# THE HAUSDORFF DIMENSION OF MEASURES WHICH CONTRACT ON AVERAGE

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Abstract. In this note we consider measures supported on limit sets of systems that contract on average. In particular, we present an upper bound on their Hausdorff dimension.

## 1. Introduction and Statement of Results

In this note we want to consider measures supported on limit sets of systems that contract on average. There have been many articles concerning finding upper bounds on the Hausdorff dimension of measures for attractors (and stationary measures for strictly contracting iterated function systems (IFS)) or the closely related case of repellers and invariant measures for expanding maps. For conformal maps there are quite comprehensive results and, even in the case of non-conformal results there are a number of strong results. For example, [2],[6] deal with the linear case and [1],[9] with the nonlinear case. In [4] iterates of random functions which contract on average are considered and this idea can be put into the framework of IFS which contract on average. In [8] with the addition of random errors the exact value of the dimension is computed almost surely. However, when we turn to the problem of estimating the Hausdorff dimension of measures, for IFS which contract on average, most previous authors have concentrated on the case when the maps are conformal. Our aim is to find upper bounds for the general case. Although there have been several papers which provide upper bounds for the Hausdorff dimension of the measures defined by such systems, including [7] and [3], our results are also new in the uniformly contracting case.

In this paper we shall consider an iterated function system in $\mathbb{R}^d$ which contracts on average. Our aim is to provide a sharp upper bound for the Hausdorff dimension of natural measures defined using such systems. Let $0 < \gamma_1^{(i)} < 1 < \gamma_2^{(i)}$, $i = 1, \cdots, m$ and $f_1, \ldots, f_m : \mathbb{R}^d \to \mathbb{R}^d$ be $C^2$ diffeomorphisms satisfying

$$0 < \gamma_1^{(i)} \leq ||df_i|| \leq \gamma_2^{(i)} \text{ for all } 1 \leq i \leq m.$$

We can denote $\gamma_1 = \min_{1\le i\le m}\gamma_1^{(i)}$ and $\gamma_2 = \max_{1\le i\le m}\gamma_2^{(i)}$. Let $\Sigma_m$ be the full shift on $m$ symbols. We shall consider ergodic probability measures $\mu$ on $\Sigma_m$ satisfying

$$\eta := \sum_i \mu(\{\underline{x} : x_0 = i\}) \log \gamma_2^{(i)} < 0. \tag{1}$$

If this is the case we say that the iterated function system *contracts on average*. The sequence

$$f_{i_1} \circ \cdots \circ f_{i_n}(0)$$

converges for $\mu$ almost all $\underline{i}$ (see [4]) and we will denote

$$\Pi(\underline{i}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0).$$

This is well defined for $\mu$-almost all $\underline{i}$ and so we can let $\nu = \mu \circ \Pi^{-1}$. If the system is uniformly contracting and the open set condition is satisfied then [9] gives an upper bound for the dimension. If there are expansions in the IFS then the limit set will include $\infty$ and in many cases is equal to be the whole of $\mathbb{R}^d$ (see [7] for examples). For the linear case where the contractions are less than $\frac{1}{2}$ typical values for the Hausdorff dimension of the attractor and stationary measures can be computed [2], [6]. If the linear maps have norm less than 1 then adding random perturbations gives an almost sure equality, [6].

For $A \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^d)$ we define the singular values

$$\alpha_1(A) \geq \cdots \geq \alpha_d(A)$$

to be the eigenvalues of $(A^*A)^{1/2}$. For $1 \le j \le d$ we define $\alpha_j(x, f_i) = \alpha_j(Df_i(x))$. For $0 \le s \le d$ choose $k = [s] + 1$ and define

$$\phi^s(f_i, x) = \log \alpha_1(x, f_i) + \ldots + (s - k + 1) \log \alpha_k(x, f_i).$$

Our first result describes the subadditive behaviour of $\phi^s(f_i, x)$.

