PROPERTIES OF ITERATED FUNCTION SYSTEMS WITH OVERLAPS

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Contents

Abstract

In the first chapter we give an introduction to dimension theory and iterated function systems. This provides a background and the necessary basic results for the rest of the thesis. The second chapter gives an account of the 'transversality' technique. This technique is used in the last three chapters of the thesis. In chapter 3 we look at bounds of the local dimension of stationary measures for some iterated function systems with overlaps.

In chapters 4,5 and 6 we use the transversality technique to investigate families of iterated function systems in the plane. We also consider the question of whether it is possible for a self-similar set to have positive measure and no interior. Chapter 4 contains work from [18]. Chapter 5 contains joint work with my supervisor Mark Pollicott which is found in [19]. Chapter 6 contains the material in [5] which is joint with Marianna Csörnyei, Mark Pollicott, David Preiss and Boris Solomyak.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Chapter 1

Introduction and Important Definitions

1.1 Content of the Thesis

The work in this thesis involves analyzing the properties of iterated function systems. These involve applying a finite (or occasionally countable) collection of functions on a metric space. The functions are usually contractions and the only metric spaces we will consider are $\mathbb{R}^n$. A compact non-empty set is called the attractor of the iterated function system if the union of all the functions applied to the set is the set itself. In the case where the metric space is $\mathbb{R}^n$ and all the functions are strictly contracting this attractor is unique. The attractor is often an uncountable set with zero Lebesgue measure.

Attractors for iterated function systems often have what are called fractal properties. There is no precise definition of fractal but fractal properties can include intricate detail at arbitrary small scales, smaller parts of the set resembling the whole set, and a non-integer dimension (for appropriately defined dimension). Two standard examples of such sets are the middle third Cantor set and the Sierpiński gasket. Figure 1.1 shows the construction of the Sierpiński gasket and figure 1.1 shows a picture of the Sierpiński gasket. Both sets can easily be represented as attractors of iterated function systems.

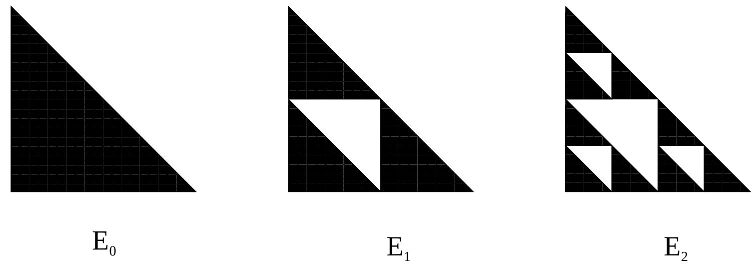

Figure 1.1: E shows the initial process. E_1 shows E split into four smaller triangles the shaded ones are retained. The process is then repeated to give E_2 and can be repeated infinitely to give the Sierpiński gasket.

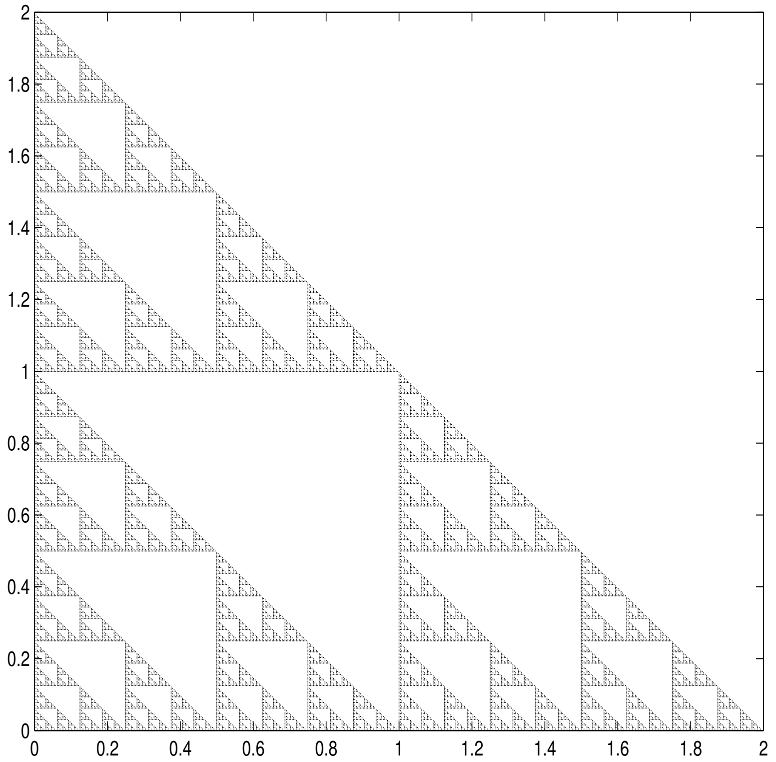

Figure 1.2: The Sierpiński gasket

Throughout the thesis we will use the notion of an iterated function system given by Hutchinson in [16]. However some of the concepts used were around for much longer. In this paper the open set condition, a condition which essentially ensures that the images of the functions only overlap in a trivial manner, is introduced. The dimension theoretic properties of the attractor are understood well in the case where the functions are conformal and the open set condition is satisfied. However the cases where either the open set condition is not satisfied or the functions are not conformal are much less well understood. We will concentrate on the case where the open set condition is not satisfied.

The main features of these attractors which we will investigate are the Lebesgue measure and the Hausdorff dimension. Hausdorff dimension is the most mathematically natural definition of fractal dimension. It is based on Hausdorff measure which is essentially a generalization of Lebesgue measure to non-integer dimensions. For any set there always exists t such that for any $s < t$ the s-dimensional Hausdorff measure is infinite and for any $t > s$ this value is zero. The Hausdorff dimension of the set is the unique value t.

In the rest of the chapter we will give the definitions and background results necessary for the rest of the thesis. In chapter 2 we describe a technique, known as transversality, for studying iterated function systems which don't satisfy the open set condition. In chapter 3 some uniform estimates of the local dimension of stationary measures on iterated function systems are found. Finally in Chapters 4,5 and 6 we use transversality to investigate families of iterated function systems in $\mathbb{R}^n$ for $n \geq 2$.

1.2 Basic Measure Theory

Measure theory plays an essential role throughout this thesis. $\mathcal{L}^n$ will refer to n dimensional Lebesgue measure and $\mathcal{L}$ will refer to 1 dimensional Lebesgue measure. The set of Borel measures supported on $\mathbb{R}^n$ will be referred to as $M(\mathbb{R}^n)$. For a measure $\mu \in M(\mathbb{R}^n)$ a result is said to hold for μ almost every (a.e.) value if the values where the result does not hold for a set X where $\mu(X) = 0$. If the measure is not specified then Lebesgue measure is meant.

Let μ and ν be two Borel probability measures supported on $\mathbb{R}^n$. μ is said to be absolutely continuous with respect to ν if and only if for any Borel set X where $\nu(X) = 0$ also satisfies $\mu(X) = 0$. μ is said to be singular with respect to ν if and only if there exists a set, X such that $\nu(X) = 0$ but $\mu(X) = 1$. A measure is said to be of pure type if is known to be either absolutely continuous or singular. The lower and upper derivatives of a measure, with respect to Lebesgue measure, supported on $\mathbb{R}$ at a point x are given by

$$\underline{D}_\mu(x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{2r} \text{ and } \overline{D}_\mu(x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{2r}.$$

The following result shows how the lower derivative can be used to determine whether a measure on $\mathbb{R}$ is absolutely continuous with respect to Lebesgue measure.

Lemma 1.1. *μ is absolutely continuous with respect to $\mathcal{L}$ if and only if*

$$\underline{D}_\mu(x) < \infty \textit{ for } \mu \textit{ almost all } x.$$

A proof of this result can be found in [29]. The Fourier transform of a measure μ on $\mathbb{R}$ is defined by the function $\hat{\mu} : \mathbb{R} \to \mathbb{R}$ where

$$\hat{\mu}(u) = \int_{\mathbb{R}} e^{iux} \mathrm{d}\mu(x).$$

The asymptotic behaviour of $\hat{\mu}$ is important in determining the properties of μ.

Let μ and ν be measures defined on $\mathbb{R}^n$ the direct product, $\mu \times \nu$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\mu \times \nu(A \times B) = \mu(A)\nu(B).$$

The convolution of μ and ν is defined such that for any Borel set A,

$$\mu * \nu(A) = \mu \times \nu\{(x,y) : x + y \in A\}.$$

An important property of convolution of measures is the behaviour of Fourier transforms. In particular

$$\widehat{\mu * \nu}(u) = \hat{\mu}(u)\hat{\nu}(u).$$

The behaviour of the Fourier transform of μ can sometimes be used to determine whether μ is absolutely continuous or not.

Lemma 1.2. *Let $\mu \in M(\mathbb{R}^n)$.*

1. *If $\hat{\mu}(u) \not\to 0$ as $|u| \to \infty$ then μ is not absolutely continuous.*

2. *If $\hat{\mu} \in L^2(\mathbb{R}^n)$ then μ is absolutely continuous with respect to $\mathcal{L}^n$.*

The first part of this Lemma follows from the classical Riemann-Lebesgue Lemma. A proof to the second part can be found in [46]. For $\mu \in M(\mathbb{R})$ if $\hat{\mu} = O(|u|^{-\tau})$ for $\tau > \frac{1}{2}$ then $\hat{\mu} \in L^2(\mathbb{R})$ and by Lemma 1.2 μ is absolutely continuous.

Two important results concerning integration which will be used in later chapters of this thesis are Fatou's Lemma and Fubini's Theorem.

Lemma 1.3 (Fatou). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative measurable functions and E a measurable set. If we let*

$$f(x) = \liminf_{n\to\infty} f_n(x)$$

then

$$\int_E f\, d\mu \leq \liminf_{n\to\infty} \int_E f_n\, d\mu.$$

For a proof see [6].

Lemma 1.4 (Fubini's Theorem). *Let (X, F_1, μ_1) and (Y, F_2, μ_2) be measure spaces and let $f : X\times Y \to \mathbb{R}$ be $F_1 \times F_2$ measurable and $\mu_1 \times \mu_2$ integrable. Then the functions $x \to f(x, y)$ and $y \to f(x, y)$ are μ_1 integrable for μ_2-almost every y and μ_2 integrable for μ_1 almost every x respectively. Furthermore the functions*

$$x \mapsto \int f(x, y)\, d\mu_2(y) \text{ and } y \mapsto \int f(x, y)\, d\mu_1(x)$$

are defined for μ_1 almost every x and μ_2 almost every y and are μ_2 and μ_1 integrable. Finally

$$\int\int f(x, y)\, d\mu_1(x)\, d\mu_2(y) = \int\int f(x, y)\, d\mu_2(y)\, d\mu_1(x).$$

A proof of this result can also be found in [6]. Another important result on measure theory which will be used later in this thesis is Ergorov's theorem.

Theorem 1.1 (Ergorov). *Let X be a measure space and let μ be a probability measure supported on X. Let $\{f_m\}_{m\in\mathbb{N}} : X \to \mathbb{R}$ be measurable. If $\lim_{m\to\infty} f_m(x) = y$ for μ almost all $x \in X$ then for any $\epsilon > 0$ there exists a set $A \subset X$ where the convergence is uniform and $\mu(A) > 1 - \epsilon$.*

1.3 Large Sets

Let X be a measure space. A measurable set $Y \subseteq X$ is considered to be 'large' with respect to a measure μ, supported on X, if $\mu(X\backslash Y) = 0$. A property is said to hold for μ-almost all values if the set of points where it does not hold has zero measure with respect to μ. In the rest of this thesis if μ is not specified then the natural Lebesgue measure is meant. There is a topological notion of a set being large. If X is a metric space then $Y \subseteq X$ is dense if for all $\epsilon > 0$ and $x \in X$ there exists $y \in Y$ such that $\text{dist}(x, y) \leq \epsilon$. A countable intersection of open sets is said to be a G_δ set. The topological notion of a large set is a residual set which is a set that contains a dense G_δ set. The following standard Lemma will be used later in the thesis to help determine whether a set is residual.

Lemma 1.5. *Let X be a metric spaces and $f : X \to \mathbb{R}$ a function. The set*

$$\{x \in X : f \text{ is continuous at } x\}$$

is a G_δ set.

This result can be deduced immediately from Theorem 7.1 in [33]. The topological and measure theoretic notions of large do not necessarily coincide. For sets with measure 0 it can still be useful to have a notion of relative size. One way of looking at this is the idea of Hausdorff dimension which is now defined.

1.4 Hausdorff Dimension for Sets and Measures

For a set $X \subseteq \mathbb{R}^n$ the s dimensional Hausdorff measure is defined for any $s > 0$ by

$$H^s(X) = \lim_{\epsilon\to 0} \inf \left\{ \sum_{i\in I} |U_i|^s : \{U_i\}_{i\in I} \text{ is a finite or countable } \epsilon \text{ -cover of } X \right\}.$$

It is straightforward to see that $H^s(X)$ is decreasing with s and takes the values ∞ or 0 with at most one exception. The Hausdorff dimension is then defined to be

$$\dim_{\mathcal{H}} X = \inf\{s : H^s(X) = 0\} = \sup\{s : H^s(X) = \infty\}.$$

The only value of s where it is possible that $H^s(X) \notin \{0, \infty\}$ is $\dim_{\mathcal{H}} X$. For a Borel probability measure, μ, the Hausdorff dimension is defined to be

$$\dim_{\mathcal{H}} \mu = \inf\{\dim_{\mathcal{H}} X : \mu(X) = 1\}.$$

A crucial result when calculating the Hausdorff dimension is the mass distribution principle.

Lemma 1.6 (Mass distribution principle). *Let $X \subseteq \mathbb{R}^n$. If there exists a measure μ such that $0 < \mu(X) < \infty$ and $R > 0$ such that for $r < R$*

$$\mu(B(x, r)) \leq Cr^s$$

for $C, s > 0$ then $\dim_{\mathcal{H}} X \geq s$.

A straightforward proof of this important result can be found in [10]. This is related to the idea of local dimension of measure. The upper and lower dimensions of a measure μ at a point x are defined as

$$\overline{d}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \text{ and } \underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

This is strongly related to the Hausdorff dimension of μ.

Lemma 1.7.

$$\dim_{\mathcal{H}} \mu = \text{ess-sup}_x\{\underline{d}_\mu(x)\}.$$

Where the essential supremum is taken with respect to μ.

A proof can be found in [6].

An important extension to the mass distribution principle are the potential theoretic methods.

Lemma 1.8.

1. *Let $A \subseteq \mathbb{R}$ be a Borel set and $\alpha, s \in (0,1]$. If there exists a mass distribution μ on A such that,*
$$\int_A \left(\int_A \frac{d\mu(x)}{|x-y|^s} \right)^\alpha d\mu(y) < \infty \tag{1.1}$$
then $\dim_{\mathcal{H}} A \geq s$ and $\dim_{\mathcal{H}} \mu \geq s$.

