Applications of the asymptotic poissonity in time and space of visits to small sets

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Point Process of visits to small sets

- Probability preserving dynamical system:
  $(\Omega, \mathcal{F}, \mu, T)$ or $(\Omega, \mathcal{F}, \mu, (Y_t)_{t\geq 0})$
  $(\Omega, \mathcal{F}, \mu)$ probability space
  $T : \Omega \to \Omega$ preserves μ (i.e. $\mu(T^{-1}(A)) = \mu(A)$)
  or $Y_t : \Omega \to \Omega$ preserves μ and $Y_{t+s} = Y_s \circ Y_t$
- Family $(A_\varepsilon)_{\varepsilon>0}$: $A_\varepsilon \in \mathcal{F}$ such that $\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0$.
- **Goal**: Behaviour of the visits of $(T^k(x))_{k\geq 0}$ to A_ε, as $\varepsilon \to 0$.

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- Time spent: $\mathcal{T}_{A_\varepsilon,t} = \sum_{k=1}^{t} \mathbf{1}_{A_\varepsilon} \circ T^k$ or $\mathcal{T}_{A_\varepsilon,t} = \int_0^t \mathbf{1}_{A_\varepsilon} \circ Y_s \, ds$.
- First hitting time: $\tau_{A_\varepsilon} = \min\{n \geq 1 \,:\, T^n \in A_\varepsilon\}$.

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- First estimates $\mathbb{E}[\mathcal{T}_{A_\varepsilon,t}] = t\mu(A_\varepsilon)$ and $\mathbb{E}[\tau_{A_\varepsilon}] = \frac{1}{\mu(A_\varepsilon)}$.

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- Examples of questions:
  - Behaviour of the first visit time: $\mu(A_\varepsilon)\tau_{A_\varepsilon} \to Exp(1)$?
    (historically first question of interest)
  - Successive visit times: increments: $\to$ i.i.d. $Exp(1)$?
  - Number of visits up to $t/\mu(A_\varepsilon)$: $\to Poisson(t)$?
  - Number of visits to one set before the first visit to another one
  - Time spent by a flow in the set
  - Number of high records, etc.
- Study of a process $\mathcal{N}_\varepsilon$ containing these informations

Some previous works

- results for cylinder sets: [Hirata1993], [Hirata,Saussol,Vaienti1999], [Bruin,Vaienti2003], [Abadi,Vergne2008], [Haydn,Vaienti2004].
- Uniformly expanding maps [Collet,Galves1995].
- Non-uniformly expanding maps (intermittent maps) [Collet,Galves1993], [Bruin,Saussol2003], [Bruin,Vaienti2003], [Collet2001], [Freitas,Freitas,Todd2010], [Holland,Nicol,Török].
- Partially hyperbolic systems [Dolgopyat2004]
- Sinai billiard, Axiom A attractors with one-dimensional unstable manifolds [Collet,Chazottes2013].
- Polynomial mixing[Haydn,Wasilewska2016]
- Billiard stadium: [Freitas,Haydn,Nicol2014] and [Pène,Saussol2016]

Spatio-temporal Point Process of visits to small sets

- $(\Omega, \mathcal{F}, \mu, T)$ or $(\Omega, \mathcal{F}, \mu, (Y_t)_t)$; $A_\varepsilon \in \mathcal{F}$, $\lim_{\varepsilon\to 0} \mu(A_\varepsilon) = 0$.
- Point process of visits to A_ε: $\sum_{n\geq 1\,:\, T^n(x)\in A_\varepsilon} \delta_{(n,T^n(x))}$
- Normalized point process:

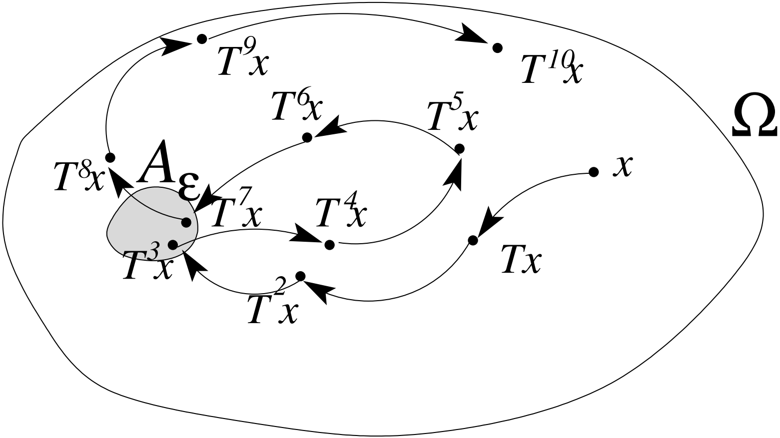
$$\mathcal{N}_\varepsilon(x) = \sum_{n\geq 1\,:\, T^n(x)\in A_\varepsilon} \delta_{(nh_\varepsilon, H_\varepsilon(T^n(x)))}\,,$$

$$\mathcal{N}_\varepsilon(x) = \sum_{t>0\,:\, Y_t(x) \text{ enters } A_\varepsilon} \delta_{(th_\varepsilon, H_\varepsilon(Y_t(x)))}\,,$$

with $h_\varepsilon \to 0$ and $H_\varepsilon : A_\varepsilon \to V \subset \mathbb{R}^D$ normalization function.

- $\mathcal{N}_\varepsilon([a,b]\times U)(x) = \#\{n \in [\frac{a}{h_\epsilon}, \frac{b}{h_\varepsilon}] \ :\ T^n x \in H_\varepsilon^{-1}(U)\}$
- **Goal**: convergence of $\mathcal{N}_\varepsilon$ as $\varepsilon \to 0$
- For T: $\mathbb{E}_\mu[\mathcal{N}_\varepsilon([a,b]\times V)] = \frac{b-a}{h_\varepsilon}\mu(A_\varepsilon)$. So $h_\varepsilon \sim \mu(A_\varepsilon)$.
- For Y: use special flow representation : $h_\varepsilon \sim \nu(\Pi(A_\varepsilon))$
  $M \subset \Omega$ s.t. $\tau_M > 0$,
  $\Pi(A_\varepsilon) = \{y \in M \ :\ \exists s \in [0, \tau_M(x)),\ Y_s(y) \in A_\varepsilon\}$,
  $\mu(A) = \int_M \int_0^{\tau_M(y)} \mathbf{1}_A(Y_s(y))\, ds\, d\nu(y)$

Normalized Point Process of visits to small sets



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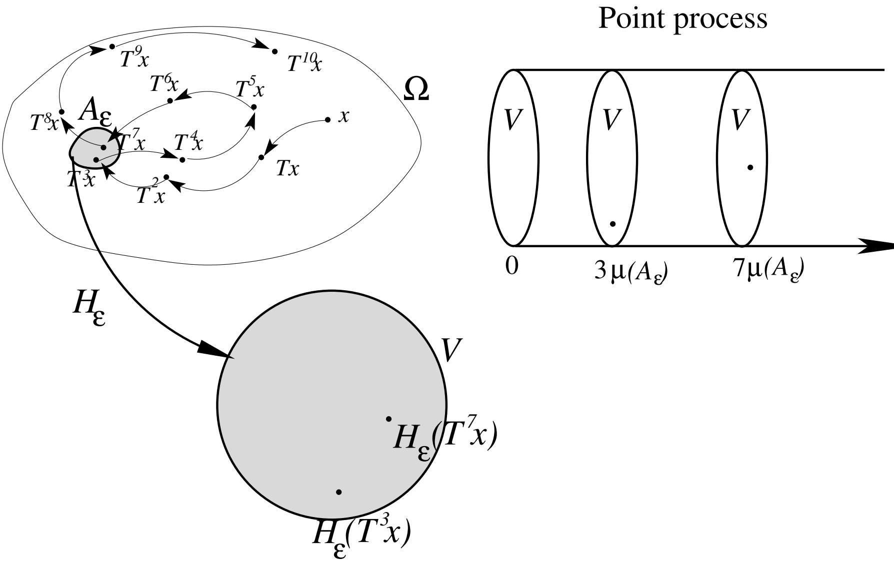