**Lemma 1.** *For any $0 \le s \le d$ the function $\phi^s$ satisfies the following subadditive property*

$$\phi^s(f_{i_1} \circ f_{i_2}, x) \le \phi^s(f_{i_1}, f_{i_2}(x)) + \phi^s(f_{i_2}, x)$$

*Proof.* This can be proved using Lemma 2.1 in [2] which states that for $T, U \in \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^d)$ we have that

$$\phi^s(TU) \le \phi^s(T)\phi^s(U), \tag{2}$$

where $\phi^s(T)$, etc., have the obvious interpretation. By the chain rule we have that $D_x(f_{i_1} \circ f_{i_2}) = D_{f_{i_2}(x)}(f_{i_1}) D_x(f_{i_2})$ and the result follows from the definition of $\phi^s(f, x)$ and (2). $\square$

Let $g^s : \Sigma_m \to \mathbb{R}$ be defined by

$$g^s(\underline{i}) = \phi^s(f_i, \Pi(\sigma(\underline{i}))).$$

It follows by the Birkhoff Ergodic Theorem that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g^s(\sigma^j \underline{i}) = \int g^s(\underline{i}) \mathrm{d}\mu(\underline{i}) =: g^s(\mu) \tag{3}$$

for $\mu$ almost all $\underline{i} \in \Sigma_m$.

Given a Borel set $X$ we define $H^s_\rho(X)$ to be the infimum of the summations $\sum_j r_j^s$ where $\cup_j B(z_j, r_j) \supseteq X$, is a finite cover by balls $B(z_j, r_j)$ with radii satisfying $r_j \le \rho$. The Hausdorff dimension of $X$ is then given by

$$\dim_{\mathcal{H}}(X) = \inf \left\{ s : \lim_{\rho\to 0} H^s_\rho(X) = 0 \right\}.$$

We recall that the Hausdorff dimension of measure is the infimum of the dimensions of Borel sets of full measure. We now have our first upper bound for the Hausdorff dimension of $\nu$.

**Lemma 2.** *Let $s$ satisfy*

$$g^s(\mu) = -h(\mu)$$

*then we have that*

$$dim_{\mathcal{H}}(\nu) \le s.$$

However, it is possible to improve on this bound. For $\underline{i} \in \Sigma_m$ we consider the values

$$\frac{1}{n} \phi^s(f_{i_1} \circ \cdots \circ f_{i_n}, \Pi(\sigma^n \underline{i})).$$

By the sub-additive ergodic theorem [5] this converges almost surely to

$$f^s(\mu) := \inf_{n\ge 1} \left\{ \frac{1}{n} \int \phi^s(f_{i_1} \circ \cdots \circ f_{i_n}, \Pi(\sigma^n \underline{i})) d\mu(\underline{i}) \right\}.$$

Now we consider the iterated function system formed by taking the iterates $f_{i_1} \circ \cdots \circ f_{i_n}$ and the same measure $\mu$. In this case we can define $g_n^s(\mu)$ by

$$g_n^s(\mu) = \int \phi^s(f_{i_1} \circ \cdots , \circ f_{i_n}, \Pi(\sigma^n \underline{i})) \mathrm{d}\mu(\underline{i})$$

and note that considering the system of $n$th level iterates will cause the entropy to be multiplied by $n$. Thus applying Lemma 2 gives that

$$\dim_{\mathcal{H}}(\nu) \le s_n$$

where $s_n$ satisfies
$$\frac{1}{n} g_n^{s_n}(\mu) = -h(\mu).$$
Moreover, by the subadditive ergodic theorem we have that
$$\inf_{n \geq 1} \left\{ \frac{1}{n} g_n^s(\mu) \right\} = f^s(\mu).$$
Hence
$$\dim_{\mathcal{H}}(\nu) \leq s$$
where $s = \inf_{n\geq 1} s_n$ satisfies
$$f^s(\mu) = -h(\mu),$$
by the subadditivity of $\phi^s$.

Our next step is to show how $f^s(\mu)$ can be written in terms of Lyapunov exponents.

**Lemma 3.** *There exist constants*
$$0 > \lambda_1(\mu) \geq \cdots \geq \lambda_d(\mu)$$
*where for $\mu$ almost all $\underline{i}$*
$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_j(D_{\Pi(\sigma \underline{i})} f_{i_1} \cdots D_{\Pi(\sigma^n \underline{i}))} f_{i_n}) = \lambda_j$$

*Proof.* This follows from Oseledec's Multiplicative Ergodic Theorem [5]. □

We now come to our main result.