2. *Let F be a Borel set such that $H^s(F) > 0$. Then there exists a mass distribution μ such that $\mu(F) > 0$ and*
$$\int_F \int_F \frac{d\mu(x) d\mu(y)}{|x-y|^s} < \infty.$$

Proof. Let $\phi_\mu(y) = \left(\int \frac{\mathrm{d}\mu(x)}{|x-y|^s} \right)$. If the inequality (1.1) holds for a measure μ on a set A then it follows that $(\phi_\mu(y))^\alpha$ is integrable with respect to μ. This means that there exists M such that

$$A_M = \{y : (\phi_\mu(y))^\alpha \leq M\}$$

satisfies $\mu(A_M) > 0$. Thus we can define a measure ν simply by the restriction of μ to A_M. Hence for any $x \in A$

$$M^{\frac{1}{\alpha}} \geq \int_A \frac{\mathrm{d}\nu(y)}{|x-y|^s} \geq \int_{B(x,r)} \frac{\mathrm{d}\nu(y)}{|x-y|^s} \geq \frac{1}{r^s} \nu(B(x,r)).$$

Thus for any $x \in A$, $\nu(B(x,r)) \leq M^{\frac{1}{\alpha}} r^s$ and by the mass distribution principle $\dim_{\mathcal{H}} A \geq s$ and $\dim_{\mathcal{H}} \mu \geq s$. This completes the proof of part 1, a proof of part 2 can be found in [10] (Theorem 4.13). □

More details of this method can be found in [10]. The links between the use of the exponent α and generalized dimensions is given in [15]. Calculating the Hausdorff dimension of sets in $\mathbb{R}^n$ when $n > 1$ can be extremely difficult. The following method enables lower bounds for the Hausdorff dimension to be calculated for sets in $\mathbb{R}^2$. This is done by looking at the dimension of 1 dimensional sections. For a point $y \in \mathbb{R}$ let

$$L_y = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = y\}.$$

Lemma 1.9 (Marstrand's slicing theorem). *Let $F \subseteq \mathbb{R}^2$ and let E be a subset of the y-axis. If $\dim_{\mathcal{H}}(F \cap L_y) \geq s$ for all $y \in E$ then*

$$\dim_{\mathcal{H}} F \geq s + \dim_{\mathcal{H}} E.$$

This result was introduced in [28]. A proof can be found in [3] and it is listed as corollary 7.12 in [10]. The obvious generalizations to higher dimensions hold.

An alternative definition of dimension is the box counting dimension. It is often easier to compute than Hausdorff dimension, particularly numerically. However it is less satisfactory mathematically and less sensitive to the geometry of the set. Let $X \subseteq \mathbb{R}^N$ and let $N_\delta(X)$ be the smallest number of balls of radius δ which can cover X. The upper and lower box counting dimensions of X can be defined as

$$\overline{\dim_B}(X) = \limsup_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta} \text{ and } \underline{\dim_B}(X) = \liminf_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta}.$$

When the limit exists the box counting dimension is said to be the common value of the lower and upper box counting dimensions.

1.5 Iterated Function Systems

We now formally define iterated function systems. Let $F = \{f_0, \ldots, f_{m-1}\}$ be C^1 functions on $\mathbb{R}^n$ where $||f_i'||_\infty < 1$ for all i. In [16] it is shown using the contraction mapping theorem that there exists a unique non-empty compact set E such that

$$E = \cup_i f_i(E).$$

F is called an iterated function system (IFS) and E is called the attractor of F. It is often of interest to calculate the Hausdorff or box counting dimension of the attractor. To compute the Hausdorff dimension it is nearly always useful to define a measure supported on the attractor. For an IFS $F = \{f_0, \ldots, f_{m-1}\}$ we can create a measure by defining a probability vector $(p_0, \ldots, p_{m-1})$. It is shown in [16] that there then exists a unique probability measure, μ such that for any Borel set $A \subseteq \mathbb{R}^n$

$$\mu(A) = \sum_{i=0}^{m-1} p_i \mu(T_i^{-1}(A)).$$

μ is often referred to as the stationary or invariant measure for the IFS or, in the specific case where all of the functions are similarities, a self-similar measure.

If there exists an open set $V \subseteq \mathbb{R}^N$ such that

$$V \supset \cup_i f_i(V) \text{ and } f_i(V) \cap f_j(V) = \emptyset \text{ for all } i \neq j$$

then the IFS is said to satisfy the open set condition (OSC). This essentially means that the images of the maps only overlap in a trivial manner.

For an IFS, F, where each of the maps is a similarity and the OSC is satisfied the Hausdorff dimension of the attractor E is well known. In the case where the OSC is not satisfied it is only normally possible to give an upper bound.

Theorem 1.2. *Let $F = \{f_0, \ldots, f_{m-1}\}$ be an IFS on $\mathbb{R}^n$ where each f_i is defined by $f_i(x) = r_i x + b_i$ for $0 < r_i < 1$ and $b_i \in \mathbb{R}^n$. Let s be the unique solution of*

$$\sum_{i=0}^{m-1} r_i^s = 1.$$

1. $\dim_{\mathcal{H}} F = dim_B F \leq \min\{s, n\}$.

2. *If F satisfies the OSC then* $\dim_{\mathcal{H}} E = dim_B E = s$.

For a proof of this classical result see [10].

A C^2 map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be conformal if the derivative $f'(x)$ is a similarity transformation for all $x \in \mathbb{R}^n$. For an IFS where each map is conformal and the OSC is satisfied it is possible to generalize the methods in Theorem 1.2 using the thermodynamic formalism to compute the Hausdorff dimension of the attractor. For more details of this see [11].

To calculate the dimension of an IFS when the OSC is not satisfied is a difficult problem. Some work on $\mathbb{R}$ is mentioned in Chapter 2 and some specific cases in $\mathbb{R}^n$ are dealt with later in this thesis. If the maps are not conformal even when the OSC is satisfied the Hausdorff dimension is not easy to compute. In this case the box counting dimension and the Hausdorff dimension do not necessarily agree. The following self-affine sets were considered in both [2] and [30]. Let $n, m \in \mathbb{N}$ and let

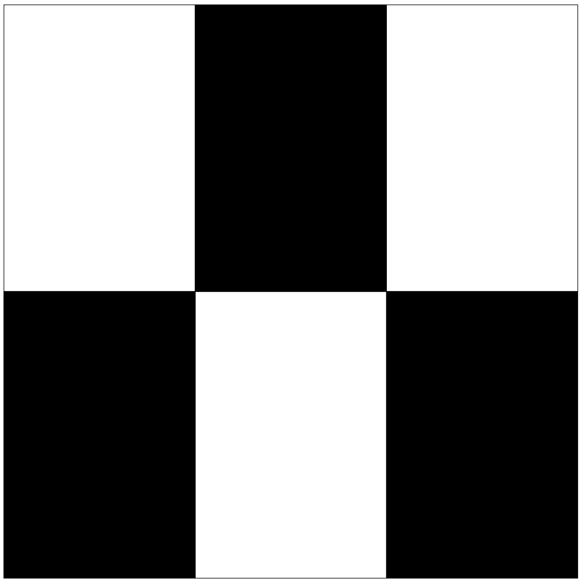

Figure 1.3: The first level construction of one of the sets investigated by Bedford and McMullen. Here $R = \{(0,0),(1,1),(2,0)\}$.

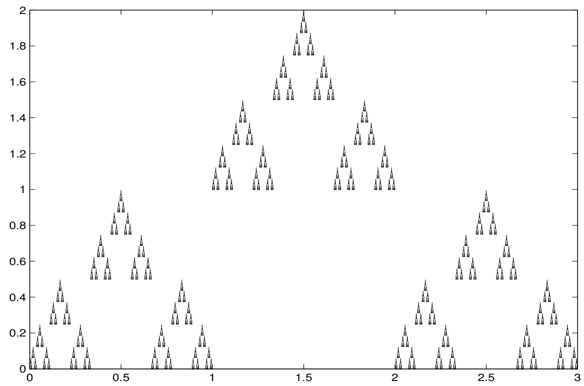

Figure 1.4: A picture of one of the self-affine sets investigated by Bedford and McMullen.

R consist of a subset of the set of all pairs of integers (i,j) such that $0 \le i \le n-1$ and $0 \le j \le m-1$. Let

$$\overline{R} = \left\{ \left(\sum_{k=1}^{\infty} \frac{x_k}{n^k}, \sum_{k=1}^{\infty} \frac{y_k}{m^k} \right) : (x_k, y_k) \in R \text{ for all } k \right\}.$$

Let t_j be the number of i such that $(i,j) \in R$, s be the number of j such that $(i,j) \in R$ and $r = |R|$.

Theorem 1.3 (Bedford, McMullen, 1984).

1. $\dim_{\mathcal{H}} \overline{R} = \log_m \left(\sum_{j=0}^{m-1} t_j^{\log_n m} \right)$.

2. $dim_B \overline{R} = \log_m S + \log_n (\frac{r}{s})$.

Observe that the Hausdorff dimension and box counting dimension of $\overline{R}$ is only the same if either $n = m$ or all values of t_j are the same. A slight generalisation of this set that allows the rectangles to overlap will be studied in Chapter 5. A generic formula for the Hausdorff dimension of more general classes of self-affine set is given in [12].

1.6 Subshifts and Entropy

A useful tool for studying IFS's are subshifts of finite type. Let A be a $n \times n$ matrix where each entry is 0 or 1. Let

$$X_A = \{(x_m)_{m \in \mathbb{N}_0} : 0 \le x_m \le n-1, A(x_m, x_{m+1}) = 1 \text{ for all } m \in \mathbb{N}_0\}.$$

Here $\mathbb{N}_0$ refers to the natural numbers with the addition of 0. The shift map $\sigma : X_A \to X_A$ is defined by

$$(\sigma x)_m = x_{m+1} \text{ for } m \ge 0.$$

The case we will use in this thesis is the full shift. This is where each value of A is 1. Thus all sequences are allowed. If A is the $n \times n$ matrix where each element is 1 then X_A is denoted Σ_n and called the full shift on n symbols. The kth level cylinder is defined by

$$\omega_k = [x_0, \ldots, x_{k-1}] = \{\{i_m\}_{m \in \mathbb{N}_0} \in \Sigma_n : i_m = x_m \text{ for } 0 \le m \le k-1\}$$

and we denote by W_k the set of all kth level cylinders. For a sequence $\underline{i} \in \Sigma_n$ and $0 \leq r \leq n-1$ we define

$$k_r(\underline{i}) = \mathrm{card}\{0 \leq m \leq k-1 : i_m = r\}.$$

We will also slightly abuse this notation and use it for mth level cylinder sets where $k \leq m$. For two sequences $\omega, \tau \in \Sigma_n$ we define

$$|\omega \wedge \tau| = \min\{k : \omega_k \neq \tau_k\}.$$

For the study of self-similar sets an important class of measures are the Bernoulli measures. This involves introducing a probability vector $\underline{p} = (p_0, \ldots, p_{n-1})$ and defining the measure of any kth level cylinder to be

$$\mu([i_0, \ldots, i_{k-1}]) = {p_0}^{k_0(\underline{i})} {p_1}^{k_1(\underline{i})} \cdots {p_{n-1}}^{k_{n-1}(\underline{i})}.$$

Any general probability measure μ on σ_n is said to be invariant under the shift map if for any measurable set X,

$$\mu(X) = \mu(\sigma^{-1}(X))$$

and ergodic if for any measurable set $X \subseteq \Sigma_n$ such that $\sigma^{-1}(X) = X$ satisfies $\mu(X) = 0$ or $\mu(X) = 1$. A Bernoulli measure is both invariant and ergodic. A crucial result is the Birkhoff Ergodic Theorem. We state it just in the setting of shift spaces however it is true in far more general settings.

Theorem 1.4 (Birkhoff's Ergodic Theorem). *Let X_A be a shift space as defined above. Let μ be an ergodic invariant measure supported on X_A such that $\mu(X_A) < \infty$. If $f \in L^1(\mu)$ then*

$$\frac{1}{n} \sum_{i=0}^{n-1} f(\sigma^i(x)) \to \int f \, d\mu$$

for μ almost all $x \in X_A$.

We now introduce the concept of entropy. We start by defining entropy for general ergodic systems before going back to shift spaces. All of the details given here can be found in Chapter 4 in [45]. Let (X, B, μ) be a probability space and let $T : X \to X$ be a measure preserving transformation. A partition of (X, B, μ) is a finite or countable collection of elements in B whose union is X. For example if we take $X = \Sigma_n$, B to be the Borel sigma algebra for Σ_n and μ to be evenly weighted Bernoulli measure the cylinders $\{[0], \ldots, [n-1]\}$ form a partition of Σ_n. For a finite partition $A = \{A_1, \ldots, A_m\}$ we define

$$H_\mu(A) = -\sum_{i=1}^{m} \mu(A_i) \log(\mu(A_i)).$$

For two partitions $A = \{A_1, A_2, \ldots, A_m\}$ and $C = \{C_1, \ldots, C_l\}$ we define the join to be

$$A \vee C = \{A_i \cap C_j : 1 \leq i \leq m, 1 \leq j \leq l\}.$$

This definition also holds for countable partitions. The partition $T^{-k}(A)$ for $k \in \mathbb{N}$ is defined as

$$T^{-k}(A) = \{T^{-k}(A_1), T^{-k}(A_2), \ldots, T^{-k}(A_m)\}.$$

We define the entropy of T with respect to A to be

$$h_\mu(T, A) = \lim_{k \to \infty} \frac{1}{k} H\left(\vee_{i=0}^{k-1} T^{-i} A\right).$$

This limit is shown to exist in [45]. The entropy of the transformation can now be defined as

$$h_\mu(T) = \sup_{A : h_\mu(T,A) < \infty} h(T, A).$$

To directly calculate the entropy of a transformation using this definition is usually extremely difficult. The idea of a strong generator often makes the calculation much easier. A countable partition A is said to be a strong generator if

$$\bigvee_{k=-\infty}^{\infty} T^k(A) = B$$

where B is the sigma algebra for the measure space. If A is a strong generator and $H_\mu(A) < \infty$ then $h_\mu(T) = h_\mu(T, A)$.

We now return to Σ_n. In this case the set of cylinders $\{[0], \ldots, [n-1]\}$ is a generating partition. Thus for an ergodic and invariant measure μ on Σ_n the entropy with respect of the shift map is defined as

$$h_\mu(\sigma) = h_\mu = \lim_{k \to \infty} -\frac{1}{k} \sum_{\omega_k \in W_k} \mu(\omega_k) \log(\mu(\omega_k)).$$

For a Bernoulli measure μ defined by a probability vector $\underline{p} = (p_0, \ldots, p_{n-1})$ the entropy can be calculated to be ([45])

$$h_\mu = -\sum_{i=0}^{n-1} p_i \log p_i.$$

An important result when using entropy for dimension theory is the Shannon-McMillan-Brieman Theorem ([45] page 99).

Theorem 1.5 (Shannon-McMillan-Brieman). *Let μ be an ergodic shift invariant measure on Σ_n. For μ almost all $\underline{i} \in \Sigma_n$ we have*

$$-\lim_{k\to\infty} \frac{1}{k} \log \mu([i_0, \ldots, i_{k-1}]) = h_\mu.$$

The dynamics of shift spaces can be extremely helpful when studying IFS. This is because when $F = \{f_0, \ldots, f_{m-1}\}$ is an IFS with attractor E it is possible to define a natural projection, $\Pi : \Sigma_m \to E$, by

$$\Pi(\underline{i}) = \lim_{k\to\infty} f_{i_0} \cdots f_{i_k}(0).$$

This allows any measure supported on Σ_m to be projected onto E. This is simply done by defining $\nu = \mu \circ \Pi_\lambda^{-1}$. In the case where μ is $\underline{p}$-Bernoulli measure the measure ν satisfies

$$\nu(A) = \sum_{i=0}^{m-1} p_i \nu(f_i^{-1}(A)).$$

Thus it is the same measure as the stationary measure for the probability vector $\underline{p}$ defined in the section on iterated function systems.

In the case where all the functions in the IFS's are similarities ν is a self-similar measure. In this case it is possible to use the Shannon-McMillan-Brieman Theorem to get an upper bound on the Hausdorff dimension of ν. Let each f_i have contraction ratio $r_i < 1$ and let

$$\chi = \sum_{i=0}^{m-1} p_i \log r_i.$$

The following Theorem is standard in the theory of iterated functions systems we include a proof for completeness. A proof can also be found in [6]. It is not clear when it was first proved.