Convergence in distribution to a Poisson Point Process

- Poisson Point Process (PPP) on $E = [0, +\infty) \times V$ of intensity $\overline{m} = Leb \times m$, m probability measure on V
  $PPP(\bar{m})$: $\mathcal{P} = \sum_i \delta_{(t_i, x_i)}$ ((t_i, x_i) random) such that
  - $\forall B \in \mathcal{B}([0, +\infty) \times V)$, $\mathcal{P}(B) \rightsquigarrow Poisson(\overline{m}(B))$
  - $\forall K \geq 1$, $\forall B_1, ..., B_K \in \mathcal{B}([0, +\infty) \times V)$ pairwise disjoint, $\mathcal{P}(B_1), ..., \mathcal{P}(B_K)$ are independent

  Interpretation: $\mathcal{P} = \sum_i \delta_{(t_i, x_i)}$ with $P = \sum_i \delta_{t_i}$ PPP of intensity 1 and x_i iid with distribution m, $\perp\!\!\!\perp P$.

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  Interpretation: $\mathcal{P}=\sum_i \delta_{(t_i,x_i)}$ with $P=\sum_i \delta_{t_i}$ PPP of intensity 1 and x_i iid with distribution m, $\perp\!\!\!\perp P$.
- Convergence $\mathcal{N}_\varepsilon\Rightarrow\mathcal{P}$ means:
  - $\forall f\in C_c(E\to[0,+\infty))$, $\int_E f\,d\mathcal{N}_\varepsilon\Rightarrow\int_E f\,d\mathcal{P}$.
  - $\mathcal{N}_\varepsilon(B)\Rightarrow\mathcal{P}(B)$
    $\forall B\subset E$ open, relatively compact, s.t. $\bar{m}(\partial B)=0$.

General resuls

Theorem (F.P.,B.Saussol2016, maps)

For (Ω, μ, T): under general assumptions:

$$\mathcal{N}_\varepsilon = \sum_{n \geq 1 \,:\, T^n(\cdot) \in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), H_\varepsilon(T^n(\cdot)))} \Rightarrow PPP(Leb \times m)$$

Applications to dispersive billiards: Sinai, Bunimovich Stadium, corners (with B. Saussol); cusps (with P. Jung and H.-K. Zhang)

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Applications to dispersive billiards: Sinai, Bunimovich Stadium, corners (with B. Saussol); cusps (with P. Jung and H.-K. Zhang)

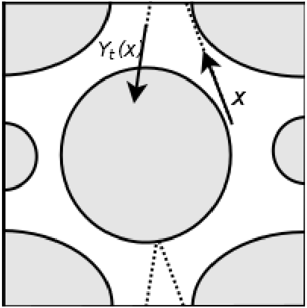
Theorem (F.P., B. Saussol 2016, special flows)

$(\Omega, \mu, (Y_t)_t)$; $M \subset \Omega$, $\tau_M > 0$; $\Pi(x) = y \in M$ *s.t.* $x \in Y_{[0,\tau_M(y))}(y)$
$(M, \nu, S = Y_{\tau_M(\cdot)})$ *p.p.d.s.*, $\mu(A) \approx \int_M \int_0^{\tau_M(y)} \mathbf{1}_A(Y_s(y))\, ds\, d\nu(y)$
Assume at most one entrance in A_ε between two visits to M. Then

$$\sum_{t>0\,:\, Y_t(\cdot) \text{ enters } A_\varepsilon} \delta_{(th_\varepsilon/\mathbb{E}_\nu[\tau_M], G_\varepsilon\circ\Pi(Y_t(x)))} \sim \sum_{n\ge 1\,:\, S^n(\cdot)\in\Pi(A_\varepsilon)} \delta_{(nh_\varepsilon, G_\varepsilon(S^n(\cdot)))}$$