**Theorem 1.** *Let $\nu = \mu \circ \Pi^{-1}$ be the stationary measure for an iterated function system as defined above. We have that*
$$dim_{\mathcal{H}} \nu \leq \min_{1 \leq k \leq d} \left\{ k - 1 - \frac{h(\mu) + \sum_{j=1}^{k-1} \lambda_j}{\lambda_k} \right\}. \tag{4}$$

*Proof of Theorem 1 (assuming Lemma 2).* The proof of Lemma 2 will be given in the remainder of the paper. Fix $0 \leq s \leq d$. It follows by the definitions of $g_n^s(\mu)$ and $f^s(\mu)$ and by Lemma 3 that
$$f^s(\mu) = \sum_{j=1}^{k-1} \lambda_j + (s - k + 1)\lambda_k \text{ for } k - 1 \leq s \leq k.$$
Let $s_0$ be the solution in $s$ to the identity $-h(\mu) = f^s(\mu)$ and $k$ where $k - 1 \leq t \leq k$. We have that
$$-h(\mu) = \lambda_1 + \ldots + \lambda_{k-1} + (s_0 - k + 1)\lambda_k$$

and thus

$$s_0 = (k-1) - \frac{h(\mu) + \lambda_1 + \ldots + \lambda_{k-1}}{\lambda_k}.$$

A routine, if long, calculation shows that this is the minimum given in (4). □

In the case of conformal maps we always have an equality, although there exist examples of affine contractions for which there is a strict inequality. However, with random perturbations to affine contractions we can recover the equality in the context of random attractors [6].

## 2. Calculating Hausdorff Measure and Hausdorff dimension

The key to our proof of Lemma 2 is estimating how one iteration of each map effects the Hausdorff measure. For this purpose we need a simple result regarding the derivative of a diffeomorphism.

**Lemma 4.** *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a $C^1$ diffeomorphism. For any $\epsilon > 0$, $r > 0$ we can find $\rho$ such that for $z, y \in B(0,r)$ with $||z-y|| \leq \rho$ we have*

$$||f(z) - f(y) - D_z f(z-y)|| \leq \epsilon ||z-y|| \tag{5}$$

*and*

$$\text{for } 1 \leq j \leq d \text{ we have that } |\alpha_j(f,z) - \alpha_j(f,y)| \leq \epsilon. \tag{6}$$

*Proof.* Let $\epsilon > 0$ we can find $\rho_1$ such that condition (5) is satisfied by Frechet differentiability of $f$. The Frechet derivative $Df : \mathbb{R}^d \to \mathbb{R}^d$ is a linear map. Since $Df$ is uniformly continuous we can find $\rho_2$ such that for $||y-z|| \leq \rho_2$ and any $x \in B(0,1)$ we have $||D_y f(x) - D_z f(x)|| \leq \epsilon$. Thus $D_y f(B(0,1)) \subset B(D_z f(B(0,1)), \epsilon)$ where $B(D_z f(B(0,1)), \epsilon)$ denotes an $\epsilon$-neighbourhood of the image $D_z f(B(0,1))$. Since the singular values of $D_z f$ are given by the principal axes of the ellipsoid $D_z f(B(0,1))$ it follows that $D_y f(B(0,1))$ will be contained inside the ellipsoid with principal axes $\alpha_1(f,z) + \epsilon, \ldots, \alpha_d(f,z) + \epsilon$. Similarly, $D_z f(B(0,1))$ will be contained inside the ellipse with axes $\alpha_1(f,y) + \epsilon, \ldots, \alpha_d(f,y) + \epsilon$. Thus for each $1 \leq j \leq d$, we have $|\alpha_j(f,z) - \alpha_j(f,y)| \leq \epsilon$. □

We can now prove a result estimating the effect on Hausdorff measure of an iteration of $f$. This is very similar in nature to Lemma 3 and Corollary 1 in [9].

**Lemma 5.** *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a $C^1$ diffeomorphism. We can choose $\rho$ sufficiently small such that for $A \subset B(x, \rho) \subset \mathbb{R}^d$ we have*

$$H^s_{b\rho}(f(A)) \leq C H^s_\rho(A)$$

*where $C = 2^s d^{s/2} \exp(\phi^s(f,x))$ and $b = 2\sqrt{d}\exp(\alpha_1(f,x))$.*