Theorem 1.6.

$$\dim_{\mathcal{H}}\nu \leq -\frac{h_\mu}{\chi} = -\frac{\sum_{i=0}^{m-1} p_i \log p_i}{\chi}.$$

Proof. Recall Lemma 1.7. Thus to show $\dim_{\mathcal{H}}\nu \geq s$ for some s it is sufficient to show that

$$\liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \leq s$$

for ν-almost all x. However $\nu = \mu \circ \Pi^{-1}$ thus it is sufficient to show that for μ-almost all $\tau \in \Sigma_n$,

$$\liminf_{r\to 0} \frac{\log \nu(B(\Pi^{-1}\tau, r))}{\log r} \leq s.$$

We define a metric, d, on Σ_m as follows. Let $\omega, \tau \in \Sigma_n$ and let $k = |\omega \wedge \tau|$.

$$d(\omega,\tau) = \sum_{i=0}^{m-1} r_i^{k_i(\tau)}.$$

The important property of this metric d is that the diameter of any cylinder in the shift space is the same as the diameter of the projection of the cylinder in $\mathbb{R}^n$.

Fix $\tau \in \Sigma_m$ and let $x = \Pi^{-1}\tau$. For $r > 0$ there exists $k(r)$ such that

$$\mathrm{diam}[i_1, \ldots, i_{k(r)}, i_{k(r)+1}] \leq 2r \leq \mathrm{diam}[i_1, \ldots, i_{k(r)}]$$

and $k(r) \to \infty$ as $r \to 0$. Hence

$$\liminf_{r\to 0} \frac{\log(\nu(B(x,r)))}{\log r} \leq \liminf_{k\to\infty} \frac{\log(\mu([\tau_0, \ldots, \tau_{k-1}]))}{\log(\mathrm{diam}([\tau_0, \ldots, \tau_{k-1}]))}.$$

This inequality can be strict because in the case where the OSC is not satisfied $\nu(B(x,r))$ can be significantly bigger than $\mu([\tau_1, \ldots, \tau_{k(r)-1}])$. By the Shannon-McMillan-Brieman Theorem we have that

$$-\lim_{n\to\infty} \frac{1}{n} \log(\mu([\tau_0, \ldots, \tau_{n-1}])) \to \sum_{i=0}^{m-1} p_i \log p_i = h_\mu$$

for μ almost all τ. By the Birkhoff Ergodic Theorem we have that

$$-\lim_{n\to\infty} \frac{1}{n} \log \mathrm{diam}[\tau_0, \ldots, \tau_{n-1}] \to \sum_{i=0}^{m-1} p_i \log r_i = \chi$$

for μ almost all τ. Hence for μ almost all τ where $x = \Pi\tau$ or equivalently ν almost all x.

$$-\liminf_{r\to 0} \frac{\log(\nu(B(x,r)))}{\log r} \leq \frac{h_\mu}{\chi}.$$

$\square$

It is a standard result that this result is an equality when the OSC holds. It is also clear from the proof of Theorem 1.6 that the theorem holds if $\nu = \Pi\mu$ for any ergodic invariant measure μ on Σ_n.

1.7 Decomposition of Measures and Conditional Entropy

Let $k < n$ and let $p : \Sigma_n \to \Sigma_k$ be a surjection which maps elements from $\{[0], [1], \ldots, [n-1]\}$ surjectively onto $\{[0], [1], \ldots, [k-1]\}$. Let μ be an Ergodic invariant measure supported on Σ_n and $\overline{\mu}$ be the projection of μ under p to Σ_k. It is possible to uniquely decompose μ into a family of probability measures supported on fibres $p^{-1}\xi$ for $\xi \in \Sigma_k$ (Rohlin [39]). For any set $A \subseteq \Sigma_k$

$$\mu(A) = \int_{\Sigma_k} \mu_\xi(A \cap p^{-1}\xi) \mathrm{d}\overline{\mu}(\xi)$$

where μ_ξ are supported on the fibres $p^{-1}\xi$. This decomposition will be used in Chapter 5 in combination with Lemma 1.9 to compute the dimension of certain self-similar sets.

It is extremely useful to have an analogue of entropy for the measures supported on $p^{-1}\xi$. We use the idea of conditional entropy ([45]). Let $B(\Sigma_n)$ be the Borel sigma algebra for Σ_n and $B(\Sigma_k)$ be the Borel sigma algebra for σ_k. $\beta = p^{-1}B(\Sigma_k)$ is the sub-sigma algebra of $B(\Sigma_n)$ which fails to distinguish between symbols in $\{0, 1, \ldots, n-1\}$ which are projected under p to the same element in $\{0, 1, \ldots, k-1\}$. Let $A = A(B(\Sigma_n)) = \{[0], [1], \ldots, [n-1]\}$ be the standard generating partition for Σ_n and $\eta = \eta(\beta) = \{p^{-1}[0], \ldots, p^{-1}[k-1]\}$. We let

$$H_\mu(A|\eta) = -\sum_{j=0}^{k-1} \mu(p^{-1}[j]) \sum_{i=0}^{n-1} \frac{\mu([i] \cap p^{-1}[j])}{\mu(p^{-1}[j])} \log\left(\frac{\mu([i] \cap p^{-1}[j])}{\mu(p^{-1}[j])}\right).$$

and the conditional entropy of A with respect to η is defined by

$$h_\mu(A|\eta) = \lim_{N\to\infty} \frac{1}{N} H_\mu(\vee_{i=0}^{N-1} \sigma^i A|\eta).$$

We now give an analogue of the Shannon-McMillan-Brieman Theorem in the case of conditional entropy. This result was proved by Ledrappier and Young in [25] and will play an important role in Chapter 5.

Theorem 1.7 (Ledrappier-Young). *For μ almost every $x \in \Sigma_n$,*

$$\lim_{N\to\infty} -\frac{\log \mu_\xi([x_0, \ldots, x_{N-1}] \cap p^{-1}(\xi))}{N} = h(\mu|A)$$

This result can be particularly useful when using Lemma 1.9. In Lemma 1.9 a lower bound for the Hausdorff dimension of a set in $\mathbb{R}^2$ can be found by looking at the dimensions of one-dimensional sections. Theorem 1.7 can be useful when calculating the dimension of these sections. Finally a classical result relates the conditional entropy to the entropy of the projection measure $\overline{\mu}$ and the entropy of μ.

Proposition 1.1 (Abramov-Rohlin).

$$h(\mu) = h(\overline{\mu}) + h(\mu|A).$$

A proof of this result can be found in [1].

Chapter 2

Transversality

2.1 Projection Theorems

In this chapter we provide an outline of the transversality method. We also state some crucial lemma using transversality which will be used in later chapters of this thesis. For $\epsilon > 0$ a function is said to be ϵ transversal if it crosses any horizontal line within ϵ of the origin with slope at most $-\epsilon$ or at least ϵ.

One of the simplest settings where the idea can be illustrated is in the classical projection theorems. Let $A \subseteq \mathbb{R}^2$ and p_θ denote the linear projection to the line at an angle θ to the x axis. We will call this line L_θ and denote points on it by taking the distance from the origin with the distance taken to be positive for points to the right of the origin and negative to the left of the origin.

Theorem 2.1 (Marstrand). *Let $A \subseteq \mathbb{R}^2$ and $\dim_{\mathcal{H}} A = s$.*

1. *If $s \leq 1$ then for almost all θ, $\dim_{\mathcal{H}} p_\theta(A) = \dim_{\mathcal{H}} A$.*

2. *If $s > 1$ then for almost all θ, $\mathcal{L}^1(p_\theta(A)) > 0$.*

This result was first proved by Marstrand in [27] in 1954. However in [20] an alternative method was introduced which later became known as the transversality technique. We sketch the proof of the first part of this theorem to outline the method. Let $A \subseteq \mathbb{R}^2$ where $\dim_{\mathcal{H}} A = s < 1$, let $\epsilon > 0$ and $t = s - \epsilon$. $H^t(A) > 0$ and thus by part 2 of Lemma 1.8 there exists a mass distribution μ such that $\mu(A) > 0$ and,

$$\int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^t} < \infty. \tag{2.1}$$

We define μ_θ to be the projection of μ by p_θ. Hence

$$\mu_\theta([a,b]) = \mu(p_\theta^{-1}([a,b])) = \mu\{x : a \leq x \cdot \underline{\theta} \leq b\}$$

where $\underline{\theta}$ denotes the unit vector in direction θ and $\cdot$ denotes the usual vector scalar product.

By Lemma 1.8 Part 1 it suffices to show that

$$\int_0^\pi \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\mathrm{d}\mu_\theta(u)\mathrm{d}\mu_\theta(v)\mathrm{d}\theta}{|u-v|^t} < \infty.$$

We begin by lifting to $\mathbb{R}^2$ and using Fubini's Theorem.

$$\begin{aligned} I &= \int_0^\pi \int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)\mathrm{d}\theta}{|x\cdot\underline{\theta} - y\cdot\underline{\theta}|^t} \\ &= \int_A \int_A \int_0^\pi \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)\mathrm{d}\theta}{|(x-y)\cdot\underline{\theta}|^t}. \end{aligned}$$

Next we introduce a constant unit vector τ and note that since

$$\int_0^\pi \frac{\mathrm{d}\theta}{|(x-y)\cdot\underline{\theta}|^t} = \frac{1}{|x-y|^t}\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\theta}\cdot\tau|^t}$$

we have

$$I = \int_0^\pi \frac{\mathrm{d}\theta}{|\tau\cdot\theta|^t} \int_A \int_A \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^t}.$$

We know immediately by (2.1) that the second part of this product is finite. Thus it remains to show that

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\theta}.\underline{\tau}|^t} < \infty.$$

We know that

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\underline{\tau}\cdot\underline{\theta}|^t} = \int_0^\pi \frac{\mathrm{d}\theta}{|\cos(\tau-\theta)|^t}.$$

The derivative of $\cos(\tau-\theta)$ is bounded away from 0 when $\cos(\tau-\theta)$ is equal to 0 so when $\cos(\tau-\theta)^t$ is close to 0 it can be bounded below by Cx^t for some $C > 0$. Since $t < 1$ this means

$$\int_0^\pi \frac{\mathrm{d}\theta}{|\cos(\tau-\theta)|^t} < \infty.$$

This is the basic idea of transversality. Thus $I < \infty$ for any $t < s$ and so the proof is complete.

2.2 The $\{0, 1, 3\}$-Problem

A similar technique to the one outlined in the previous section can be used to help calculate the dimension of certain self-similar sets where the similarities overlap. Let $F = \{f_0, f_1, f_2\}$ be an IFS on $\mathbb{R}$ where,

$$\begin{aligned} f_0(x) &= \lambda x \\ f_1(x) &= \lambda x + 1 \\ f_2(x) &= \lambda x + 3. \end{aligned}$$

For $\lambda \leq \frac{1}{4}$ the OSC applies and the Hausdorff dimension of the attractor $\Lambda(\lambda)$ is thus $-\frac{\log 3}{\log \lambda}$. When $\lambda \in (\frac{1}{4}, \frac{1}{3})$ the OSC can not be applied and the problem of whether $\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$ is still not fully solved. A generic solution was found by Pollicott and Simon in [38].

Theorem 2.2 (Pollicott-Simon, 1994). *For almost all $\lambda \in (\frac{1}{4}, \frac{1}{3}]$,*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\lambda}.$$

The method of proof is extremely similar to that of the projection theorem. Let μ be $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on $\{0, 1, 3\}^{\mathbb{N}_0}$. A projection $\Pi_\lambda : \{0, 1, 3\}^{\mathbb{N}_0} \to \mathbb{R}$ can be defined by

$$\Pi_\lambda(\underline{i}) = \sum_{k=0}^{\infty} i_k \lambda^k.$$

Thus on each possible attractor $\Lambda(\lambda)$ a self-similar measure ν_λ can be defined by $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. Let $\epsilon > 0$ and $s_\epsilon(\lambda) = -\frac{\log 3}{\log(\lambda + \epsilon)}$. Note that the proof is complete, using Lemma 1.8 if it can be shown that

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mathrm{d}\nu_\lambda(x) \mathrm{d}\nu_\lambda(y) \mathrm{d}\lambda}{|x - y|^{s_\epsilon(\lambda)}} < \infty$$

for all $\epsilon > 0$. Using the projection Π_λ the inner two integrals can be transferred to Σ_3.

$$I = \int_{\frac{1}{4}}^{\frac{1}{3}} \int_{\Sigma_3} \int_{\Sigma_3} \frac{\mathrm{d}\mu(\underline{i}) \mathrm{d}\mu(\underline{j}) \mathrm{d}\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}}.$$

We then turn I into a product of two expressions using Fubini's theorem. We let $t = \max_{\frac{1}{4}\leq\lambda\leq\frac{1}{3}} s_\epsilon(\lambda)$ and note that $t<1$. It can be seen that,

$$\begin{aligned}|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)} &= \lambda^{|\underline{i}\wedge\underline{j}|s_\epsilon(\lambda)} \left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^{s_\epsilon(\lambda)} \\ &\geq \left(\frac{1}{3}+\epsilon\right)^{s_\epsilon(\lambda)|\underline{i}\wedge\underline{j}|} \left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^{t}\end{aligned}$$

where $\{a_k\}_{k\in\mathbb{N}_0}$ is a sequence such that $a_k \in \{0,\pm1,\pm2,\pm3\}$ and $a_0 \neq 0$. Substituting this back into I and using Fubini's Theorem we get

$$I \leq \int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t} \int_{\Sigma_3}\int_{\Sigma_3} \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}}.$$

By simple integration on Σ_3 it can be seen that,

$$\begin{aligned}\int\int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\mu(\underline{j})}{\left(\frac{1}{3}+\epsilon\right)^{|\underline{i}\wedge\underline{j}|}} &\leq \sum_{k=0}^{\infty} \sum_{[i_0,i_1,\ldots,i_{k-1}]} \frac{\mu([i_0,i_1,\ldots,i_{k-1}])^2}{\left(\frac{1}{3}+\epsilon\right)^k} \\ &= \sum_{k=0}^{\infty} \frac{\frac{1}{3}^{k+1}}{\left(\frac{1}{3}+\epsilon\right)^k} < \infty.\end{aligned}$$

Thus to show that $I<\infty$ it remains to show that

$$\int \frac{\mathrm{d}\lambda}{\left(\sum_{k=0}^{\infty} a_k\lambda^k\right)^t} < \infty$$

for any sequence $\{a_k\}_{k\in\mathbb{N}_0}$ where each $a_k \in \{0,\pm1,\pm2,\pm3\}$ and $a_0\neq 0$. Let $g(\lambda)$ be a power series of that form. In Lemma 1 of [38] it is shown that whenever such a power series $g(\lambda)$ is close to 0 its derivative is bounded away from 0. Thus a transversality condition is satisfied. The integral can be shown to be finite by splitting it into two parts, one part where $g(\lambda)$ is bounded away from 0 and one where $g(\lambda)$ is close to zero but the derivative is bounded away from 0. The first part is clearly finite and the second part is finite because $t<1$ and $g(\lambda)$ can be bounded below by linear functions in this region. This method will be used again in chapters 4, 5 and 6.

In [38] a general result about when specific power series satisfy a transversality condition is given. Let

$$F_b = \left\{ f(\lambda) = \sum_{k=0}^{\infty} g_k\lambda^k : g_k \in [-b,b] \text{ and } g_0 \neq 0 \right\}.$$

Note that since $\lambda < 1$ and the g_k are bounded all elements of F_b are differentiable. As in [36] we now define

$$y(b) = \min\{x > 0 : \exists f \in F_b \text{ where } f(x) = f'(x) = 0\}.$$

Theorem 2.3 (Peres-Solomyak,1996). *The function* $y : [1, \infty) \to [0, 1]$ *is strictly decreasing, continuous and piecewise algebraic.* $y(1) \approx 0.649$, $y(2) = 0.5$ *and* $y(b) \geq (\sqrt{b} + 1)^{-1}$ *with equality when* $b \geq 3 + \sqrt{8}$.