Sinai billiard flow

Billiard domain: $Q = \mathbb{T}^2 \setminus \bigcup_{i=1}^{I} \mathcal{O}_i$ (Q is in white)
$\mathcal{O}_i$ open convex, boundary C^3 with non null curvature (the $\mathcal{O}_i$ are in grey)
closures of $\mathcal{O}_i$ pairwise disjoint



point particle moving at unit speed in Q
straight + elastic collisions off ∂Q

Finite horizon: any trajectory meets ∂Q

space of configurations $\Omega = Q \times S^1$
$Y_t(x)$ = configuration at time t

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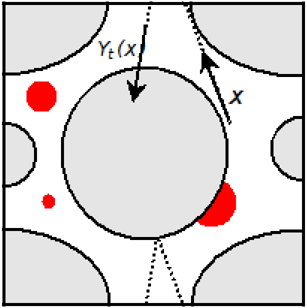

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$$\text{Take } A_\varepsilon = \bigcup_{j=1}^{J} B(q_j, \varepsilon r_j) \times S^1$$

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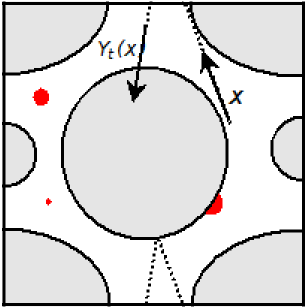

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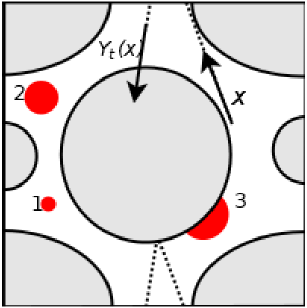
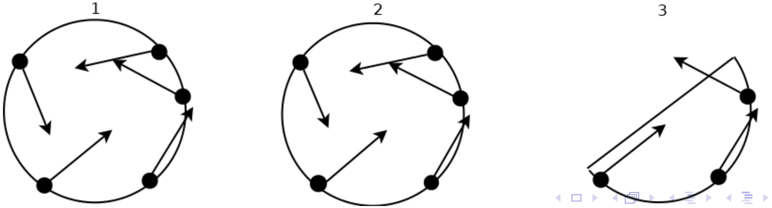
Results for the Sinai billiard flow [F.P., B. Saussol 2020]



$A_\varepsilon = \bigcup_{j=1}^{J} B(q_j, \varepsilon r_j) \times S^1$
$V = \{1, ..., J\} \times S^1 \times S^1$
$H_\varepsilon(q, \vec{v}) = (j, \frac{\overrightarrow{q_j q}}{r_j \varepsilon}, \vec{v})$ if $q \in \mathcal{C}(q_j, \varepsilon r_j)$
$\mathcal{N}_\varepsilon = \sum_{s>0:\, Y_s(\cdot) \text{ enters in } A_\varepsilon} \delta_{(sh_\varepsilon, H_\varepsilon(Y_s(\cdot)))} \Rightarrow PPP(Leb \times m)$
$h_\varepsilon = \frac{\sum_{j=1}^{J}(2 - \mathbf{1}_{q_j \in \partial Q}) r_j \varepsilon}{|Q|}$
m probability measure with density proportional to $(j, p, v) \to r_j \langle (-p), v \rangle^+ \mathbf{1}_{\{\langle p, n_{q_j} \rangle \geq 0\}}$
n_{q_j}: normal to ∂Q at q_j ($n_{q_j} = 0$ if $q_j \notin \partial Q$)



Results for the Sinai billiard flow: First consequences

$A_\varepsilon = \bigcup_{j=1}^J B(q_j,\varepsilon r_j)\times S^1$, $dm(j,p,v)\approx r_j\langle(-p),v\rangle^+\mathbf{1}_{\{\langle p,n_{q_j}\rangle\ge 0\}}$

$$\mathcal{N}_\varepsilon = \sum_{s>0\,:\,Y_s(\cdot)\text{ enters in }A_\varepsilon} \delta_{(sh_\varepsilon, H_\varepsilon(Y_s(\cdot)))} \Rightarrow PPP(Leb\times m)$$

$H_\varepsilon(q,\vec v) = (j, \frac{\overrightarrow{q_jq}}{r_j\varepsilon}, \vec v) \in V = \{1,...,J\}\times S^1\times S^1$ if $q\in\mathcal{C}(q_j,\varepsilon r_j)$