*Proof.* We choose $\epsilon > 0$ to be sufficiently small such that $(1+\epsilon)e^\epsilon < 2$. By Lemma 4 we can find $\rho$ such that both (5) and (6) are satisfied. Let $H^s_\rho(A) = h$. It follows that for $\delta > 0$ we can find a finite set of balls $B(z_i, r_i)$ where $\cup_i B(z_i, r_i) \supseteq A$, $r_i \leq \rho$ for all $j$ and $\sum_i r_j^s < h + \delta$. By definition, the sets $f(B(z_i, r_i))$ cover $f(A)$. Furthermore, due to our choice of $\rho$ and Lemma 4 these will be contained in ellipses with principal axes $(1+\epsilon) r_i \exp(\alpha_j(f,x) + \epsilon)$, $j = 1, \cdots, d$. More precisely, by (5) $f(B(z_i, r_i))$ is contained in an ellipse with principal axes

$$(1+\epsilon) r_j \exp(\alpha_j(f, z_j))$$

and we can then apply (6). Choose $k$ such that $k - 1 \leq s \leq k$. We can cover $f(B(z_i, r_i))$ with

$$\left[\frac{\exp(\alpha_1(f,x) + \ldots + \alpha_{k-1}(f,x) + (k-1)\epsilon)}{\exp((k-1)\alpha_k(f,x) + \epsilon)}\right] + 1$$

balls of radius $(1+\epsilon)\sqrt{d}\exp(\alpha_k + \epsilon) r_i$. Thus we have that

$$\begin{aligned} H^s_{b\rho}(f(A)) &\leq \frac{\exp(\alpha_1(f,x) + \ldots + \alpha_{k-1}(f,x) + (k-1)\epsilon)}{\exp((k-1)(\alpha_k(f,x) + \epsilon))} \\ &\quad \times d^{s/2} \exp(s(\alpha_k + \epsilon))(1+\epsilon)^s \sum_i r_i^s \\ &\leq e^{s\epsilon}(1+\epsilon)^s d^{s/2} \exp(\phi^s(f,x)) \sum_i r_i^s \\ &\leq C(h + \delta). \end{aligned}$$

Since $\delta$ was arbitrary the proof is complete. □

The next lemma provides a simple method for giving an upper bound to the Hausdorff dimension of a measure.

**Lemma 6.** *Let $\mu$ be a probability measure on $\mathbb{R}^d$. If we can find a sequence of sets $A_n$ such that*

(1) $\lim_{n\to\infty} \mu(A_n) = 1$
(2) $\lim_{n\to\infty} H^s_{\beta_n}(A_n) = 0$ *for a sequence* $\{\beta_n\}_{n\in\mathbb{N}}$ *where* $\lim_{n\to\infty} \beta_n = 0$

*then it follows that*

$$dim_{\mathcal{H}}(\mu) \leq s.$$

*Proof.* We can choose a subsequence $\{B_n\}_{n\in\mathbb{N}}$ where $\mu(B_n) > 1-\left(\frac{1}{2}\right)^n$ for all $n$. Fix $t\in\mathbb{N}$ and let $Y_t = \cap_{n\geq t} B_n$. Observe that

$$\mu(Y_t) \geq 1-\sum_{n=t}^{\infty}\left(\frac{1}{2}\right)^n = 1-\left(\frac{1}{2}\right)^{t-1}.$$

For any $n\geq t$ a cover of $B_n$ is also a cover of $Y_t$ and so $H^s(Y_t)=0$ thus implying $\dim_{\mathcal{H}} Y_t \leq s$. Furthermore $\mu(\cup_{t\in\mathbb{N}} Y_t) = 1$ and $\dim_{\mathcal{H}}(\cup_{t\in\mathbb{N}} Y_t) \leq s$ which is sufficient to complete the proof. □

## 3. Proof of Lemma 2.

The method of proof of Lemma 2 involves applying Lemma 6. To begin we need to define a suitable sequence of sets. This will be done by defining suitable subsets on the shift space, $\Sigma_m$ and then projecting to $\mathbb{R}^d$. Recall the definition of $\eta$ given in (1). Fix $\epsilon > 0$ such that $\eta+\epsilon < 0$. We then choose $t$ such that $g^t(\mu)+h(\mu) = -3\epsilon$. It is clear that as $\epsilon\to 0$ we have $t\to s$ from above. Let $C_0 > 2^t d^{e/2}$ and choose $N$ such that

$$C_0 e^{N\epsilon} < 1 \text{ and } e^{N(\eta+\epsilon)} > 2\sqrt{d}. \tag{7}$$

We would next like to choose sets $X_n \subset \Sigma_m$ such that any $\underline{i} \in X_n$ satisfies

(1) $e^{-nN(h(\mu)+\epsilon)} \leq \mu([i_1,\ldots,i_{nN}]) \leq e^{-nN(h(\mu)-\epsilon)}$
(2) $nN(g^s(\mu)-\epsilon) \leq \sum_{i=0}^{nN-1} g(\sigma^i(\underline{i})) \leq nN(g^s(\mu)+\epsilon)$
(3) $\log \|df_{i_1}\circ\cdots\circ df_{i_{kN}}\| \leq kN(\eta+\epsilon)$ for all $k\geq[\log n]$.
(4) Let $r_n = n^2$, then we have that $\Pi(\sigma^{nN}(\underline{i})) \in B(0,r_n)$.