A proof can be found in [36]. The following corollary is crucial when trying to use the transversality technique to calculate the dimension or measure of self-similar sets.

Corollary 2.1. *Let* $f \in F_b$. *We have that,*

1. *for any* $s < 1$ *there exists* $K(s) > 0$ *where*
$$\int_0^{y(b)} \frac{d\lambda}{|f(\lambda)|^s} < K(s).$$
2. *There exists* $C > 0$ *such that for all* ϵ *sufficiently small,*
$$\mathcal{L}\{\lambda \in (0, y(b)) : |f(\lambda)| \leq \epsilon\} \leq C\epsilon.$$

Sketch of Proof

We give a quick sketch of the second part from which the first part can be easily deduced. Let $f \in F_b$ and $0 \leq t < y(b)$. Thus there exists $\delta > 0$ such that for any $0 \leq x \leq t$ we have $|f(x)| \leq \delta$ implies that $|f'(x)| \geq \delta$. We assume that $\epsilon \leq \delta$. It is clear that $|f'(x)|$ is bounded above so there are at most a finite number of intervals $I_1, \ldots, I_m \subset [0, t]$ where for any $x \in I_j$ $|f(x)| < \epsilon$ and because f is monotone whenever $|f(x)| \leq \delta$ m stays fixed as ϵ decreases. For any one of these intervals $|f'(x)| \geq \delta$ for any $x \in I_i$ and hence $\text{diam}(I_i) \leq 2\frac{\epsilon}{\delta}$. Putting this together we have that

$$\mathcal{L}\{\lambda \in (0, t) : |f(\lambda)| \leq \epsilon\} \leq 2m\frac{\epsilon}{\delta}.$$

This is true for any $t < y(b)$ and so is sufficient to complete the proof.

The first part is extremely useful when proving theorems of a similar type to Theorem 2.2. The second part is useful in the case when we wish to show that a class of self-similar sets have positive Lebesgue measure for almost all parameter values. We will now look at the $\{0,1,3\}$ problem in the region $\lambda \in [\frac{1}{3}, y(3)]$ to outline how this method works. Let μ and ν_λ be defined exactly as in the proof of Theorem 2.2.

Theorem 2.4. *For a.e.* $\lambda \in [\frac{1}{3}, y(3)]$ ν_λ *is absolutely continuous and hence* $\mathcal{L}(\Lambda(\lambda)) > 0$.

This result was proved in [44]. The method of proof relies on Lemma 1.1 which states that ν_λ being absolutely continuous is equivalent to

$$\int \liminf_{r\to 0} \frac{\nu_\lambda(B(x,r))}{2r} \mathrm{d}\nu_\lambda(x) < \infty.$$

Thus to show that ν_λ is absolutely continuous for a.e. $\lambda \in (\frac{1}{3}, y(3))$ it is sufficient to show for any $\epsilon > 0$

$$I = \int_{\frac{1}{3}+\epsilon}^{y(3)} \int \liminf_{r\to 0} \frac{\nu_\lambda(B(x,r))}{2r} \mathrm{d}\nu_\lambda(x)\mathrm{d}\lambda < \infty.$$

The first step is to apply Fatou's Lemma and lift to the shift space. Thus

$$\begin{aligned} I &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\frac{1}{3}+\epsilon}^{y(3)} \int \nu_\lambda(B(x,r))\mathrm{d}\nu_\lambda(x)\mathrm{d}\lambda \\ &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\frac{1}{3}+\epsilon} \int_{\Sigma_3} \int_{\Sigma_3} \mathbb{1}_{\{\omega,\tau:|\Pi_\lambda(\omega)-\Pi_\lambda(\tau)|\leq r\}} \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau)\mathrm{d}\lambda. \end{aligned}$$

Applying Fubini's Theorem bounds I by an expression which allows part (ii) of Corollary 2.1 to be used. This gives

$$I \leq \liminf_{r\to 0} \frac{1}{2r} \int_{\Sigma_3} \int_{\Sigma_3} \mathcal{L}\left\{\lambda \in \left(\frac{1}{3}+\epsilon, y(3)\right) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\right\} \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau).$$

It can be seen that

$$|\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| = \lambda^{|\omega\wedge\tau|} g(\lambda)$$

where $g(\lambda) \in F_\lambda$ for all $\omega, \tau \in \Sigma_3$. Thus part (ii) of Corollary 2.1 gives that

$$\mathcal{L}\left\{\lambda \in \left(\frac{1}{3}+\epsilon, y(3)\right) : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\right\} \leq 2C\left(\frac{1}{3}+\epsilon\right)^{-|\omega\wedge\tau|} r$$

for some $C > 0$. This gives

$$I \leq C \int_{\Sigma_3} \int_{\Sigma_3} \left(\frac{1}{3} + \epsilon\right)^{-|\omega \wedge \tau|} \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tau)$$

which can be seen to be finite by simply integrating on the shift space as in Theorem 2.2. This is the standard method for using transversality that for a.e parameter a family of measures are absolutely continuous. This method has been successfully used in many contexts. These include self-affine sets ([12]), Bernoulli convolutions ([44],[36],[34]), non linear hyperbolic IFS ([42]), Parabolic IFS and random continued fraction expansions ([42]).

Chapter 3

Properties of Self Similar Measures

In this chapter we study some properties of self-similar measures supported on $\mathbb{R}$. We do not assume that the OSC is satisfied. We look at the local dimension of the measures.

3.1 Bernoulli Convolutions

Let $0 < \lambda_0, \lambda_1 < 1$ and let $T_0, T_1 : [0,1] \to [0,1]$ be defined by

$$\begin{aligned} T_0(x) &= \lambda_0 x \\ T_1(x) &= \lambda_1 x + (1 - \lambda_1). \end{aligned}$$

For a probability vector, $\underline{p} = (p_0, p_1)$ we let $\nu = \nu_{\lambda_0,\lambda_1}^{p_0,p_1}$ be the usual self-similar measure defined by

$$\nu = p_0(\nu \circ T_0^{-1}) + p_1(\nu \circ T_1^{-1}).$$

The case we are particularly interested in is when $\lambda_0 + \lambda_1 > 1$. In this case the attractor of the system can be seen to be $[0,1]$. However the OSC is not satisfied so the properties of the measure ν are not straightforward to investigate.

When $\lambda_0 = \lambda_1$ and $p_0 = p_1 = \frac{1}{2}$ the measure ν is called a Bernoulli convolution. The properties of this family of measures have been studied since the 1930s. In [44] it was shown that for almost every $\lambda \in (\frac{1}{2}, 1)$ ν_λ is absolutely continuous. This was shown using the transversality method. We study the local dimension of ν_λ at all values x for all values of λ.

The case where $\lambda_0 \neq \lambda_1$ is more difficult to deal with as there is no explicit form for the Fourier transform. However it is possible to use transversality to give a partial result on the absolute continuity of ν. This was done independently by Neunhäuserer and Ngai-Wang in [31] and [32].

Proposition 3.1 (Neunhäuserer, Ngai-Wang). *Fix a probability vector (p_0, p_1). For almost all $(\lambda_0, \lambda_1) \in (0, y(1))^2$ the measures $\nu = \nu_{\lambda_0,\lambda_1}^{p_0,p_1}$ are absolutely continuous when*

$$(\lambda_1 p_0)^{p_0} (\lambda_0 p_1)^{p_1} \leq \lambda_0 \lambda_1.$$

This proposition is definitely not true when

$$(\lambda_1 p_0)^{p_0} (\lambda_0 p_1)^{p_1} > \lambda_0 \lambda_1$$

because in this case

$$\frac{p_0 \log p_0 + p_1 \log p_1}{p_0 \log \lambda_0 + p_1 \log \lambda_1} < 1$$

and hence by Theorem 1.6 $\dim_{\mathcal{H}} \nu < 1$ and thus ν is singular. It is unknown whether this result holds for almost all (λ_0, λ_1). It is not possible to use convolutions to extend the result to the whole interval as was done in the case where $\lambda_0 = \lambda_1$.

We will refer to the self-similar measures when $\lambda_0 \neq \lambda_1$ or when $p_0 \neq p_1$ as biased Bernoulli convolutions. From now on the term Bernoulli convolution will refer to all these cases. The case where $\lambda_0 = \lambda_1$ and $(p_0, p_1) = \left(\frac{1}{2}, \frac{1}{2}\right)$ will be called unbiased Bernoulli convolutions. Neither $\lambda_0 = \lambda_1$ or $p_0 = p_1$ will be assumed.

3.2 Local Dimension for Bernoulli Convolutions

In this section we obtain estimates for uniform bounds for the local dimension of a Bernoulli convolution at any point. Recall that the lower pointwise dimension of a measure μ at the point x is defined by

$$\underline{d}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Note that to show $\underline{d}_\mu(x) \geq s$ it is sufficient to show there exists a constant $C > 0$ such that $\mu(B(x,r)) \leq Cr^s$.

There has been work on estimating the Hausdorff dimension of ν when $\lambda_0 = \lambda_1 = \lambda$ and λ is Pisot ([23],[24]). Recall that from Lemma 1.7 if $\dim_{\mathcal{H}}\nu = s$ then $d_\nu(x) \leq s$ for ν-almost all x. We will be wishing to obtain values s such that for all x, $\underline{d}_\nu(x) \geq s$. In the case where $\lambda_1 = \lambda_2$ and this value is the reciprocal of the golden ratio this problem was considered in [13]. Our main result is the following.

Theorem 3.1. *Let ν be a Bernoulli convolution. There exists $C > 0$ and $0 < \delta < 1$ such that for any subinterval $J \subset [0,1]$ we can bound*

$$\nu(J) \leq c(diam(J))^\delta.$$

Hence for all $x \in [0,1]$

$$\underline{d}_\nu(x) \geq \delta.$$

We can prove Theorem 3.1 in a more general setting. Let $T_0, \dots, T_{k-1} : [0,1] \to [0,1]$ be distinct C^1 contractions where $T_i'(x) > 0$ for all i and all $x \in [0,1]$. We also assume that $\cup_{i=0}^{k-1} T_i(I) = I$. Let $(p_0, \dots, p_{k-1})$ be a probability vector with no zero elements and μ the standard self-similar probability measure where for $J \subset [0,1]$

$$\mu(J) = \sum_{i=0}^{k-1} p_i \mu(T_i^{-1}(J)).$$

We denote $q = \min_{0 \leq i \leq k-1}\{p_i\}$. For any $n \geq 1$ we can consider a string $\underline{i} = (i_0, \dots, i_{n-1}) \in \{0, \dots, k-1\}^n$ and define $T_{\underline{i}} = T_{i_0} \circ \cdots \circ T_{i_{n-1}} : [0,1] \to [0,1]$.

Lemma 3.1. *There exists $N \geq 1$ and $\epsilon > 0$ such that for any interval $J \subseteq [0,1]$ with* $\mathrm{diam}(J) < \epsilon$ *we have that*

$$\mu(J) \leq (1 - q^N) \max_{|\underline{i}|=N} \mu(T_{\underline{i}}^{-1} J).$$

Proof. We have that $\cup_{i=0}^{k-1} T_i(I) = I$ and $||T_i'||_\infty < 1$ for all i. Thus for any string $\underline{i} \in \{0, \dots, k-1\}^{\mathbb{N}}$ it follows that

$$\lim_{n \to \infty} \mathrm{diam}(T_{i_0} \circ \cdots T_{i_{n-1}}([0,1])) \to 0.$$

Hence there must exist N where there are two strings $\underline{i} = (i_0, \ldots, i_{N-1})$ and $\underline{j} = (j_0, \ldots, j_{N-1})$ such that $T_{\underline{i}}([0,1])$ and $T_{\underline{j}}([0,1])$ are disjoint. Let $\epsilon = \mathrm{dist}(T_{\underline{i}}([0,1]), T_{\underline{j}}([0,1]))$. Hence any interval J such that $\mathrm{diam}(J) < \epsilon$ cannot intersect both $T_{\underline{i}}([0,1])$ or $T_{\underline{j}}([0,1])$. Thus J can only intersect at most $k^N - 1$ of the images $\{T_{\underline{i}}([0,1]) : \underline{i} \in \{0, \ldots, k-1\}^{\mathbb{N}}$. In other words the set

$$A(N, J) = \{\underline{i} = (i_0, \ldots, i_{N-1}) : T_{\underline{i}}([0,1]) \cap J \neq \emptyset\}$$

has cardinality at most $k^N - 1$. Applying the definition of μ iteratively we have that

$$\begin{aligned} \mu(J) &= \sum_{|\underline{i}|=N} p_{i_0} \cdots p_{i_{N-1}} \mu(T_{\underline{i}}^{-1}(J)) \\ &\leq \sum_{\underline{i} \in A(N,J)} p_{i_0} \cdots p_{i_{N-1}} \max_{|\underline{j}|=N} \mu(T_{\underline{j}}^{-1}(J)) \\ &\leq \max_{|\underline{i}|=N} \mu(T_{\underline{i}}^{-1}(J))(1 - q^N) \end{aligned}$$

as required. □

This lemma can be used to obtain the following result concerning the local dimension of μ.

Proposition 3.2. *There exists $C, \delta > 0$ such that for any interval $J \subset [0,1]$ with $\mathrm{diam}(J) < \epsilon$ we have $\mu(J) \leq C(\mathrm{diam}(J))^{\delta}$. Hence*

$$\underline{d}_{\mu}(x) \geq \beta.$$

Proof. Since for all i, $|T_i'(x)| > 0$ for all x we can choose $0 < \rho < |T_i'(x)|$ for all $x \in [0,1]$ and $0 \leq i \leq k-1$. We take ϵ, N as defined in Lemma 3.1. For any subinterval we can choose $m \geq 1$ such that

$$\epsilon \rho^{mN} \leq \mathrm{diam}(J) \leq \epsilon \rho^{(m-1)N}.$$

We note that whenever $|\underline{i}| = rN$ for $1 \leq r \leq m-1$ we have that

$$\mathrm{diam}(T_{\underline{i}}^{-1}(J)) \leq \frac{\mathrm{diam}(J)}{\rho^{rN}} < \epsilon.$$

Hence we can apply Lemma 3.1 repeatedly to get

$$\begin{aligned}
\mu(J) &\leq (1-q^N)\max_{|\underline{i}|=N}\mu(T_{\underline{i}}^{-1}(J)) \\
&\leq (1-q^N)^2\max_{|\underline{i}|,|\underline{i}'|=N}\mu(T_{\underline{i}'}^{-1}T_{\underline{i}}^{-1}(J)) = (1-q^N)^2\max_{|i|=2n}\mu(\{T_{\underline{i}}^{-1}(J)\}) \\
&\quad \ldots \\
&\leq (1-q^N)^m\max_{|\underline{j}|=rN}\mu(\{T_{\underline{j}}^{-1}(J)\}) \leq (1-q^N)^m.
\end{aligned}$$

If we write $\delta = \frac{\log(1-q^N)}{N\log\rho}$ and $C = \epsilon^{-\delta}$. In particular we get

$$\mu(J) \leq \rho^{Nm\delta} \leq \left(\frac{1}{\epsilon^\delta}\right)(\mathrm{diam}(J))^\delta = C\mathrm{diam}(J)^\delta$$

and the proof is complete. □

Theorem 3.1 follows as a simple corollary to Proposition 3.2. We are able to obtain explicit estimates of δ we now do this in the case of Bernoulli convolutions.