- **Numbers of visits** $N^{(i,\varepsilon)}(t)$ **to** $B(q_i,r_i\varepsilon)\times S^1$ **before** t/h_ε:

  $\boxed{(N^{(i,\varepsilon)}(t))_i \Rightarrow (N_t^{(i)})_i}$ *independent Poisson* $\left(\frac{(2-\mathbf{1}_{q_i\in\partial Q})r_i}{\sum_{j=1}^J(2-\mathbf{1}_{q_j\in\partial Q})r_j}t\right)$

  Proof: $N^{(i,\varepsilon)}(t) = \mathcal{N}_\varepsilon([0,t)\times\{i\}\times S^1\times S^1)$

- **time** $\mathcal{T}^{(\varepsilon)}(t)$ **spent in** A_ε **before time** t/h_ε:

$$\boxed{\left(\varepsilon^{-1}\mathcal{T}^{(\varepsilon,j)}(t)\right)_j \Rightarrow 2\left(r_j\sum_{k=1}^{N_t^{(j)}} Z_k^{(j)}\right)_j} \quad Z_k^{(j)} \text{ iid pdf } \frac{y\mathbf{1}_{[0,1](y)}}{\sqrt{1-y^2}},\ \perp\!\!\!\perp (N_t^{(i)})_i.$$

Proof: $\mathcal{T}^{(\varepsilon)}(t) = \int_{[0,t)\times V} D_\varepsilon\, d\mathcal{N}_\varepsilon$, $\varepsilon^{-1}D_\varepsilon(j,p,v)\to 2r_j\langle -p,v\rangle$.

Results for the Sinai billiard flow: Stopped process

$$A_\varepsilon = \bigcup_{j=1}^J B(q_j, \varepsilon r_j) \times S^1,\ dm(j,p,v) \approx r_j \langle (-p), v\rangle^+ \mathbf{1}_{\{\langle p, n_{q_j}\rangle > 0\}}$$

$$\mathcal{N}_\varepsilon = \sum_{s>0\,:\, Y_s(\cdot) \text{ enters in } A_\varepsilon} \delta_{(sh_\varepsilon, H_\varepsilon(Y_s(\cdot)))} \Rightarrow PPP(Leb \times m)$$

$$H_\varepsilon(q, \vec{v}) = (j, \frac{\overrightarrow{q_j q}}{r_j \varepsilon}, \vec{v}) \in V = \{1, ..., J\} \times S^1 \times S^1 \text{ if } q \in \mathcal{C}(q_j, \varepsilon r_j)$$

Inspired by [Kifer,Rapaport,2019]: $\tau_\varepsilon^{(2)}$: first visit time to$B(q_2, \varepsilon r_2) \times S^1$

- **Number $N^{(1<2,\varepsilon)}$ of visits to $B(q_1, \varepsilon r_1) \times S^1$ before $\tau_\varepsilon^{(2)}$:**

$$\boxed{\mu(N^{(1<2,\varepsilon)} \geq k) \to \mathbb{P}(X \geq k) = \frac{d_1^k}{(d_1 + d_2)^k}}\quad d_j = (2 - \mathbf{1}_{q_j \in \partial Q}) r_j.$$

  Proof: $N^{(1<2,\varepsilon)} = N^{(1,\varepsilon)}(\tau_\varepsilon^{(2)} = \inf\{s > 0 \,:\, N_s^{(2,\varepsilon)} \neq 0\})$.