We then write $\Lambda_n = \Pi X_n$ . It remain to show that we can choose sets $X_n$, and in such a way that $\Lambda_n$ satisfies the conditions of Lemma 6.

**Lemma 7.** *We can find sets $X_n$ satisfying the above hypotheses and thus*

$$\lim_{n\to\infty}\nu(\Lambda_n) = 1.$$

*Proof.* By the definition of $\nu$ and $\Lambda_n$, to get that $\nu(\Lambda_n)\to 1$ it suffices to show that $\mu(X_n)\to 1$. Thus it suffices to show that as $n\to\infty$ the $\mu$ measure of sequences satisfying each of the four conditions above will converge to 1. The fact that sequences satisfying conditions (1) and (2) have measure tending to 1 follows from the Shannon-Macmillan-Brieman Theorem and the Birkhoff Ergodic Theorem [5], respectively. For condition (3), note by the Birkhoff Ergodic Theorem applied to $\log\gamma_2^{(i_1)}$ we have that for $\mu$-almost all $\underline{i}$

$$\lim_{k\to\infty}\frac{1}{k}\log\gamma_2^{(i_1)}\cdots\gamma_2^{(i_{kN})} = \int\log\gamma_2^{(j_k)}\mathrm{d}\mu(\underline{j} = \eta. \tag{8}$$

The result then follows since

$$\log ||df_{i_1} \circ \cdots \circ df_{i_{kN}}|| \leq \log \gamma_2^{(i_1)} \cdots \gamma_2^{(i_{kN})}.$$

For condition (4) we first note that

$$\mu\{\underline{i} : \Pi(\underline{i}) \in B(0, r_n)\} \to 1 \text{ as } n \to \infty.$$

Since $\mu$ is shift invariant it then follows that

$$\mu\{\underline{i} : \Pi(\sigma^{nN}\underline{i}) \in B(0, r_n)\} \to 1 \text{ as } n \to \infty$$

which is sufficient to complete the proof. □

We now need to consider the second condition from Lemma 6. We define a sequence $\{\beta_n\}$ by

$$\beta_n = 2\sqrt{d}e^{nN(\eta+\epsilon)}.$$

Fix $\rho$ as in Lemma 5. For a sequence $\underline{i} \in X_n$ we want to consider the following set

$$B_{nN}(\underline{i}, \rho) = \left\{ \underline{j} \in \Sigma_m : (i_1, \ldots, i_{nN}) = (j_1, \ldots, j_{nN}) \text{ and } d\left(\Pi\sigma^{nN}(\underline{i}), \Pi\sigma^{nN}(\underline{j})\right) \leq \frac{\rho}{\gamma_2^{Nk}} \right\}$$

where $k = \log n$. An important property of these sets is that for $\underline{l} \in B_{nN}(\underline{i}, \rho)$ and $0 \leq j \leq nN$ we have that

$$\Pi(\sigma^j \underline{l}) \in B(\Pi(\sigma^j \underline{i}), \rho).$$

For notational convenience we will write

$$B_0(\underline{i}) = \left\{ \underline{j} \in \Sigma_m : d(\Pi\underline{i}, \Pi\underline{j}) \leq \frac{\rho}{\gamma_2^{Nk}} \right\}.$$

**Lemma 8.** *We can find a finite set $Y_n \subset \Sigma_m$ with at most*

$$\left[ \frac{2\sqrt{d}n^2 n^{N\log\gamma_2}}{\rho} e^{nN(h(\mu)+\epsilon)} \right] + 1$$

*points such that*

$$\cup_{\underline{i}\in Y_n} B_{nN}(\underline{i}, \rho) \supseteq X_n.$$

*Proof.* By property (1) of $X_n$ each $\underline{i} \in X_n$ satisfies

$$\mu([i_1, \ldots, i_{nN}]) \geq e^{-nN(h(\mu)+\epsilon)}$$

and hence since $\mu$ is a probability measure it follows that there are at most $e^{nN(h(\mu)+\epsilon)}$ choices for the first $nN$ elements of a sequence in $X_n$. Fix one of these choices $[i_1, \ldots, i_{nN}]$. Consider

$$\Pi(\sigma^{-nN}(X_n \cap [i_1, \ldots, i_{nN}]))$$

and note that we can find a centred covering with at most $\frac{2\sqrt{d}n^2 n^{N\log\gamma_2}}{\rho}$ balls of size less than $\frac{\rho}{\gamma_2^{Nk}}$. □

We now use Lemma 5 to estimate the Hausdorff measure of one of these sets.

