammadi, and deduce *equidistribution* and *orbit closure statement*, and

D) *more* later.

Uniform expansion of random walks (1)

Let V be finite dimensional $\mathbb{R}$-vector space and μ a probability measure on $\mathrm{GL}(V)$ with finite first moment (i.e. $\int \log ||g|| d\mu(g) < \infty$). Let b_i's be iid random variables with distribution μ and denote $L_n = b_n \dots b_1$.

It follows from results of Hennion and Furstenberg–Kifer that there exist constants $\beta_1 > \dots > \beta_r$ and a filtration
$V = L_1 > L_2 > \dots > L_r > L_{r+1} = 0$ such that for every $v \in L_i \setminus L_{i+1}$, we have

$$\frac{1}{n} \log ||L_n v|| \underset{n \to \infty}{\longrightarrow} \beta_i$$

almost surely. Moreover the filtration is invariant by the group Γ_μ generated by the support of μ. In particular, when Γ_μ acts irreducibly on V, there exists a single exponent (which is the top Lyapunov exponent) and the filtration is trivial.

Uniform expansion of random walks (2)

When the Zariski closure of Γ_μ is a semisimple group H without compact factors, then by an important result of Furstenberg, all these exponents are positive (of course if $V^H = \{0\}$).

We say that a μ is **uniformly expanding** if $\beta_r > 0$. There are several useful equivalent formulations:

1. for every $v \neq 0$, almost surely $||L_n v|| \to \infty$,
2. there exists $C > 0$ and $N \in \mathbb{N}$ such that for every $v \in V \setminus \{0\}$, we have $\int \log \frac{||gv||}{||v||} d\mu^{*N}(g) > C$. (useful and more likely to be checkable)

H-expansion

Let H be a connected semisimple Lie group with finite center and μ a probability measure on H. We say that μ is H-**expanding** if for every representation (V, ρ) of H with $V^H = \{0\}$, the measure $\rho_* \mu$ is uniformly expanding.

Examples (1-2): Zariski dense – parabolic subgroups

1. Zariski dense case: H semisimple without compact factors and Γ_μ is Zariski dense (Furstenberg).

2. Proper parabolic subgroups: We can give a checkable criterion as in 1. to ensure H-expansion using the notion of **expanding cone** introduced and studied by Shi (particular forms in earlier works of Shah, Weiss, Mohammadi and Salehi Golsefidy):

Given a proper parabolic subgroup $Q < H$ and a maximal $\mathbb{R}$-split torus $A < Q$, Shi shows that there exists a cone A_Q^+ in A such that any $a \in A_Q^+$ expands the U-fixed vectors in any representation. This cone can be explicitly described in terms of the root data.

Proposition

Any μ supported on $Q < H$ such that $\overline{\mathrm{Ad}(\Gamma_\mu)}^Z$ contains $\mathrm{Ad}(U)$ and the central-toral Lyapunov vector $A(\mu)$ belongs to A_Q^+ is H-expanding.

Examples (3): Epimorphic split solvable subgroups

Let H be connected semisimple without compact factors. A subgroup $F < H$ is called **epimorphic** if in any linear representation of H any F fixed vector is H-fixed. (H-expansion $\implies$ epimorphic property)

For algebraic subgroups F, dynamical consequences of this representation-theoretic property was first investigated by Mozes and subsequently by Shah and Weiss.

3. Split solvable epimorphic groups: we show that any such algebraic group $F = TU < H$ with one dimensional torus T supports (many) H-expanding measures (using some results of Weiss and Shah-Weiss).

$--$ $>$ On the other hand, by a result of Bien–Borel (also Kollár), based on work by Kostant, any $\mathbb{R}$-split simple Lie group contains such epimorphic subgroups (even three-dimensional ones).

Question

Does any epimorphic and algebraic subgroup F of H support a H-expanding measure?

Negative if F is not epimorphic or algebraic.

Measure rigidity for *H*-expanding measures

Theorem

Let G be a Lie group, $\Lambda < G$ a lattice and $H < G$ be a connected semisimple Lie group with finite center. Let μ be a probability measure on H that is H-expanding and has a finite first moment. Then any μ-ergodic μ-stationary probability measure ν on G/Λ is Γ_μ-invariant and homogeneous. Moreover, the connected component of $\mathrm{Stab}_G(\nu)$ is normalized by H.

Using some further properties of H-expanding measures, this result is deduced by an iterative application of Eskin–Lindenstrauss' measure classification theorem.

A consequence

Here is a result that we can deduce using the previous theorem①, Margulis arithmeticity② and some further analysis of stationary measures charging an orbit of the centralizer of Γ_μ ③:

Corollary

Let G be connected semisimple and $\Lambda < G$ be an irreducible lattice. Let $H \trianglelefteq G$ of positive dimension and let μ be an H-expanding probability on H with finite first moment.