- **time $\mathcal{T}^{(\varepsilon,1<2)}$ spent in $B(q_1, \varepsilon r_1) \times S^1$ before $\tau_\varepsilon^{(2)}$:**

$$\boxed{\varepsilon^{-1}\mathcal{T}^{(\varepsilon,1<2)} \Rightarrow 2r_1 \sum_{i=1}^X Z_i}\quad Z_i \text{ iid, pdf } \frac{y\mathbf{1}_{[0,1](y)}}{\sqrt{1-y^2}},\ \perp\!\!\!\perp X.$$

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- **time** $\mathcal{T}^{(\varepsilon,1<2)}$ **spent in** $B(q_1,\varepsilon r_1)\times S^1$ **before** $\tau_\varepsilon^{(2)}$:

$$\boxed{\varepsilon^{-1}\mathcal{T}^{(\varepsilon,1<2)}\Rightarrow 2r_1\sum_{i=1}^X Z_i}\quad Z_i \text{ iid, pdf } \frac{y\mathbf{1}_{[0,1](y)}}{\sqrt{1-y^2}},\ \perp\!\!\!\perp X.$$

<u>Proof</u>: $\mathcal{T}^{(\varepsilon,1<2)}=\int_{[0,\tau_\varepsilon^{(2)})\times\{1\}\times S^1\times S^1} D_\varepsilon\, d\mathcal{N}_\varepsilon$, $\varepsilon^{-1}D_\varepsilon(1,p,v)\to 2r_1\langle -p,v\rangle$.

Results for the Sinai billiard flow: High records ($J = r_1 = 1$)

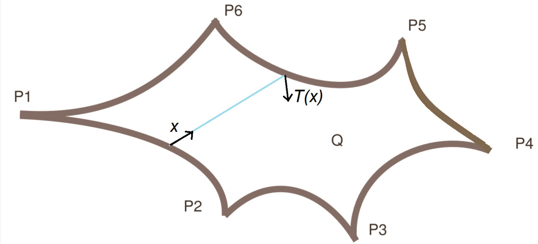
$A_\varepsilon = B(q_j, \varepsilon) \times S^1$, $dm(p, v) \approx \langle(-p), v\rangle^+ \mathbf{1}_{\{\langle p, n_{q_1}\rangle > 0\}}$

$$\mathcal{N}_\varepsilon = \sum_{s>0\,:\,Y_s(\cdot)\text{ enters in }A_\varepsilon} \delta_{(sh_\varepsilon, H_\varepsilon(Y_s(\cdot)))} \Rightarrow PPP(Leb \times m)$$

$H_\varepsilon(q, \vec{v}) = (\frac{\overrightarrow{q_1 q}}{\varepsilon}, \vec{v}) \in V = S^1 \times S^1$.

- Given $\varepsilon > 0$, we say that a **high record** happens at time t if $d(q_1, Y_t) = \min d(q_1, Y_{[0,t]}) < \varepsilon$ and if $d(q_1, Y_t)$ is a local minimum of $s \mapsto d(q_1, Y_s)$.
- **The number $\mathcal{R}_\varepsilon(t)$ of high records until time t/h_ε** converges in distribution to $\sum_{k=1}^{N_t} Z_k$, with N_t Poisson(t), and Z_k independent $Bernoulli(1/k)$, $\perp\!\!\!\perp N_t$.
  Proof: We write $\mathcal{N}_\varepsilon = \sum_i \delta_{(t_i, x_i)}$. Then
  $\mathcal{R}_\varepsilon(t) \approx \sum_{i\,:\,t_i \le t/h_\varepsilon} \mathbf{1}_{\{\forall t_j < t_i,\ d_\varepsilon(x_i) < d_\varepsilon(x_j)\}}$
  $d_\varepsilon(p, v)$= distance between q_1 and the orbit of the flow before exiting A_ε, $\varepsilon^{-1} d_\varepsilon \to |\sin \angle(-p, v)|$.

More applications: dispersing billiards with cusps



Cusps $z = \pm c_{\pm} s^{\beta} + \mathcal{O}(s^{2\beta-1})$, $\beta > 2$: $\beta^* = \max \beta$

$$\left(\frac{1}{n^{\frac{\beta^*-1}{\beta^*}}} \sum_{k=0}^{\lfloor nt \rfloor} f \circ T^k \right)_t \Rightarrow (\mathcal{Z}_t)_t \ \text{càdlàg}$$

[Jung,Zhang,2018], [Jung,F.P.,Zhang,2019], [Melbourne,Varandas,2020], [Jung,Melbourne,Pène,Varandas,Zhang] using a general criteria by [Tyran-Kamińska2010] based on PP $\mathcal{N}_\varepsilon$.