**Lemma 9.** *Fix $\rho$ as in Lemma 5. Let $\underline{i} \in X_n$. We have that*

$$H^t_{b_n\rho}(\Pi(B_{nN}(\underline{i}))) \leq C_0 C_1^n \exp\left(\sum_{j=0}^{nN-1} g^t(\sigma^j(\underline{i})\right).$$

*Where*

$$\begin{aligned}
b_n &= (2\sqrt{d})^n \exp\left(\sum_{j=0}^{nN-1} \alpha_1(f_{\sigma^j(\underline{i})_1}, \Pi(\sigma^{j+1}\underline{i}))\right)\\
C_0 &= \sup_{x\in\mathbb{R}^d} H^t_\rho(B(x,\rho))\\
C_1 &= 2^s d^{s/2}
\end{aligned}$$

*Proof.* For $1 \leq k \leq N$ consider the sets

$$f_{i_1} \circ \cdots \circ f_{i_{kN}}(\Pi(B_{nN}(\underline{i},\rho)))$$

and note that they all have diameter less than $\rho$. Thus we can apply Lemma 5 iteratively $n$ times to get

$$\begin{aligned}
H^s_{b_n\rho}(\Pi(B_{nN}(\underline{i},\rho))) \leq \quad & (2^t d^{t/2})^n \exp\left(\sum_{j=0}^{n-1} \phi^t(f_{i_{jN+1}} \circ \cdots \circ f_{i_{(j+1)N}}, \Pi(\sigma^{(j+1)N}(\underline{i}))\right)\\
& \times H^t_\rho(\Pi(B_0(\sigma^{nN}\underline{i},\rho)))
\end{aligned}$$

where $b_n$ is as in the statement of the Lemma. By the subadditivity of $\phi^t$ and the definition of $g^t$ we have that

$$\exp\left(\sum_{j=0}^{n-1} \phi^t(f_{i_{jN+1}} \circ \cdots \circ f_{i_{(j+1)N}}, \Pi(\sigma^{(j+1)N}(\underline{i}))\right) \leq \exp\left(\sum_{j=0}^{nN-1} g^t(\sigma^j(\underline{i})\right).$$

So if we let

$$C_0 = \sup_{x\in\mathbb{R}^d} H^t_\rho(B(x,\rho))$$

then we have

$$H^s_{b_n\rho}(\Pi(B_{nN}(\underline{i},\rho))) \leq C_0 C_1^n \exp\left(\sum_{j=0}^{nN-1} g^s(\sigma^j(\underline{i})\right)$$

and the proof is complete. □

By applying Lemmas 8 and 9 we get a result which shows that the sets $\Lambda_n$ satisfy the conditions to apply Lemma 6.

**Lemma 10.** *We have that*

$$H^t_{\rho c_n}(\Lambda_n) = O(n^{2+N\log\gamma_2} C_1^n e^{-nN\epsilon})$$

*where* $e^{N\epsilon} > C_1$,

$$c_n = (2\sqrt{d})^n(\eta+\epsilon)^N$$

*and* $c_n \to 0$ *as* $n \to \infty$.

*Proof.* From Lemma 8 it follows that we can find a subset $Y_n = \{\underline{i}^{(1)}, \ldots, \underline{i}^{(j)}\} \subset X_n$ where $\cup_{Y_n}\Pi(B_{nN}(\underline{i},\rho)) \supseteq \Lambda_n$ and $j = O(n^{2+N\log\gamma_2})e^{nN(h(\mu)+\epsilon)}$. Fixing $1 \le k \le j$ and applying Lemma 9 to $B_nN(\underline{i}^{(k)},\rho)$ gives that

$$H^t_{b_n\rho}(\Pi(B_{nN}(\underline{i}^{(k)}))) \le C_0 C_1^n \exp\left(\sum_{j=0}^{nN-1} g^t(\sigma^j(\underline{i}^k)\right).$$

