Cover time and hitting time

H T H H T H T T H H
$n=2$

Hitting time to HH = 4

Cover time = 8

Questions: – expected hitting time
$n \to \infty$?
– expected cover time?
$n \to \infty$?
– unfair coin?

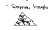
② Iterated function systems

$\mathcal{F} = \{ S_i : \mathbb{R}^d \to \mathbb{R}^d \}_{i=1}^{N}$ (IFS)

$\exists$ unique non-empty compact set
$F = \bigcup S_i(F)$ (attractor)

Examples

– Self-similar sets
– Middle $\frac{1}{3}$ Cantor set.
– Sierpinski triangle



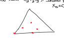
③ Chaos game

IFS for Sierpinski Δ:

$\{ S_i(\underline{x}) = \frac{1}{2}\underline{x} + \underline{t}_i \}_{i=1}^{3}$

$\underline{t}_1 = (0,0)$ $\underline{t}_2 = (\frac{1}{2}, 0)$ $\underline{t}_3 = (\frac{1}{4}, \frac{\sqrt{3}}{4})$

Prob vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Initial pt $x_0 = 0$



Let $F = \bigcup_{i=1}^{N} S_i(F)$. Choose $(p_1, \dots, p_N)$, $p_i > 0$

① Pick initial pt $x_0 \in F$

② $\forall n \geq 1$ choose $i_n \in \{1, \dots, N\}$ independently at random according to $(p_1 \dots p_N)$.

③ $x_n := S_{i_n} x_{n-1}$

$(x_n)_{n=0}^{\infty}$ is as dense in F.

– How long till sequence reaches certain "resolution" in F?
– What probs make chaos game 'most efficient' at rendering F?

Def δ-cover time (from x_0)

$W_{\delta, x_0}(\underline{i}) := \inf\{ n \geq 1 : \{ S_{i_1} x_0, \dots, S_{i_n} \dots S_{i_1} x_0 \}$ is δ-dense in $F \}$

$\Sigma = \{1 \dots N\}^{\mathbb{N}}$

⑤ Self-similar sets (joint work Ian Morris)

$\delta > 0$

$P_\delta = \{ \underline{i} \in \Sigma^* : r_{\underline{i}} \leq \delta < r_{\underline{i}^-} \}$

state space

$r_{\underline{i}} = r_{i_1} \dots r_{i_n}$

$\underline{i} \overset{i \in \{1 \dots N\}}{\longmapsto} \underline{j}$

$\underline{i} \in P_\delta$, $\underline{j} \in P_\delta$ if $[i\underline{i}] \subseteq [\underline{j}]$

Transition matrix

$$a_{\underline{i}\underline{j}} := \begin{cases} p_i & \text{if } [i\underline{i}] \subseteq [\underline{j}] \\ 0 & \text{otherwise.} \end{cases}$$

Matthews method:

$$\mathbb{E}(W_{\delta, x_0}) \approx \mathbb{E}(\tau_{cov}) \approx \max_{\underline{i}, \underline{j}} \mathbb{E}_{\underline{i}} \tau_{\underline{j}} \cdot \underbrace{\left(1 + \frac{1}{2} + \dots + \frac{1}{\# P_\delta}\right)}_{\approx \log(\frac{1}{\delta})}$$

first time visits all states in P_δ

hitting j from i

$$\mathbb{E}_{\underline{i}} \tau_{\underline{j}} \approx \frac{1}{p_{\underline{j}}} = \frac{1}{r_{j_1}^{\frac{\log p_{j_1}}{\log r_{j_1}}} \dots r_{j_n}^{\frac{\log p_{j_n}}{\log r_{j_n}}}} = \delta^{-t}$$

maximised when exponents are all

$$(*) \quad = \max_{1 \leq i \leq N} \frac{\log p_i}{\log r_i} =: t$$

Thm

(a) If max in $(*)$ is attained at unique value of i

$$C^{-1} \delta^{-t} \log\log(\tfrac{1}{\delta}) \leq \mathbb{E}(W_{\delta, x_0}) \leq C \delta^{-t} (\log\log(\tfrac{1}{\delta}))^2$$

(b) Otherwise

$$C^{-1} \delta^{-t} \log(\tfrac{1}{\delta}) \leq \mathbb{E}(W_{\delta, x_0}) \leq C \delta^{-t} \log(\tfrac{1}{\delta})$$

Optimiser $\sum r_i^s = 1$. $t \geq s$ equality $\iff$ $p^* = (r_1^s, \dots, r_N^s)$

⑥ Minkowski dimension of measure

Def (Fraser-Falconer-Kaenmaki)

$$\dim_M \mu := \lim_{r \to 0} \frac{\log \min_{y \in X} \mu(B(y, r))}{\log r}$$

on X

$\dim_M X = \min\{ \dim_M \mu$: fully supported Borel measure on $X \}$

Optimiser: projection of invariant measure which minimises $\dim_M$.

$(\Sigma \to \mathbb{R}^d)$

eg projection of $(r_1^s, \dots, r_N^s)^{\mathbb{N}}$ Bernoulli measure on Σ

⑦ Generalised chaos game (joint B. Barany, I. Kolossvary).

IFS $(S_1, \dots, S_N)$

Consider μ on Σ

$i_1 \dots i_n$ chosen with prob $\mu([i_1 \dots i_n])$

Def Say μ satisfies one-sided exponential decay of correlations if $\exists \varepsilon > 0, \kappa > 0$ s.t. $\forall \underline{i}, \underline{j}$ finite

$$\mu([\underline{i}] \cap \sigma^{-k}([\underline{j}])) \leq (1 + \kappa 2^{-\varepsilon(k-n)}) \mu([\underline{i}]) \mu([\underline{j}])$$

eg. Gibbs for Hölder potentials
some Kaenmaki measures

Thm I (a.s. version)

Suppose $\dim_L F > 0$. and μ has exp decay.
Then for μ a.e. $\underline{i}$ $\forall x_0 \in F$

$$\lim_{\delta \to 0} \frac{\log W_{\delta, x_0}(\underline{i})}{-\log \delta} = \dim_M(\overleftarrow{\mu} \circ \Pi^{-1}) =: \alpha$$

$\overleftarrow{\mu}([i_1 \dots i_n]) = \mu([i_n \dots i_1])$

Thm II (expectation)

$$l(\delta) \delta^{-\alpha + O(\delta)} \leq \mathbb{E}(W_{\delta, x_0}) \leq L(\delta) \delta^{-\alpha + O(\delta)}$$

logarithmic in δ

$$O(\delta) := \frac{\log \min_x \overleftarrow{\mu} \circ \Pi^{-1}(B(x, \delta))}{-\log \delta} - \alpha$$