1. *If $H \neq G$, then the Haar measure m_X on $X = G/\Lambda$ is the unique μ-stationary probability measure on X.*
2. *If $H = G$, then the only μ-ergodic μ-stationary probability measures on X are uniform measures on finite Γ_μ-orbits and the Haar measure m_X on X. Moreover, m_X is the only non-atomic μ-stationary probability measure on X.*

Equidistribution and orbit closures

Theorem (Prohaska-S-Shi)

$\Lambda < G > H$ with same meanings as in the previous theorem. Let μ be a H-expanding probability measure on H. Then for every $x \in X$ there is a Γ_μ-ergodic homogeneous subspace $Y_x \subset G/\Lambda$ with corresponding homogeneous probability measure ν_x such that the following hold:

1. *the orbit closure $\overline{\Gamma_\mu x}$ equals Y_x.*
2. *If μ has a finite exponential moment, then $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu^{*k} * \delta_x = \nu_x$, and*
3. *for $\mu^{\mathbb{N}}$-a.e. $(b_1, b_2, \dots)$ one has*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \delta_{b_n \cdots b_1 x} = \nu_x.$$

Using Theorem 3, the general strategy follows that of Benoist–Quint which in turn is an adaptation of Ratner's strategy.

Proof strategy

To simplify, we suppose that $C_G(H) = \{\mathrm{id}\}$.

A) Let $x \in G/\Lambda$ and consider a limit ν of $\frac{1}{N}\sum_{n=1}^{N} \mu^{*n} * \delta_x$. This is a μ-stationary measure.

B) Prove a result implying positive recurrence of the random walk to deduce that ν is a probability measure. (adaptation of Benoist–Quint's result who generalized Eskin–Margulis recurrence)

C) Consider Y_0: Γ_μ-invariant homogeneous subspace of minimal dimension containing x. Prove a result implying positive unstability of homogeneous proper subspaces to deduce that for every Y with empty interior in Y_0, we have $\nu(Y) = 0$. (we construct these height functions inspired by the construction of Eskin–Mirzakhani–Mohammadi)

Proof roadmap continued

D) Prove a countability result for such subspaces.

E) Deduce that ν gives zero mass to the union of these subdimensional homogeneous spaces. This is also true of a.e. ergodic component ν_ξ of ν.

F) By measure classification theorem applied to ergodic components ν_ξ, each ν_ξ is Γ_μ-invariant and homogeneous. This implies that a.e. ergodic component ν_ξ is "the" homogeneous measure on Y_0. So all ergodic components are the same and equal to ν. This shows 1. and 2. of the theorem.

G) The last point now follows from Breiman law of large numbers.

Birkhoff genericity

Developing on an idea of Simmons–Weiss, from the previous theorems we deduce a Birkhoff genericity result for diagonal flows, which in turn extends related works of Simmons–Weiss and Shi.

Let $F = (a_t)_{t \in \mathbb{R}}$ be a one-parameter Ad-diagonalizable subgroup of H and F^+ its positive ray. Let ν a F-invariant probability measure on $X = G/\Lambda$. We say that a measure η on H is *F^+-Birkhoff generic at $x \in X$ with respect to ν* if

$$\frac{1}{T} \int_0^T \delta_{a_t h x} dt \to \nu$$

as $T \to \infty$ for η-almost every $h \in H$.

We are interested in measures η concentrated on the strong stable submanifolds of F^+ through x (for every $x \in X$). Namely:

Measures generated by expanding random walks

Let U be an a_1-expanding subgroup contained in the unstable horospherical subgroup $H^+_{a_1}$ of a_1 in H (e.g. the full horospherical). Let $P = F_{a_1}U < H$ (one can also add a suitable compact factor K to P).

Definition (Lemma)

Let μ be a compactly supported probability on $P = KFU$ such that

1. F-average of μ is > 0.
2. the Zariski closure of $\mathrm{Ad}(\Gamma_\mu)$ contains $\mathrm{Ad}(U)$.

Then, $\mu^{\mathbb{N}}$ almost surely the U-component of the product $b_n \dots b_1$ in its KFU decomposition converges to some $u_b \in U$. Let η be the pushforward of $\mu^{\mathbb{N}}$ by this map. Such measures will be called **generated by a_1-expanding random walks.**

Measures generated by expanding random walks

Examples include a piece of the Haar measure on U, some fractal measures supported on self-similar and self-affine (of type Bedford–McMullen carpets) sets in U.

Birkhoff genericity result

Theorem (Prohaska-S-Shi)

Let $\Lambda < G > H > FU$ with the same meanings — $F = \{a_t \mid t \in \mathbb{R}\}$ a one-parameter Ad*-diagonalizable subgroup of H and U an a_1-expanding subgroup of H contained in $H_{a_1}^+$. Suppose that η is a probability measure on U generated by a_1-expanding random walks. Then, given any $x \in G/\Lambda$, for η-almost every $u \in U$, we have*

$$\frac{1}{T}\int_0^T \delta_{a_t ux}dt \underset{T\to\infty}{\longrightarrow} \nu_{\overline{Hx}},$$

where $\nu_{\overline{Hx}}$ is the homogeneous measure on the orbit closure of Hx i.e. η is (F^+)-Birkhoff generic at every $x \in X$ with respect to $\nu_{\overline{Hx}}$.

Diophantine approximation on fractals

In view of the Dani–Kleinbock correspondance principle, this result has direct applications to diophantine approximation problems on self-similar/affine fractals, namely in showing smallness of their set of badly approximable (or Dirichlet improvable) points with respect to self-similar/affine measures.

But many questions frustratingly remain open in this direction.

Thank you

Many thanks for your attention!