However by applying condition 3 of the definition of $X_n$ it follows that $b_n \le c_n$. We can also apply condition 2 to get that

$$H^t_{c_n\rho}(\Pi(B_{nN}(\underline{i}^{(k)}))) \le C_0 C_1^n \exp(nN(g^s(\mu)+\epsilon).$$

This gives us that

$$\begin{aligned} H^t_{c_n\rho}(\Lambda_n) &\le \sum_{k=1}^{j} H^t_{b_n\rho}(\Pi(B_{nN}(\underline{i}^{(k)}))) \\ &\le O(n^{2+N\log\gamma_2})C_0 C_1^n e^{nN(h(\mu)+g^t(\mu)} = O(n^{2+N\log\gamma_2})C_0 C_1^n e^{-nN\epsilon}. \end{aligned}$$

Since $N$ satisfies the conditions specified by (7) the convergence to 0 of the Hausdorff measure and $c_n$ is easy to see. □

Combining Lemmas 10 and 7 completes the proof of Lemma 2.

## 4. Examples

We will now give some simple examples in $\mathbb{R}^2$ to which our results can be applied.

**Example 1.** *Let* $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$\begin{aligned} f_1(x,y) &= \left(\frac{x}{2}, \frac{y}{3}\right) \\ f_2(x,y) &= (x+1, y+1) \end{aligned}$$

*If we take* $\mu$ *to be* $\left(\frac{1}{2},\frac{1}{2}\right)$*-Bernoulli measure on* $\Sigma_2$ *and* $\nu$ *to be the natural projection of* $\mu$ *then condition (1) is clearly satisfied. Moreover we can* *easily calculate* $\lambda_1(\mu) = \frac{1}{2}\log\frac{1}{2}$ *and* $\lambda_2(\mu) = \frac{1}{2}\log\frac{1}{3}$. *Thus applying, Theorem 1 we get that*

$$dim_{\mathcal{H}}(\nu) \leq \min\left\{-\frac{h(\mu)}{\lambda_1(\mu)}, 1 - \frac{h(\mu)+\lambda_1(\mu)}{\lambda_2(\mu)}\right\} = 1 + \frac{\log 2}{\log 3},$$

*since* $h(\mu) = \log 2$. *In this case since both the matrices were diagonal the Lyapunov exponents were easy to calculate. Moreover the upper bounds given by Lemma 2 and Theorem 1 are identical. The limit set is the sector* $\{(x,y) : 0 \leq y \leq x\}$ *which clearly has Hausdorff dimension* 2.

*We can consider a small non-linear perturbation of this example. Let* $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ *now be defined by*

$$\begin{aligned} f_1(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right) \\ f_2(x,y) &= (x(1+\epsilon\sin y)+1, y(1+\epsilon\sin x)+1) \end{aligned}$$

*where* $|\epsilon| > 0$ *is small. If we again take* $\mu$ *to be* $\left(\frac{1}{2},\frac{1}{2}\right)$*-Bernoulli measure on* $\Sigma_2$ *and* $\nu$ *to be the natural projection of* $\mu$ *then* $h(\mu) = \log 2$ *and we can use the trivial bounds* $|\lambda_1(\mu) - \frac{1}{2}\log\frac{1}{2}| < \frac{1}{2}\log(1+\epsilon)$ *and* $|\lambda_2(\mu) - \frac{1}{2}\log\frac{1}{3}| < \frac{1}{2}\log(1+\epsilon)$. *Thus applying Theorem 1 we get that*

$$dim_{\mathcal{H}}(\nu) \leq \min\left\{-\frac{h(\mu)}{\lambda_1(\mu)}, 1 - \frac{h(\mu)+\lambda_1(\mu)}{\lambda_2(\mu)}\right\} \leq 1 + \frac{\log 2 + \log(1+\epsilon)}{\log 3 - \log(1+\epsilon)}$$