Let $p_0 = p_1 = \frac{1}{2}$ and $\frac{1}{2} < \lambda_1, \lambda_2 < \frac{1}{\sqrt{2}}$. In the proof of Proposition 3.2 we can choose $N = 2$ because here $T_0 \circ T_0([0,1])$ and $T_1 \circ T_1([0,1])$ are clearly disjoint. $\rho^N > \frac{1}{4}$ and $(1-q^N) = \log\left(\frac{3}{4}\right)$. Thus we can take

$$\delta = \frac{\log\frac{3}{4}}{\log\frac{1}{4}} = 0.2075\ldots.$$

Moreover we can take δ arbitrary close to $\frac{\log\frac{3}{4}}{\log\frac{1}{2}}$ by choosing λ_1, λ_2 sufficiently close to $\frac{1}{\sqrt{2}}$.

If we take $\lambda_0 = \lambda_1 = \frac{\sqrt{5}-1}{2}$. In other words λ_0 and λ_1 are both the reciprocal of the golden mean. In this case $\nu = \nu_{\lambda_0,\lambda_1}^{p_0,p_1}$ is singular and has Hausdorff dimension less than 1 ([23]). We can estimate $\delta = \frac{\log\frac{3}{4}}{2\log(\frac{\sqrt{5}-1}{2})}$. In this case Hu has used the algebraic properties of the golden ratio to establish the much better estimate $\delta = 0.9404$. In [23] and [24] the Hausdorff dimension of the measures $\nu_{\lambda_0,\lambda_1}^{p_0,p_1}$ are considered where $\lambda_0 = \lambda_1 = \frac{\sqrt{5}-1}{2}$. Lalley calculates the Hausdorff dimension of these measures for various different values of p_0. Recall the Hausdorff dimension gives a lower bound on δ for almost all x. We can compare these values to estimates of δ for all values given in Theorem 3.1

p_0	$\dim_{\mathcal{H}}\nu$	δ
0.5	$0.9954\ldots$	$0.2989\ldots$
0.4	$0.9848\ldots$	$0.1811\ldots$
0.3	$0.9501\ldots$	$0.0979\ldots$
0.2	$0.8499\ldots$	$0.0424\ldots$
0.1	$0.6085\ldots$	$0.0104\ldots$
0.05	$0.3877\ldots$	$0.0026\ldots$

The values of δ are only lower bounds so they could in actual fact be much closer to $\dim_{\mathcal{H}}\nu$. The method of calculating $\dim_{\mathcal{H}}\nu$ relied on the algebraic properties of the golden mean so cannot be generalized to general λ_0, λ_1. Our next result looks at the behaviour of δ when λ_1, λ_2 are close to $\frac{1}{2}$.

Proposition 3.3. *If we fix $p_0 = p_1 = \frac{1}{2}$ then we have that $\delta = \delta(\lambda_0, \lambda_1) \to 1$ as $\lambda_0, \lambda_1 \to \frac{1}{2}$.*

Proof. Let

$$\Lambda_k = \{(\lambda_0, \lambda_1) : 0 \leq (\lambda_0 + \lambda_1) - 1 < \min\{\lambda_0^{k-1}(1-\lambda_1), \lambda_1^{k-1}(1-\lambda_0)\}\}.$$

If $(\lambda_0, \lambda_1) \in \Lambda_k$ then

$$T_0([0,1]) \cap T_1 \circ T_0^{k-1} \circ T_1([0,1]) = \emptyset \text{ and } T_1([0,1]) \cap T_0 \cap T_1^{k-1} \circ T_0([0,1]) = \emptyset.$$

Hence the only string $\underline{i}$ of length $k+1$ where the first element is 1 such that

$$T_{\underline{i}}([0,1]) \cap T_0([0,1]) \neq \emptyset$$

is $(1, \{0\}^k)$. Similarly the only string $\underline{j}$ of length $k+1$ where the first element is 0 such that

$$T_{\underline{j}}([0,1]) \cap T_1([0,1]) \neq \emptyset$$

is $(0, \{1\}^k)$. We shall use this to see that there exist at most two distinct $\underline{i}, \underline{j} \in \{0,1\}^{k+1}$ such that $x \in T_{\underline{i}}([0,1]) \cap T_{\underline{j}}([0,1])$.

We fix $x \in [0,1]$ and assume there are two distinct strings $\underline{i}, \underline{j} \in \{0,1\}^{k+1}$ such that $x \in T_{\underline{i}}([0,1]) \cap T_{\underline{j}}([0,1])$. We shall show that there can be no more strings $\underline{\tau} \in \{0,1\}$

where $x \in T_\tau([0,1])$. If $0 = i_0 \neq j_0 = 1$ then it is immediate from the above that $\underline{j} = (1, \{0\}^k)$ and $\underline{i} = (0, \{1\}^k)$ and it immediately follows from the definition of Λ_k that there are no more strings. If for some $1 \leq l \leq k$ $(i_1, \ldots, i_l) = (j_1, \ldots, j_l)$ and $0 = i_{l+1} \neq i_{l+1} = 1$ then, clearly,

$$(i_{l+1}, \ldots, i_{k+1}) = (0, 1, 1, \ldots, 1) \text{ and } (j_{l+1}, \ldots, j_{k+1}) = (1, 0, 0, \ldots, 0).$$

Thus there are no possible other sequences $\tau \in \{0,1\}^{k+1}$ where

$$(\tau_1, \ldots, \tau_l) = (i_1, \ldots, i_l) \text{ and } x \in T_\tau([0,1]).$$

Working backwards we can see that the first l terms must also be distinct.

Since the intervals are closed, we can choose $\epsilon > 0$ such that for any $x \in [0,1]$ there are still at most two distinct $\underline{i} \in \{0,1\}^{k+1}$ such that $B(x,\epsilon) \cap T_{\underline{i}} \neq \emptyset$. We let $\rho = \min\{\lambda_1, \lambda_2\}$. Given $0 < r < \epsilon$ we can choose n such that $\rho^{n(k+1)} \leq r \leq \rho^{(n-1)(k+1)}$. If we denote $J_n(x) = \{\underline{i} \in \{0,1\}^{(k+1)n} : T_{\underline{i}}([0,1]) \cap B(x, \rho^{(n-1)(k+1)}) \neq \emptyset\}$, then the self-similar measure satisfies,

$$\nu(B(x,r)) = \frac{1}{2^n r} \sum_{\underline{i} \in J_n(x)} \nu(T_{\underline{i}}^{-1} B(x,r)) \leq \frac{\mathrm{Card}(J_n(x))}{2^{n(k+1)}}. \tag{3.1}$$

We claim that $J_n = \sup_{x \in I} \mathrm{Card}(J_n(x)) \leq 2^n$. The earlier argument showed that $J_1 \leq 2$. Proceeding inductively, we can assume that for $1 \leq l < n$ we have that $J_n \leq 2^l$. Let $\underline{i} \in \{0,1\}^{(l+1)(k+1)}$ satisfy $T_{\underline{i}}([0,1]) \cap B\left(x, \frac{\epsilon}{2}\rho^k\right) \neq \emptyset$. Let $\underline{i}' = (i_1, \ldots, i_{k+1})$ be the truncation to the first k terms, then clearly $T_{\underline{i}}([0,1]) \cap B(x, \frac{\epsilon}{2}\rho^k) \neq \emptyset$. Thus there are at most 2 choices of $\underline{i}'$. Setting $\underline{i}'' = (i_{k+2}, \ldots, i_{(k+1)(l+1)}) \in \{0,1\}^{l(k+1)}$ we see that $T''_{\underline{i}}([0,1]) \cap B(T_{i_k}^{-1} \circ \cdots \circ T_{i_1}^{-1}) \neq \emptyset$. Thus by the inductive hypothesis $\underline{i}''$ is also specified up to at most J_n choices. Thus $\underline{i}$ is specified up to $J_{n+1} \leq 2J_n \leq 2^{n+1}$ choices as claimed. Finally we can see from (3.1) that $\nu(B(x,r)) \leq 2^{-k}$ and thus

$$\underline{d}_\nu(x) = \liminf_{r \to \infty} \frac{\log \nu(B(x,r))}{\log r} \geq \frac{k \log\left(\frac{1}{2}\right)}{(k+1)\log \rho}.$$

If we let $k \to \infty$ then $\rho \to \frac{1}{2}$ and hence $\frac{k \log\left(\frac{1}{2}\right)}{(k+1)\log \rho} \to 1$. Thus the proof is complete. □

Corollary 3.1. *If $p_0 = p_1 = \frac{1}{2}$ then as $(\lambda_0, \lambda_1) \to \left(\frac{1}{2}, \frac{1}{2}\right)$ we have*

$$\dim_{\mathcal{H}} \nu_{\lambda_1, \lambda_2} \to 1.$$

Proof. This is immediate from Lemma 1.7. □

In [13] it is shown that when $\lambda_0 = \lambda_1$ δ has an upper bound which is always obtained. It is straightforward to generalise this result to the case where $\lambda_0 \neq \lambda_1$.

Proposition 3.4. *If we fix λ_0, λ_1 and p_0, p_1 we have that*

$$\sup\{\underline{d}_\nu(x)\} = \sup\{\overline{d}_\nu(x)\} = \max\left\{\frac{\log p_0}{\log \lambda_0}, \frac{\log p_1}{\log \lambda_1}\right\}.$$

Proof. We assume without loss of generality that $\frac{\log p_0}{\log \lambda_0} \geq \frac{\log p_1}{\log \lambda_1}$ and let $\lambda = \min\{\lambda_0, \lambda_1\}$. We first show that $\overline{d}_\nu(x) \leq \frac{\log p_0}{\log \lambda_0}$ for all $x \in [0,1]$. Fix $x \in [0,1]$ and note that there exists a sequence $\underline{i} \in \{0,1\}^{\mathbb{N}}$ such that $T_{\underline{i}}(0) = x$. For $r > 0$ we choose $n = n(r)$ such that

$$\operatorname{diam}(T_{i_0} \circ \cdots \circ T_{i_{n-2}}([0,1])) \geq r > \operatorname{diam}(T_{i_0} \circ \cdots \circ T_{i_{n-1}}([0,1])).$$

Thus

$$\begin{aligned}
\frac{\log \nu(B(x,r))}{\log 2r} &\leq \frac{\log \nu(T_{i_0} \circ \cdots \circ T_{i_{n-1}}([0,1]))}{\log 2r} \\
&\leq \frac{\log(p_{i_1} \cdots p_{i_n})}{\log(2\lambda^{-1}\operatorname{diam}(T_{i_0} \circ \cdots \circ T_{i_{n-1}}([0,1])))} \\
&= \frac{\sum_{j=1}^{n} \log p_{i_j}}{\log 2\lambda^{-1} + \sum_{j=0}^{n-1} \log \lambda_{i_j}}.
\end{aligned}$$

Thus

$$\limsup_{r \to 0} \frac{\log \nu(B(x,r))}{\log 2r} \leq \frac{\log p_0}{\log \lambda_0}.$$

We complete the proof by showing that $\underline{d}_\nu(0) \geq \frac{\log p_0}{\log \lambda_0}$. For $n \in \mathbb{N}$ consider the interval $[0,r]$ where

$$\frac{(1-\lambda_1)\lambda_0^{n-1}}{2} < r < (1-\lambda_1)\lambda_0^{n-1}.$$

The only string, $\underline{i} \in \{0,1\}^n$ such that $T_{\underline{i}}([0,1]) \cap [0,r]$ is $(\{0\}^n)$. Thus

$$\nu([0,r]) \leq p_0^n = (\lambda_0)^{n\frac{\log p_0}{\log \lambda_0}} \leq C r^{\frac{\log p_0}{\log \lambda_0}}$$

where $C = \frac{\lambda_0^{\frac{\log p_0}{\log \lambda_0}}}{(2(1-\lambda_1))^{\frac{\log p_0}{\log \lambda_0}}}$. Hence the proof is complete. □

Chapter 4

Fat Sierpiński Gaskets

4.1 Definition and Results

In this chapter we investigate the properties of a family of IFS's in $\mathbb{R}^2$ with overlaps. This family is a generalization of the well known Sierpiński gasket called the fat Sierpiński gaskets. The methods we use to investigate the dimension are transversality and Marstrand's slicing theorem. The fat Sierpiński gaskets were used as an example by Simon and Solomyak in [41]. It is defined to be the attractor, $\Lambda(\lambda) \subset \mathbb{R}^2$ of the IFS, $F = \{T_0, T_1, T_2\}$ where,

$$\begin{aligned} T_0(x,y) &= \lambda(x,y) \\ T_1(x,y) &= \lambda(x,y) + (1,0) \\ T_2(x,y) &= \lambda(x,y) + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \end{aligned}$$

for $\lambda > \frac{1}{2}$. The following simple result was proven in [41].

Theorem 4.1 (Simon-Solomyak). *There exists a dense set of* $\lambda \in [\frac{1}{2}, \frac{1}{\sqrt{3}}]$ *such that,*

$$\dim_{\mathcal{H}} \Lambda(\lambda) < -\frac{\log 3}{\log \lambda}. \tag{4.1}$$

The values of λ which satisfy inequality 4.1 are the countable set of algebraic numbers which satisfy an equation of the form

$$\sum_{k=0}^{n} a_k \lambda^k = 0$$

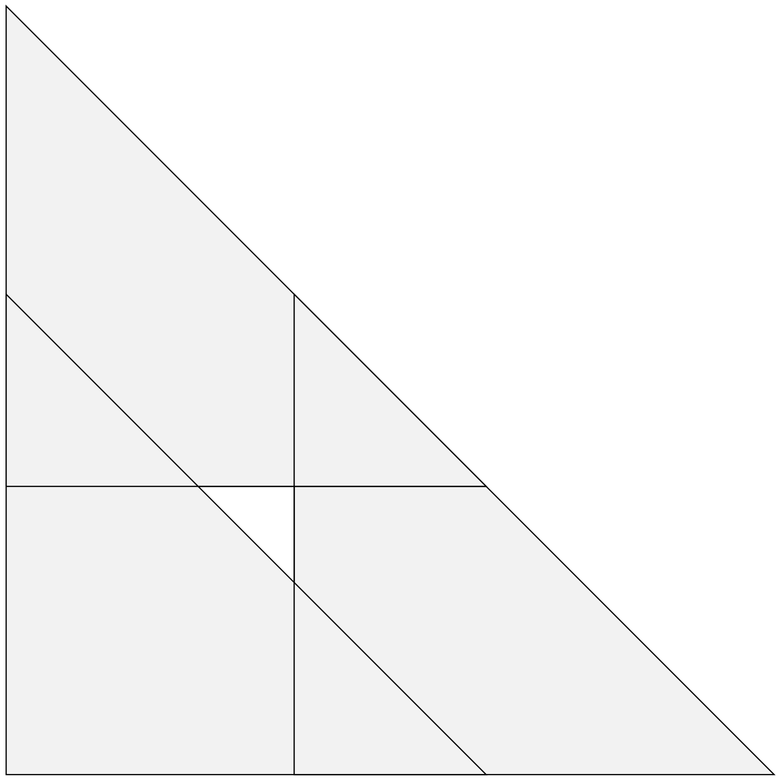

Figure 4.1: The first level construction of a fat Sierpiński gasket

where each $a_k \in \{-1, 0, 1\}$. When λ satisfies such an equation the semi-group generated by F, acting on $\mathbb{R}^2$, is not free and it is straight forward to show that inequality 4.1 holds.

A systematic investigation of the Hausdorff dimension of $\Lambda(\lambda)$ was started by Broomhead, Montaldi and Sidorov in [4]. They were able to compute the exact Hausdorff dimension of $\Lambda(\lambda)$ when λ is in a special class of algebraic numbers they call the multinacci numbers. These are the positive solutions, ω_n, to the equations $\sum_{k=1}^{n} \lambda^k = 1$. In particular ω_2 is equal to the reciprocal of the golden ratio. They obtain the following result,

Theorem 4.2 (Broomhead, Montaldi, Sidorov).

$$\dim_{\mathcal{H}}(\Lambda(\omega_n)) = \frac{\log \tau_n}{\log \omega_n},$$

where τ_n is the smallest positive root of the polynomial $3z^{n+1} - 3z + 1$.