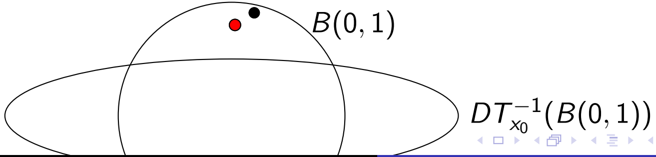
More applications: around hyperbolic periodic points

(Ω, μ, T) C^2 Anosov, Ω Riemmannian manifold, μ SRB measure
x_0 hyperbolic p-periodic point of T, $\mu(B(x_0, 2\varepsilon))\varepsilon^b = o(\mu(B(x_0, \varepsilon)))$
$A_\varepsilon = B(x_0, \varepsilon) \setminus \bigcup_{j=1}^{q_0} T^{-jp} B(x_0, \varepsilon)$ (q_0 such that $\tau_{A_\varepsilon} > c|\log \varepsilon|$).
$\mu(x_0 + \varepsilon \cdot |A_\varepsilon) \to m$. Then

$$\mathcal{N}_\varepsilon = \sum_{n\,:\,T^n x \in A_\varepsilon} \delta_{(n\mu(A_\varepsilon),(T^n x - x_0)/\varepsilon)} \Rightarrow \mathcal{P} \quad PPP(Leb \times m)\,,$$

$$\mathcal{V}_\varepsilon = \sum_{n\,:\,T^n x \in B(x_0,\varepsilon)\setminus\{x_0\}} \delta_{(n\mu(A_\varepsilon),(T^n x - x_0)/\varepsilon)} \Rightarrow \Psi(\mathcal{P})\,,$$

$$\text{with } \psi\left(\sum_n \delta_{(t_n,x_n)}\right) = \sum_n \sum_{k\geq 0\,:\,DT_{x_0}^{-kp}(x_n)\in B(0,1)} \delta_{(t_n, DT_{x_0}^{-kp}(x_n))}$$



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x_0 hyperbolic p-periodic point of T, $\mu(B(x_0, 2\varepsilon))\varepsilon^b = o(\mu(B(x_0, \varepsilon)))$

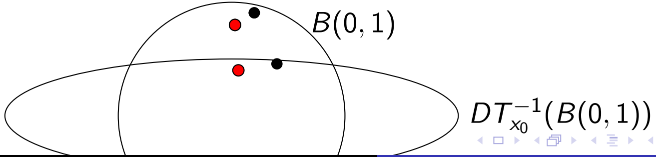
$A_\varepsilon = B(x_0, \varepsilon) \setminus \bigcup_{j=1}^{q_0} T^{-jp} B(x_0, \varepsilon)$ (q_0 such that $\tau_{A_\varepsilon} > c|\log \varepsilon|$).

$\mu(x_0 + \varepsilon \cdot |A_\varepsilon) \to m$. Then

$$\mathcal{N}_\varepsilon = \sum_{n\,:\,T^n x \in A_\varepsilon} \delta_{(n\mu(A_\varepsilon), (T^n x - x_0)/\varepsilon)} \Rightarrow \mathcal{P} \quad PPP(Leb \times m)\,,$$

$$\mathcal{V}_\varepsilon = \sum_{n\,:\,T^n x \in B(x_0,\varepsilon)\setminus\{x_0\}} \delta_{(n\mu(A_\varepsilon), (T^n x - x_0)/\varepsilon)} \Rightarrow \Psi(\mathcal{P})\,,$$

$$\text{with } \psi\left(\sum_n \delta_{(t_n, x_n)}\right) = \sum_n \sum_{k\geq 0\,:\,DT_{x_0}^{-kp}(x_n)\in B(0,1)} \delta_{(t_n, DT_{x_0}^{-kp}(x_n))}$$



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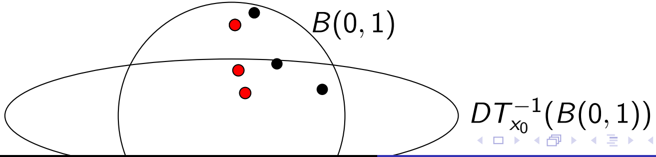