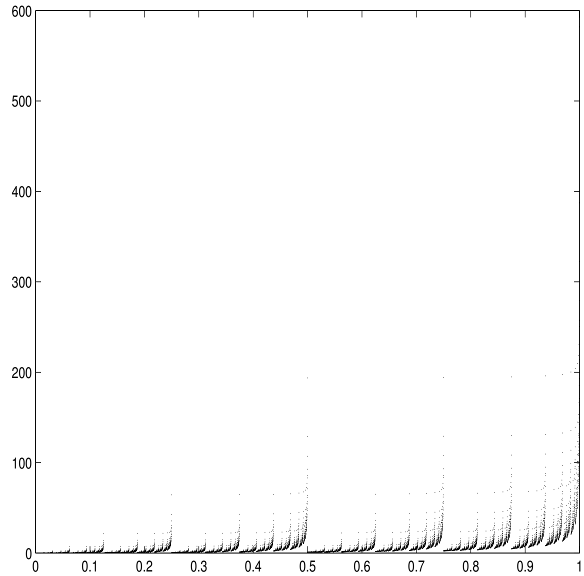
**Example 2.** *Let* $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$\begin{aligned} f_1(x,y) &= \left(\frac{x}{2}, \frac{y}{2}\right) \\ f_2(x,y) &= \left(\frac{x+1}{2}, \frac{3y}{2}+1\right) \end{aligned}$$

*If we again take* $\mu$ *to be* $\left(\frac{1}{2},\frac{1}{2}\right)$*-Bernoulli measure on* $\Sigma_2$ *and* $\nu$ *to be the natural projection of* $\mu$ *then condition (1) is clearly satisfied. Moreover we can easily calculate* $\lambda_1(\mu) = \log\frac{\sqrt{3}}{2}$ *and* $\lambda_2(\mu) = \log\frac{1}{2}$. *Thus applying Theorem 1 we get that*

$$dim_{\mathcal{H}}(\nu) = 1 + \frac{\log 3}{2\log 2} = 1 \cdot 34417 \cdots .$$

*In this case the limit set can be viewed as a measurable graph over the interval* $[0, 1]$.




*Remark.* In Example 1, if we change $\mu$ to the $(p, 1-p)$-Bernoulli measure, then as $p \to 0$ the upper bound becomes larger than 2, and thus gives no useful information. On the other hand, if $p \to 1$ then the upper bound converges to 0. In the case of Example 2, the system only contracts on average if $p < \frac{\log 2}{\log 3}$. As $p \to 0$ the upper bound converges to 0.

**Example 3.** *Let* $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$\begin{aligned} f_1(x, y) &= (0.3x + 0.2y, 0.2x + 0.3y) \\ f_2(x, y) &= (1.2x + 0.2y + 1, 0.1x + 1.2y + 1) \end{aligned}$$

*Let* $\mu$ *on the shift space be the* $\left(\frac{1}{2}, \frac{1}{2}\right)$*-Bernoulli Measure and let* $\nu$ *be the natural projection. In many cases, calculating the Lyapunov exponents can be an extremely difficult problem, but the upper bound in Lemma 2 remains easier to calculate. By taking iterates of the function it is possible to improve this estimate and eventually the values will converge to that given in Theorem 1. For this example, we give below the upper bounds* $s_n$ *given by the argument following Lemma 2 for different values of* $n$.

| *Value of n* | *Upper bound $s_n$ on dimension* |
|---|---|
| 1 | 1.4412 |
| 2 | 1.4412 |
| 6 | 1.4410 |
| 10 | 1.4409 |
| 18 | 1.4408 |

## References


[1] K. Falconer, Bounded distortion and dimension for nonconformal repellers, Math. Proc. Cambridge Philos. Soc. 115 (1994), no. 2, 315–334.

[2] Falconer, K. J. The Hausdorff dimension of self-affine fractals, Math. Proc. Cambridge Philos. Soc., 103 (1988), no. 2, 339–350.

[3] A. Fan, K. Simon and H. Tóth, Contracting on average random IFS with repelling fixed point., J. Stat. Phys. 122 (2006), no. 1, 169–193.

[4] P. Diaconis and D. Freedman, Iterated random functions, SIAM Rev, 41 (1999), no. 1, 45–76.

[5] U. Krengel, Ergodic Theorems, de Gruyter, Berlin, 1985

[6] T. Jordan, M. Pollicott and K. Simon, Hausdorff dimension for randomly perturbed self affine attractors, Preprint available at http://www.maths.warwick.ac.uk/∼tjordan/random.pdf.

[7] M. Nicol, N. Sidorov and D. Broomhead, On the fine structure of stationary measures in systems which contract-on-average, J. Theoret. Probab., 15 (2002), no. 3, 715–730.

[8] Y. Peres, K. Simon and B. Solomyak, Absolute continuity for random iterated function systems with overlaps, Preprint available at www.math.bme.hu/∼simonk/papers/PSS3.pdf

[9] Y. Zhang, Dynamical upper bounds for Hausdorff dimension of invariant sets, Ergodic Theory Dynam. Systems, 17 (1997), no. 3, 739–756.