It should be noted that $\frac{\log \tau_n}{\log \omega_n} < -\frac{\log 3}{\log \omega_n}$.

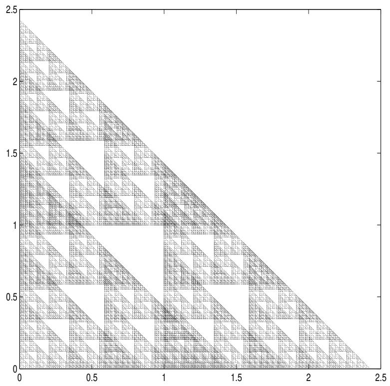

Figure 4.2: $\Lambda(\lambda)$ for $\lambda = 0.59$. Theorem 4.3 states that for almost all $\lambda > 0.5853$, $\mathcal{L}^2(\Lambda(\lambda)) > 0$.

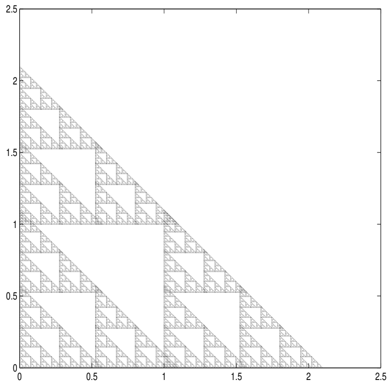

Figure 4.3: $\Lambda(\lambda)$ for $\lambda = 0.521$. Theorem 4.3 shows that for almost all $\lambda \in [0.5, 0.529]$ $\dim_{\mathcal{H}}\Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$.

In this chapter we continue the investigation into the Hausdorff dimension of $\Lambda(\lambda)$. The following is our main result.

Theorem 4.3.

1. *For almost all* $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3} \approx 0.529]$,

$$\dim_{\mathcal{H}}\Lambda(\lambda) = -\frac{\log 3}{\log \lambda}.$$

2. *For almost all* $\lambda \geq 0.5853$,

$$\dim_{\mathcal{H}}\Lambda(\lambda) = 2 \textit{ and } \mathcal{L}^2(\Lambda(\lambda)) > 0.$$

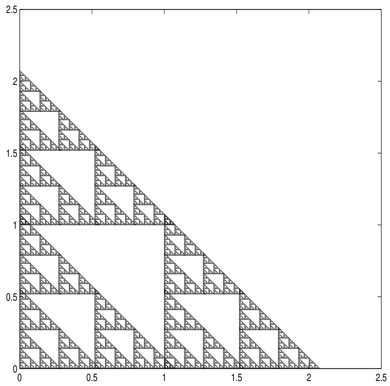

Figure 4.4: $\Lambda(\lambda)$ for $\lambda = \omega_4 \approx 0.519$. It is shown in [4] that $\dim_{\mathcal{H}}\Lambda(\lambda) = \frac{\log \tau_4}{\log \omega_4} \approx 1.654 < -\frac{\log 3}{\log \omega_4}$. Theorem 4.3 shows that this is an exceptional value.

Our methods only enable us to show that $\mathcal{L}^2(\Lambda(\lambda)) > 0$ for almost all $\lambda \leq 0.649$. However it is clear that for all $\lambda \geq \frac{2}{3}$, $\mathcal{L}^2(\Lambda(\lambda)) > 0$ and in [4] it is shown that for all $\lambda \geq 0.648$, $\Lambda(\lambda)$ has non-empty interior and hence positive Lebesgue measure. It should be noted that the results in [41] and [4] mean that the equality in Theorem 4.3 certainly does not hold for all λ. It would be interesting to know whether the region of λ for which Theorem 4.3 is true can be extended to a larger region. However the method used in this chapter only provides almost sure lower bounds for $\lambda \in (0.529, 0.5853]$ which are strictly less than $-\frac{\log 3}{\log \lambda}$. Theorem 4.3 has the following topological analogue.

Corollary 4.1.

1. *There exists a residual set* $A \subset [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$ *such that for any* $\lambda \in A$

$$\dim_{\mathcal{H}}\Lambda(\lambda) = -\frac{\log 3}{\log \lambda}.$$

2. *There exists a residual set* $B \subset [0.5853, 1]$ *such that for any* $\lambda \in B$

$$\dim_{\mathcal{H}}\Lambda(\lambda) = 2.$$

Proof. We shall only prove part 1. of Corollary 4.1 because the proof of part 2 can be done using exactly the same method. Recall that a set is said to be residual if it contains a dense G_δ set . The method is similar to the proof of Theorem 2.3

in [41]. From Theorem 4.3 we know there exists a dense set of $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$ such that $\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$. Let $\mathcal{F}$ be the set of all IFS's, $\{S_i\}_{i=0}^{2}$ in $\mathbb{R}^2$ such that $S_i(x) = \lambda x + b_i$ for $b_i \in \mathbb{R}^2$. We define a topology on F by the natural bijection from F to $[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}] \times \mathbb{R}^6$.

From Theorem B in [17] we know that the function $\alpha(F) = \dim_{\mathcal{H}}(\Lambda(F))$ is lower semi-continuous. However since for a fixed λ, $\dim_{\mathcal{H}}(\Lambda(F))$ is constant. The function, $\bar{\alpha}(\lambda) = \dim_{\mathcal{H}}(\Lambda(\lambda))$ is also lower semi-continuous. If we let $\beta(\lambda) = -\frac{\log 3}{\log \lambda}$ then we have that β is continuous and $\alpha'(\lambda) \leq \beta(\lambda)$. We now show that

$$\left\{\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right] : \bar{\alpha}(\lambda) = \beta(\lambda)\right\} = \left\{\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right] : \bar{\alpha} \text{ is continuous at } \lambda\right\}. \tag{4.2}$$

Firstly consider $\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right]$ such that $\bar{\alpha}(\lambda) = \beta(\lambda)$. We know that β is continuous and $\bar{\alpha} \leq \beta$ is lower semi continuous thus $\bar{\alpha}$ is continuous at λ. On the other hand $\bar{\alpha}$ cannot be continuous at λ if $\bar{\alpha}(\lambda) \neq \beta(\lambda)$ because $\bar{\alpha}(\lambda) = \beta(\lambda)$ for a.e. $\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right]$. This completes the proof of (4.2).

By Lemma 1.5 the set of continuity points for any function is a G_δ set. Hence the set of points where $\bar{\alpha}(\lambda) = \beta(\lambda)$ contains a dense G_δ set. □

Hence the results found in [4] and [41] in the above region were exceptional cases both in a topological and measure theoretic sense.

It is notationally more convenient to look at a slightly different IFS. This is defined by the similarities,

$$\begin{aligned} T_0(x) &= \lambda x \\ T_1(x) &= \lambda x + (1,0) \\ T_2(x) &= \lambda x + (0,1). \end{aligned}$$

However the attractor of this IFS can be obtained by an affine transformation applied to the set $\Lambda(\lambda)$ and hence has the same Hausdorff dimension. Note that the figures in this chapter are made using this iterated function system. Our proof of part 1 of Theorem 4.3 proceeds by calculating the dimension of two families of measures in one dimension. The first family are supported on the projection of the iterated function system to the y-axis and the second family are supported on horizontal slices of the attractor. The result is then deduced using Lemma 1.9. Part 2 is proved using a similar method where the absolute continuity of the measures is considered.

4.2 Dimension of Biased Bernoulli Convolutions

To prove Theorem 4.3 we first obtain some results about the properties of Bernoulli convolutions. In particular we wish to know whether they are absolutely continuous or singular and in the case where they are singular we try and estimate their Hausdorff dimension. Let $\lambda \in [0.5, 0.649\ldots]$ and $\underline{p} = (p_0, p_1)$ be a probability vector. We let,

$$\begin{aligned} T_0(x) &= \lambda x \\ T_1(x) &= \lambda x + 1. \end{aligned}$$

Let $\nu = \nu_\lambda^{p_0,p_1}$ be the self-similar measure such that for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$

$$\nu(J) = p_0\nu(T_0^{-1}(J)) + p_1\nu(T_1^{-1}(J)).$$

We will also let $\mu = \mu_{p_0,p_1}$ be (p_0, p_1)-Bernoulli measure defined on the full shift, Σ_2. We let $\Pi_\lambda : \Sigma_2 \to \mathbb{R}$ be defined by

$$\Pi_\lambda(\underline{i}) = \sum_{n=0}^{\infty} i_n \lambda^n.$$

This gives $\nu_\lambda^{(p_0,p_1)} = \mu^{(p_0,p_1)} \circ \Pi_\lambda^{-1}$.

Proposition 4.1. *Fix* (p_0, p_1).

1. *For almost all* $\lambda \in [0.5, 0.649\ldots]$,

$$\dim_{\mathcal{H}} \nu_\lambda^{(p_0,p_1)} = \min\left(\frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}, 1\right).$$

2. *For almost all* $\lambda \in [p_0^{p_0} p_1^{p_1}, 0.649]$ ν_λ *is absolutely continuous.*

This result could be deduced as a Corollary to Theorem 7.2 in [42]. However a more elementary proof of part 2 is given in [36]. We show that a similar method can be used to prove part 1. This method uses the same technique as the proof of Theorem 2.2 which was outlined in Chapter 2.

Proof of Proposition 4.1 Part 1

The proof of the upper bound follows immediately from Theorem 1.6. For the lower bound the following lemma is needed. It involves the use of an exponent $\alpha \in (0,1]$. The idea to use this exponent came from [36].

Lemma 4.1. *Fix (p_0, p_1). For all $\alpha \in (0,1]$ we have that for almost all $\lambda \in [0.5, y(1)]$*

$$\dim_{\mathcal{H}} \nu_\lambda^{(p_0,p_1)} \geq \min\left(\frac{\log((p_0^{\alpha+1}+p_1^{\alpha+1})^{\frac{1}{\alpha}})}{\log \lambda}, 1\right).$$

Proof. Fix (p_0, p_1) and let $\epsilon > 0$. For simplicity denote

$$d(\alpha,\epsilon) = (p_0^{\alpha+1} + p_1^{\alpha+1} + \epsilon)^{\frac{1}{\alpha}}.$$

We let $S_\epsilon(\lambda) = \min\left(\frac{\log(d(\alpha,\epsilon))}{\log\lambda}, 1-\epsilon\right)$. We use Lemma 1.8 together with Fubini's Theorem and Corollary 2.1.

$$\begin{aligned} I &= \int_{0.5}^{y(1)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\mathrm{d}\nu_\lambda(x)}{|x-y|^{S_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\nu_\lambda(y)\mathrm{d}\lambda \\ &= \int_{0.5}^{y(1)} \int_{\Sigma_2} \left(\int_{\Sigma_2} \frac{\mathrm{d}\mu(\underline{i})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{S_\epsilon(y)}}\right)^\alpha \mathrm{d}\mu(\underline{j})\mathrm{d}\lambda \end{aligned}$$

Apply Fubini's theorem and Hölder's inequality ($\int f^\alpha \leq C(\int f)^\alpha$ for $\alpha \in (0,1]$.) to give,

$$\begin{aligned} I &\leq C\int_{\Sigma_2}\left(\int_{0.5}^{y(1)}\int_{\Sigma_2}\frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{|\Pi_\lambda(\underline{i})-\Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\ &= C\int_{\Sigma_2}\left(\int_{0.5}^{y(1)}\int_{\Sigma_2}\frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{|\sum_{n=0}^\infty (i_n-j_n)\lambda^n|^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\ &= C\int_{\Sigma_2}\left(\int_{0.5}^{y(1)}\int_{\Sigma_2}\frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{\left(\lambda^{|\underline{i}\wedge\underline{j}|}\,|a_0+\sum_{n=1}^\infty a_n\lambda^n|\right)^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \end{aligned}$$

where $a_n \in \{-1,0,1\}$ for $n \geq 1$ and $a_0 \in \{-1,1\}$. We use Corollary 2.1 to continue,

$$\begin{aligned} I &\leq C\int_{\Sigma_2}\left(\int_{0.5}^{y(1)}\int_{\Sigma_2}\frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{\left(d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}\,|a_0+\sum_{n=1}^\infty a_n\lambda^n|\right)^{s_\epsilon(\lambda)}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\ &\leq C\int_{\Sigma_2}\left(\int_{0.5}^{y(1)}\frac{\mathrm{d}\lambda}{|a_0+\sum_{n=1}^\infty a_n\lambda^n|^{s_\epsilon(\lambda)}}\int_{\Sigma_2}\frac{\mathrm{d}\mu(\underline{i})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\ &\leq C_1/int_{\Sigma_2}\left(\int_{\Sigma_2}\frac{\mathrm{d}\mu(\underline{i})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}}\right)^\alpha \mathrm{d}\mu(\underline{j}) \\ &\leq C_1\int_{\Sigma_2}\left(\sum_{k=0}^\infty \frac{\mu(W_{\omega,k})}{d(\alpha,\epsilon)^k}\right)^\alpha \mathrm{d}\mu(\omega) \end{aligned}$$

We proceed by using the inequality $(\sum_i b_i)^\alpha \leq \sum_i b_i^\alpha$ for $b_i > 0$ and $\alpha \in (0,1]$

$$\begin{aligned} I &\leq C_1 \sum_{k=0}^{\infty} \sum_{W \in W_k} \frac{\mu(W)^{\alpha+1}}{d(\alpha,\epsilon)^{\alpha k}} \\ &\leq C_1 \sum_{k=0}^{\infty} d(\alpha,\epsilon)^{-\alpha k} (p_0^{\alpha+1} + p_1^{\alpha+1})^k. \end{aligned}$$

Thus because $d(\alpha,\epsilon)^\alpha > p_0^{\alpha+1} + p_1^{\alpha+1}$ we have $I < \infty$. Hence by Lemma 1.8 $\dim_{\mathcal{H}} \nu_\lambda \geq \min\left(\frac{d(\alpha,\epsilon)}{\log \lambda}, 1-\epsilon\right)$ for almost all λ. To complete the proof we let $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $n \to \infty$. □

To complete the proof of Proposition 4.1 we let $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}$ and observe that

$$\lim_{n\to\infty} \frac{\log(p_0^{\alpha_n+1} + p_1^{\alpha_n+1})}{\alpha_n \log \lambda} = \frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}.$$

4.3 Cross Sections of Fat Gaskets

Consider a sequence $\{i_n\} \in \Sigma_3$ we can then represent each point in $\Lambda(\lambda)$ using the expansion

$$\sum_{n=0}^{\infty} \alpha_{i_n} \lambda^n,$$

where $\alpha_0 = (0,0)$, $\alpha_1 = (1,0)$ and $\alpha_2 = (0,1)$. It should be noted that for $\lambda > \frac{1}{2}$ this expansion is not unique. Consider a sequence $\underline{x} \in \Sigma_2$. Intuitively we think of the case when $x_n = 0$ as corresponding to the bottom two triangles in the gasket and x_n=1 corresponding to the top triangle. We then define a complementary sequence $\underline{j} \in \Sigma_2$ such that $j_n = 0$ whenever $x_n = 1$. The idea of this sequence is to determine a horizontal point on the gasket corresponding to the sequence $\underline{x}$. Thus whenever $x_n = 0$ there are two choices either 0 or 1 corresponding to the bottom two triangles in the gasket. However when $x_n = 1$ there is just the one choice and j_n must equal 0. This means if we define $\underline{i} \in \Sigma_3$ such that

$$i_n = \begin{cases} 0 \text{ if } x_n = 0, j_n = 0 \\ 1 \text{ if } x_n = 0, j_n = 1 \\ 2 \text{ if } x_n = 1, j_n = 0 \end{cases}$$

then

$$\left(\sum_{n=0}^{\infty} j_n \lambda^n, \sum_{n=0}^{\infty} x_n \lambda^n\right) = \left(\sum_{n=0}^{\infty} \alpha_{i_n} \lambda^n\right) \in \Lambda(\lambda).$$

We let

$$L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda)) = \{z \in \mathbb{R} : (z, \Pi_\lambda(\underline{x})) \in \Lambda(\lambda)\}$$

be the points on the horizontal slice at height $\Pi_\lambda(\underline{x})$ which are also in the limit set $\Lambda(\lambda)$. Hence for any sequence $\underline{i} \in \Sigma_2$ such that $i_n = 0$ if $x_n = 1$ we have that $\Pi_\lambda(\underline{i}) \in L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$.