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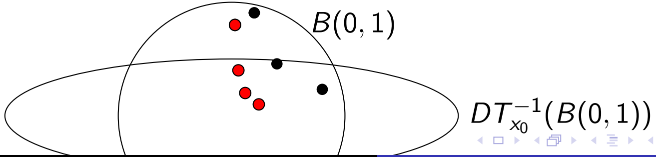

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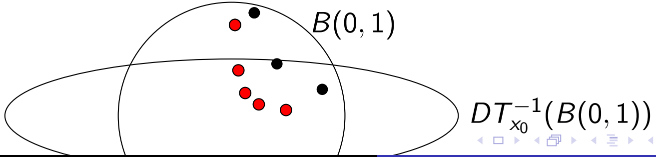

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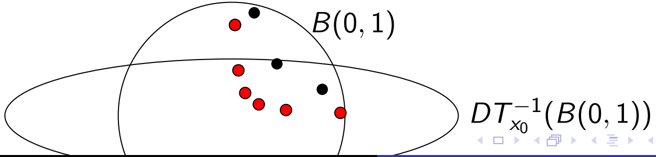

More applications: line process for geodesic flow

Let $\mathcal{S}$ be a compact surface with negative curvature. Fix $q_0 \in \mathcal{S}$. Let $(Y_t)_t$ be the geodesic flow on T^1S.

The trace of $Y_{[0,t/\varepsilon]}$ in the disk $B(q_0,\varepsilon)$ converges after normalization to a Poisson random variable (of intensity $2t/|S|$) number of chords starting with parameters $(p,v) \in S^1 \times S^1$ chosen independently in the unit disk, with pdf $\approx \langle -p, v\rangle^+ = (\cos\varphi)^+$ [Athreya,Lalley,Sapir,Wrotten], [F.P.,Saussol2020]

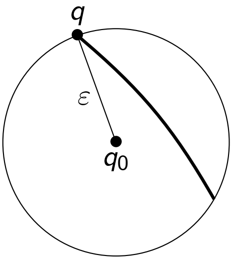

Figure: A geodesic trajectory entering the ball $B(q_0,\varepsilon)$.

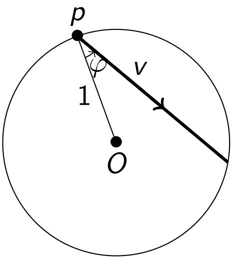

Figure: The limit line after renormalization

General resuls

Theorem (F.P., B. Saussol 2016)

Let $(\mathcal{W}_n)_n$ be a sequence of finite families of disjoint relatively compact open subsets of $[0,+\infty)\times V$ such that

- $\sigma(\mathcal{W}_m)\uparrow\sigma(\bigcup_{m\ge1}\mathcal{W}_m)=\mathcal{B}([0,+\infty)\times V)$
- $\forall F\in\bigcup_{m\ge1}\mathcal{W}_m$, $m(\partial F)=0$ *and* $\mu(H_\varepsilon^{-1}(F))\to m(F)$
- $\forall m$, $\sup_{n\ge1}\sup_{A\in H_\varepsilon^{-1}(\mathcal{W}_m),B\in\sigma(\bigcup_{k=0}^n T^{-k}(H_\varepsilon^{-1}(\mathcal{W}_m)))}|\mu(A\cap B)-\mu(A)\mu(B)|=o(\mu(A_\varepsilon))$
  note that B is the union of sets of the form $\bigcap_\ell T^{-k_\ell}B_{j_\ell}$ with $k_\ell\in\{1,...,n\}$ and $B_{j_\ell}\in H_\varepsilon^{-1}(\mathcal{W}_m)$

Then $\mathcal{N}_\varepsilon=\sum_{n\ge1\,:\,T^n(\cdot)\in A_\varepsilon}\delta_{(nh_\varepsilon,H_\varepsilon(T^n(\cdot)))}\Rightarrow PPP(Leb\times m)$.