We look at the dimension of the set $L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$. We will fix (p_0, p_1). Let $\mu = \mu_{p_0,p_1}$ be (p_0, p_1)-Bernoulli measure on Σ_2. We can define another measure $\tilde{\mu}_{\underline{x}}$ on Σ_2 such that,

$$\tilde{\mu}_{\underline{x}}(\{\underline{i} : i_n = 0\}) = \begin{cases} 1 \text{ if } x_n = 1 \\ \frac{1}{2} \text{ if } x_n = 0 \end{cases}.$$

This means for kth level cylinders,

$$\tilde{\mu}_{\underline{x}}([i_0, \ldots, i_{k-1}]) = \begin{cases} 0 \text{ if } \exists j \text{ such that } i_j = x_j = 1 \\ 2^{-k_0(\underline{x})} \text{ if for all } x_j = 1 \text{ we have } i_j = 0 \end{cases}.$$

Intuitively this means whenever $x_k = 0$ this corresponds to the bottom two triangles in the gasket and we have a choice of the two triangles but whenever $x_k = 1$ we are in the top triangle in the gasket so there is only one choice.

Let $\tilde{\nu}_{\lambda,\underline{x}} = \tilde{\mu}_{\underline{x}} \circ \Pi_\lambda^{-1}$ and note that it is supported on a subset of $L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$.

Lemma 4.2. *For almost all $\lambda \in [0.5, 0.649\ldots]$, and for ν_λ almost all $y \in \mathbb{R}$*

$$\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq \min\left(-\frac{p_0 \log 2}{\log \lambda}, 1\right).$$

Proof. We shall show that for all $\alpha \in (0, 1]$

$$\dim_{\mathcal{H}} L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda)) \geq -\frac{\log(1 - p_0(1 - 2^{-\alpha}))}{\alpha \log \lambda},$$

for almost all λ and μ almost all $\underline{x} \in \Sigma_2$. The result then follows because if we let $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}$ then

$$\lim_{n\to\infty} \frac{\log(1 - p_0(1 - 2^{-\alpha_n}))}{\alpha_n \log \lambda} = -\frac{p_0 \log 2}{\log \lambda}$$

and if $\dim_{\mathcal{H}} L_{\Pi_\lambda \underline{x}}(\Lambda(\lambda)) \geq s$ for μ-almost all $\underline{x} \in \Sigma_2$ then $\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq s$ for ν_λ almost all $y \in \mathbb{R}$. Let $\epsilon > 0$ and for simplicity let $d(\alpha,\epsilon) = (1 - p_0(1-2^{-\alpha}) + \epsilon)^{\frac{1}{\alpha}}$ and $s_\epsilon(\lambda) = \min\left(-\frac{\log d(\alpha,\epsilon)}{\log \lambda}, 1-\epsilon\right)$. We use the measure $\tilde{\nu}_{\lambda,\underline{x}}$. Using the potential theoretic method for calculating Hausdorff dimension it suffices to show that

$$I = \int_{0.5}^{y(1)} \int_{\Sigma_2} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\mathrm{d}\tilde{\nu}_{\lambda,\underline{x}}(y)\mathrm{d}\tilde{\nu}_{\lambda,\underline{x}}(z)}{|z-y|^{s_\epsilon(\lambda)}} \right)^\alpha \mathrm{d}\mu(\underline{x})\mathrm{d}\lambda < \infty.$$

We start by lifting to the shift space, using Fubini's theorem and Hölder's inequality, $\int f^\alpha \leq C\left(\int f\right)^\alpha$ for $\alpha \in (0,1]$.

$$\begin{aligned}
I &= \int_{0.5}^{y(1)} \int_{\Sigma_2} \left(\int_{\Sigma_2} \int_{\Sigma_2} \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x})\mathrm{d}\lambda \\
&\leq C \int_{\Sigma_2} \left(\int_{0.5}^{y(1)} \int_{\Sigma_2} \int_{\Sigma_2} \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})\mathrm{d}\lambda}{|\sum_{n=0}^\infty (i_n - j_n)\lambda^n|^{s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\
&\leq C_1 \int_{\Sigma_2} \left(\int_{0.5}^{y(1)} \int_{\Sigma_2} \int_{\Sigma_2} \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^\infty a_n\lambda^n|^{s_\epsilon(\lambda,\alpha)} \lambda^{|\underline{i}\wedge\underline{j}|s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\
&\leq C_1 \int_{\Sigma_2} \left(\left(\int_{0.5}^{y(1)} \frac{\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^\infty a_n\lambda^n|^{s_\epsilon(\lambda,\alpha)}} \right) \left(\int_{\Sigma_2} \int_{\Sigma_2} \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}} \right) \right)^\alpha \mathrm{d}\mu(\underline{x}),
\end{aligned}$$

where $a_0 \in \{-1,1\}$ and $a_n \in \{-1,0,1\}$ for $n \geq 1$. This means we can apply Corollary 2.1. Hence

$$\begin{aligned}
I &\leq C_2 \int_{\Sigma_2} \left(\int_{\Sigma_2} \int_{\Sigma_2} \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\
&\leq C_2 \int_{\Sigma_2} \left(\sum_{k=0}^\infty \frac{2^{-k_0(\underline{x})}}{d(\alpha,\epsilon)^k} \right)^\alpha \mathrm{d}\mu(\underline{x})
\end{aligned}$$

As in the proof of Lemma 4.1 we use the inequality $\left(\sum_i b_i\right)^\alpha \leq \sum_i b_i^\alpha$ for $b_i > 0$. We get,

$$\begin{aligned}
I &\leq C_2 \sum_{k=0}^\infty d(\alpha,\epsilon)^{-\alpha k} \int 2^{-k_0(\underline{x})\alpha}\mathrm{d}\mu(\underline{x}) \\
&\leq C_2 \sum_{k=0}^\infty d(\alpha,\epsilon)^{-\alpha k} \sum_{[i_0,\ldots,i_{k-1}]\in W_k} 2^{-k_0([i_0,\ldots,i_{k-1}])\alpha}\mu(W_k) \\
&\leq C_1 \sum_{k=0}^\infty \frac{(p_0 2^{-\alpha} + p_1)^k}{(d^{\alpha,\epsilon})^{\alpha k}}.
\end{aligned}$$

We can now see that $I < \infty$ because $p_0 2^{-\alpha} + p_1 = 1 - p_0(1-2^{-\alpha}) < d(\alpha,\epsilon)$. To finish the proof let $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $n \to \infty$. □

We now investigate when the slice $L_{\Pi_\lambda(\underline{x})} \cap \Lambda(\lambda)$ has positive measure. To do this we note that if $\tilde{\nu}_{\lambda,\underline{x}}$ is absolutely continuous then $L_{\Pi_\lambda(\underline{x})} \cap \Lambda(\lambda)$ has positive measure.

Lemma 4.3. *For almost every $\lambda \in [2^{-p_0}, y(1)]$ for $\mu^{(p_o,p_1)}$ almost every $\underline{x}$ the measure $\tilde{\nu}_{\lambda,\underline{x}}$ is absolutely continuous with respect to $\mathcal{L}^1$.*

Proof. We use the same method to show a measure is absolutely continuous with respect to almost every λ that was used in the proof of Theorem 2.4 outlined in Chapter 2. It suffices to show that for some $\alpha > 0$ and any $\delta > 0$

$$I_\alpha = \int_{2^{-p_0}+\delta}^{y(1)} \int_{\Sigma_2} \left(\int_{\mathbb{R}} \liminf_{r\to 0} \frac{\tilde{\nu}_{\lambda,\underline{x}}(B(y,r))}{2r} \mathrm{d}\tilde{\nu}_{\lambda,\underline{x}}(y) \right)^\alpha \mathrm{d}\mu(\underline{x})\mathrm{d}\lambda < \infty.$$

We start by projecting to the shift space, applying Fatou's Lemma, Hölder's inequality and Fubini's Theorem. If we let

$$G(\omega,\tau,r) = \{\lambda \in [2^{-p_0}+\delta, 0.649] : |\Pi_{\lambda,\underline{x}}(\omega) - \Pi_{\lambda,\underline{x}}(\omega)|\} \le 2r$$

this gives

$$I_\alpha \le \liminf_{r\to 0} \int_{2^{-p_0}+\delta} \int_{\Sigma_2} \left(\int_{\Sigma_2} \int_{\Sigma_2} \mathbb{1}_{\{\omega,\tau : |\Pi_{\lambda,\underline{x}}(\omega)-\Pi_{\lambda,\underline{x}}| \le 2r\}} \mathrm{d}\tilde{\mu}_{\underline{x}}(\omega)\mathrm{d}\tilde{\mu}_{\underline{x}}(\tau) \right)^\alpha \mathrm{d}\mu(\underline{x})\mathrm{d}\lambda$$
$$\le \int_{\Sigma_2} \left(\int_{\Sigma_2} \int_{\Sigma_2} \mathcal{L}(G(\omega,\tau,r)\mathrm{d}\tilde{\mu}_{\underline{x}}(\omega)\mathrm{d}\tilde{\mu}_{\underline{x}}(\tau) \right)^\alpha \mathrm{d}\mu(\underline{x})$$

By Corollary 2.1 we have that

$$\mathcal{L}(G(\omega,\tau,r)\} \le 2Cr(2^{-p_0}+\delta)^{-|\omega\wedge\tau|}$$

for some constant $C > 0$. For notational ease we will let $\lambda_0 = 2^{-p_0} + \delta$. We now integrate on the shift space and use the inequality $(\sum_i a_i)^\alpha \le \sum_i a_i^\alpha$ for $\alpha < 1$. This gives,

$$\begin{aligned} I_\alpha &\le C \int_{\Sigma_2} \left(\int_{\Sigma_2} \int_{\Sigma_2} \lambda_0^{-|\omega\wedge\tau|} \mathrm{d}\tilde{\mu}_{\underline{x}}(\omega)\mathrm{d}\tilde{\mu}_{\underline{x}}(\tau) \right)^\alpha \mathrm{d}\mu(\underline{x}) \\ &\le C \int_{\Sigma_2} \left(\sum_{k=0}^{\infty} \lambda_0^{-k} 2^{-k_0(\underline{x})} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\ &\le C \sum_{k=0}^{\infty} \int_{\Sigma_2} \lambda_0^{-\alpha k} 2^{-\alpha k_0(\underline{x})} \mathrm{d}\mu(\underline{x}) \\ &\le C \sum_{k=0}^{\infty} (\lambda_0^{-\alpha}(p_0 2^{-\alpha} + p_1))^k. \end{aligned}$$

Thus I_α is finite provided $\lambda_0^{-\alpha}(p_0 2^{-\alpha}+p_1) < 1$. For $\lambda_0 > 2^{-p_0}$ we can always choose an α such that $\lambda_0^{-\alpha}(p_0 2^{-\alpha}+p_1) < 1$ meaning that $I_\alpha < \infty$. Thus the proof is complete.

$\square$

4.4 Proof of Theorem 4.3

It follows from Theorem 1.2 that $\dim_{\mathcal{H}}\Lambda(\lambda) \leq -\frac{\log 3}{\log \lambda}$ for all λ, see, for example, [10]. Let $\underline{p} = (\frac{2}{3}, \frac{1}{3})$, let $\mu_{\underline{p}}$ be the standard $\underline{p}$-Bernoulli measure on Σ_2 and let $\nu_\lambda = \mu_{\underline{p}} \circ \Pi_\lambda^{-1}$. We know from Proposition 4.1 that for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$

$$\dim_{\mathcal{H}}\nu_\lambda = \frac{\frac{1}{3}\log\left(\frac{1}{3}\right) + \frac{2}{3}\log\left(\frac{2}{3}\right)}{\log \lambda}$$

and by Lemma 4.2 that for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$ and ν_λ almost all $y \in \mathbb{R}$

$$\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq -\frac{\frac{2}{3}\log 2}{\log \lambda}.$$

Thus using Lemma 1.9 we have that

$$\dim_{\mathcal{H}}\Lambda(\lambda) \geq \frac{\frac{1}{3}\log\left(\frac{1}{3}\right) + \frac{2}{3}\log\left(\frac{2}{3}\right)}{\log \lambda} - \frac{\frac{2}{3}\log 2}{\log \lambda} = -\frac{\log 3}{\log \lambda}$$

for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$.

To prove part 2 of Theorem 4.3 we need to take an alternative choice of probability vector. We choose $\underline{p} = (p_0, p_1)$ such that

$$p_0^{p_0} p_1^{p_1} = 2^{-p_0}. \tag{4.3}$$

This means that both the projected measure and the measures on the slice (typically) both become absolutely continuous for $\lambda \geq 2^{-p_0}$. Solving Equation (4.3) numerically gives $\underline{p} = (0.7729\ldots, 0.2270\ldots)$ and $2^{-p_0} = 0.5853\ldots$.

Formally by letting $\mu = \mu_{\underline{p}}$, $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$ we have by Proposition 4.1 that ν_λ is absolutely continuous for almost every $\lambda \in [0.5853\ldots, 0.649]$. By Lemma 4.3 $\tilde{\nu}_{\lambda,\underline{x}}$ is absolutely continuous for μ almost every $\underline{x}$ for almost every $\lambda \in [0.5853\ldots, 0.649]$. We will now assume that ν_λ is absolutely continuous and $\nu_{\lambda,\underline{x}}$ is absolutely continuous for almost every $\underline{x}$. Hence for ν_λ almost every y the slice $L_y \cap \Lambda(\lambda)$ has positive one dimensional Lebesgue measure. Since we have assumed ν_λ is absolutely continuous this is sufficient to see that $\mathcal{L}^2(\Lambda(\lambda)) > 0$. Thus for almost every $\lambda > 0.5853\ldots$ we have that $\mathcal{L}^2(\Lambda(\lambda)) > 0$.

Chapter 5

Measures Supported on Fat Sierpiński Carpets

5.1 Definition

Most of the work in this chapter is joint work with Mark Pollicott and is contained in [19]. We look at a more general family of Iterated Function systems in $\mathbb{R}^2$. These are based on the standard Sierpiński carpets. We use similar methods to the last chapter to find the dimension of their attractors but we also look at the properties of more general measures. This involves using the idea of conditional entropy from chapter 1.

Let $0 < \lambda < 1$ and $n > k \in \mathbb{N}$. We consider a family of n contractions given by

$$T_i(x, y) = (\lambda x, \lambda y) + (c_i^{(1)}, c_i^{(2)}),$$

$i = 0, \ldots, n-1$ where $(c_i^{(1)}, c_i^{(2)}) \in \{(j, l) \in \mathbb{Z}^2 : 0 \leq j, l \leq k-1\}$ are n distinct points in a $k \times k$ grid. If $\lambda \in (0, \frac{1}{k}]$ then it immediately follows that the attractor $\Lambda(\lambda)$ has dimension $-\frac{\log n}{\log \lambda}$. As was the case in the previous chapter we try to extend these results to a larger set of parameter values. We also look at the natural projection of invariant measures on Σ_n. In this case we determine whether the measure is typically singular or absolutely continuous and in the case where the measure is singular its Hausdorff dimension.

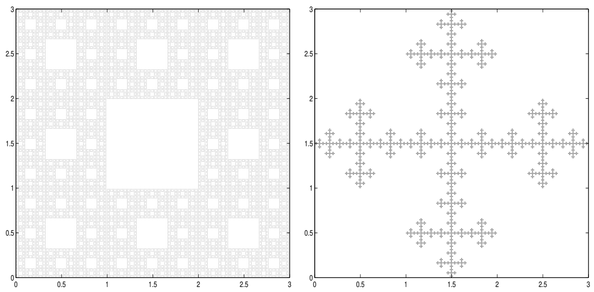

Figure 5.1: Examples of Sierpiński carpets with $\lambda = \frac{1}{k}$

5.2 Results

Our main results are similar in nature to those in the last chapter. However in this chapter we use the ideas on decomposition of measures and conditional entropy to find more general results.

Theorem 5.1. *There exists $\frac{1}{k} \leq s \leq \frac{1}{\sqrt{n}}$ such that for almost all $\lambda \in (\frac{1}{k}, s)$ we have*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log n}{\log \lambda}.$$

We can give an explicit estimate for s. We denote the number of images in the jth row by $n_j = \text{Card}\{0 \leq i \leq n-1 : c_i^{(2)} = j\}$. If we assume that each $n_j \geq 1$ then we can take

$$s = \min\left\{\frac{1}{n}\left(\prod_{j=1}^{k} n_j^{n_j}\right), \left(\prod_{j=1}^{k} n_j^{-n_j}\right)^{\frac{1}{n}}\right\}.$$

It should be noted that if all the values of n_j are the same then $s = \frac{1}{k}$ and 5.1 gives no new information. As was the case in the previous chapter we are able to prove a corollary which calculates the dimension for a large set in the topological sense.

Corollary 5.1. *There exists a residual set $G \subset [\frac{1}{k}, s]$ such that for any $\lambda \in G$*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log n}{\log \lambda}.$$

Proof. This can be proved using exactly the same method as Corollary 4.2. □

It follows immediately from Theorem 1.2 that $\dim_{\mathcal{H}} \Lambda(\lambda) \leq -\frac{\log n}{\log \lambda}$. For the sets which we consider. An argument analogous to the proof of Theorem 4.1 can show that there is a dense sets of values in $(\frac{1}{k}, \frac{1}{\sqrt{n}})$ where this inequality is strict.

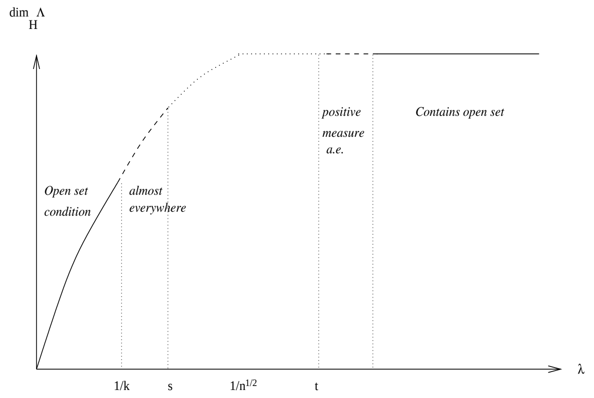

Figure 5.2: The dimension of a typical fat Sierpiński carpet. The line indicated is $min\left(-\frac{\log n}{\log \lambda}, 2\right)$. Where the line is fully region the dimension is always equal to this value, where it is dashed the dimension is almost surely equal to this value and the dotted region is where it just gives an upper bound.

The result we obtain concerning positive measure of the attractor is as follows.

Theorem 5.2. *There exists* $\frac{1}{\sqrt{n}} \leq t \leq y(k-1)$ *such that for almost all* $\lambda \in [t, y(k-1)]$ *we have that* $\mathcal{L}^2(\Lambda(\lambda)) > 0$.

We can give an explicit estimate for t by

$$t = \sup\left\{\prod_{j=1}^{k} n_j^{-q_j} : \sum_{j=1}^{k} q_j \log\left(\frac{q_j}{n_j}\right) = 0, \sum_{j=1}^{k} q_j = 1 \text{ and } q_j \geq 0\right\}.$$

It is possible that $t \geq y(k-1)$ in which case Theorem 5.2 gives no new information.

We obtain both Theorems 5.1 and 5.2 by finding stronger results about measures supported on fat Sierpiński carpets. Let μ be a shift invariant ergodic measure defined on Σ_n. We define the natural projection $\Pi_\lambda : \Sigma_n \to \Lambda(\lambda)$ by

$$\Pi_\lambda(\underline{i}) = \lim_{j\to\infty} T_{i_0} \circ \cdots \circ T_{i_{n-1}}(0,0) = \sum_{j=0}^{n} c_{i_j}\lambda^j.$$

Thus we can define a measure supported on $\Lambda(\lambda)$ by, $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. We also define a map $p : \Sigma_n \to \Sigma_k$ which corresponds to the projection of the carpets onto the y-axis. This is given by

$$p(i_0, i_1, \ldots) = (c_{i_0}^{(2)}, c_{i_1}^{(2)}, \ldots).$$

We define $\overline{\mu}$ by $\overline{\mu} = \mu \circ p^{-1}$.

We obtain the following results about the properties of ν_λ.

Theorem 5.3. *For almost all $\lambda \in \left[\frac{1}{k}, \frac{1}{\sqrt{n}}\right]$ we have that,*

$$\begin{aligned}
\dim_{\mathcal{H}}(\nu_\lambda) &= -\frac{h(\mu)}{\log \lambda} \textit{ if } \max\left\{-\frac{h(\overline{\mu})}{\log \lambda}, -\frac{h(\mu)-h(\overline{\mu})}{\log \lambda}\right\} \leq 1;\\
\dim_{\mathcal{H}}(\nu_\lambda) &\in \left[\min\left\{1-\frac{h(\overline{\mu})}{\log \lambda}, 1-\frac{h(\mu)-h(\overline{\mu})}{\log \lambda}\right\}, -\frac{h(\mu)}{\log \lambda}\right] \textit{ otherwise.}
\end{aligned}$$

Theorem 5.4. *For almost all λ in the set*

$$\left\{\lambda \in \left[\frac{1}{k}, y(k-1)\right] : \min\{h(\mu)-h(\overline{\mu}), h(\overline{\mu})\} \geq -\log \lambda\right\}$$

the measure ν_λ is absolutely continuous.

5.3 Examples

Before proving the main Theorems we give some examples of Fat Sierpiński carpets and the results we get in these individual cases. Our first example is the fat Vicsek set. Here we take $n = 5$ and $k = 3$ and $c_0 = (1,0), c_1 = (0,1), c_2 = (1,1), c_3 = (2,1), c_4 = (1,2)$. In Theorem 5.1 we can take $s = \left(\frac{3}{5}\right)^{\frac{3}{5}}\left(\frac{1}{5}\right)^{\frac{2}{5}} = 0.3866\ldots$. Thus we have that for almost all $\lambda \in [\frac{1}{3}, 0.3866]$ that

$$\dim_{\mathcal{H}}\Lambda(\lambda) = -\frac{\log 5}{\log n}.$$

In Theorem 5.2 we can take $t = 0.4541$ to give that for almost all $\lambda \in (0.4541, 0.5)$ we have $\mathcal{L}^2(\Lambda(\lambda)) > 0$.

Another example is the fat Sierpiński carpet. Here we take $n = 8$ and $k = 3$ and choose $c_0 = (0,0), c_1 = (0,1), c_2 = (0,2), c_3 = (1,0), c_4 = (1,2), c_5 = (2,0), c_6 = (2,1), c_7 = (2,2)$. In Theorem 5.1 we can take $s = 0.338\ldots$ and in Theorem 5.2 we can take t=0.357.

We use the fat Sierpiński gasket, studied in the previous chapter, to provide an example for Theorem's 5.3 and 5.4. In this case $n = 3, k = 2, c_0 = (0,0), c_1 = (0,1), c_2 = (1,0)$. Let μ be $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$-Bernoulli measure on Σ_3 and $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. Theorem's 5.1 and 5.2 can be used to obtain identical results to the previous chapter. However Theorem 5.3 can be used to give the following result about the Hausdorff

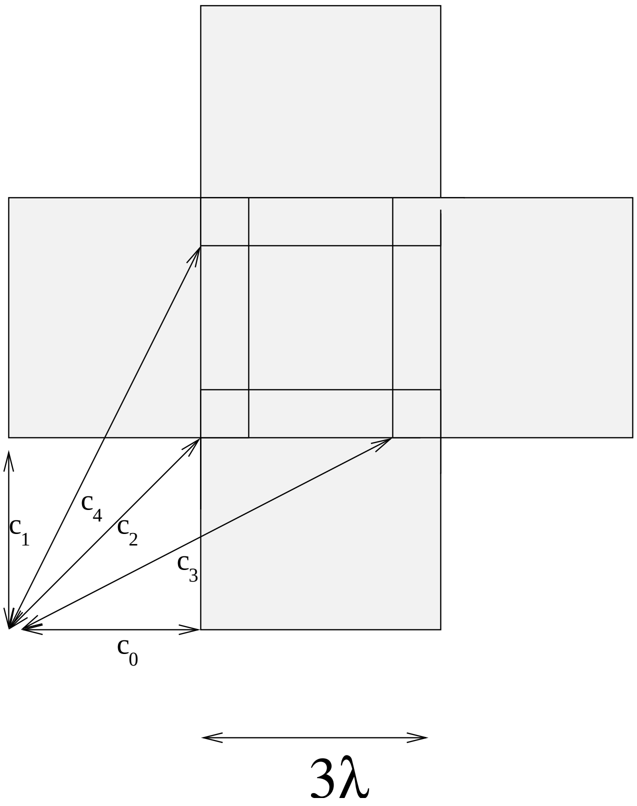

Figure 5.3: The first level construction of a fat Vicsek set

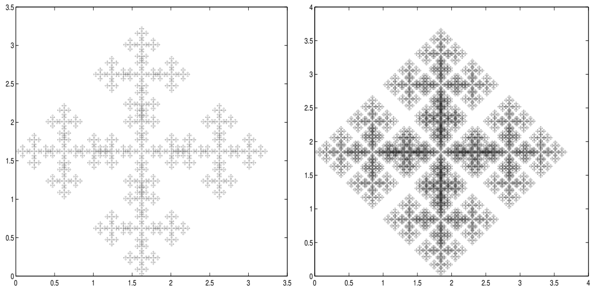

Figure 5.4: Examples of fat Vicsek sets

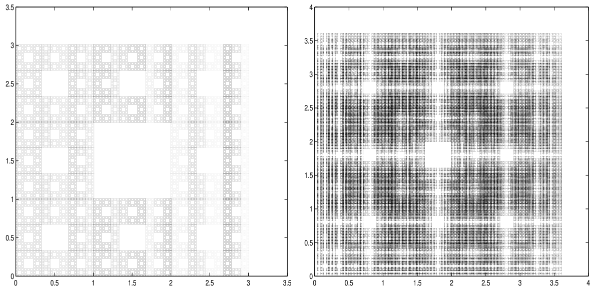

Figure 5.5: Examples of fat Sierpiński carpets

dimension of ν_λ

$$\begin{aligned}
\dim_{\mathcal{H}}\nu_\lambda &= -\frac{\log 3}{\log \lambda} \text{ for a.e. } \lambda \in [0.5, 0.529\ldots] \\
\dim_{\mathcal{H}}\nu_\lambda &\geq 1 - \frac{\log 4}{\log \lambda^3} \text{ for a.e. } \lambda \in [0.529\ldots, 2^{-2/3}].
\end{aligned}$$

Theorem 5.4 can be applied to show that ν_λ is absolutely continuous for a.e. $\lambda \in [2^{-\frac{2}{3}}, y(1) = 0.649\ldots]$. ν_λ can be thought of as a two dimensional analogue of a Bernoulli convolution.

Our final example is contrived to have a region of values of λ where the dimension is definitely not equal to $-\frac{\log n}{\log \lambda}$ for almost all λ. We take $k = 3$ and $n = 5$ and choose $c_0 = (0,0), c_1 = (1,0), c_2 = (2,0), c_3 = (0,2), c_4 = (2,2)$. In Theorem 5.1 we can take $s = 2^{-2/5} \times 3^{-3/5} = 0.3920\ldots$. However if we added to the iterated function system the map $T_5(x,y) = \lambda(x,y) + (1,2)$ the attractor would simply be the Cartesian product of an interval with the middle $(1-2\lambda)$ Cantor set and thus has dimension $1 - \frac{\log 2}{\log \lambda}$. The attractor of our original system must be contained inside this set and so the dimension must be bounded above by $1 - \frac{\log 2}{\log \lambda}$. For $\lambda > 0.4$ we have that $1 - \frac{\log 2}{\log \lambda} < -\frac{\log 5}{\log \lambda}$ and thus $\dim_{\mathcal{H}}\Lambda(\lambda) < -\frac{\log 5}{\log \lambda}$ for all $\lambda > 0.4$. In fact if we take μ to be $\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}\right)$-Bernoulli measure we can use Theorem 5.3 to show that $\dim_{\mathcal{H}}\nu_\lambda \geq 1 - \frac{\log 2}{\log \lambda}$ for almost all $\lambda \geq 0.4082$. This gives $\dim_{\mathcal{H}}\Lambda(\lambda) = 1 - \frac{\log 2}{\log \lambda}$ for almost all $\lambda \geq 0.4082$. It is not clear whether the other examples we have considered also have regions where the dimension drops below $-\frac{\log n}{\log \lambda}$ for a set of λ with positive measure.

5.4 Proofs of Theorems 5.3 and 5.1

In this section we prove Theorem 5.3 and use it to deduce Theorem 5.1. For $\xi \in \Sigma_k$ we define μ_ξ to be the conditional measure defined such that for any Borel set $A \subseteq \Sigma_n$

$$\mu(A) = \int_{\Sigma_k} \mu_\xi(A \cap p^{-1}\xi)\mathrm{d}\overline{\mu}(\xi).$$

Let $B(\Sigma_n)$ be the Borel sigma algebra for Σ_n and let $B(\Sigma_k)$ be the Borel sigma algebra for Σ_k. Let $\mathcal{A} = p^{-1}B(\Sigma_k) \subset B(\Sigma_n)$ be the corresponding σ-invariant sub-sigma algebra. That is the sigma algebra which does not distinguish between symbols in $\{0, 1, \ldots, n-1\}$ that project under p to the same symbol in Σ_k. Thus by Theorem 1.7 we have that for μ almost every $\underline{x} \in \Sigma_n$

$$\lim_{N\to\infty} -\frac{\log(\mu_\xi([x_0, \ldots, x_{N-1}]))}{N} = h(\mu|\mathcal{A}). \tag{5.1}$$

We define $\overline{\Pi}_\lambda : \Sigma_k \to \mathbb{R}$ by

$$\overline{\Pi}_\lambda(\underline{i}) = \sum_{j=0}^{\infty} c^{(2)}_{i_j}\lambda^j$$

and note that $\overline{\Pi}_\lambda$ maps sequences from Σ_k to points on the perpendicular projection of $\Lambda(\lambda)$ to the y-axis. For a sequence $\xi \in \Sigma_k$ we consider the set of points $\Pi_\lambda(p^{-1}\xi) \subset \Lambda(\lambda)$. If we let $\omega \in \Sigma_n$ satisfy $p\omega = \xi$ then we know that $c^{(2)}_{\omega_i} = \xi_i$. Thus if we consider

$$\Pi_\lambda(\omega) = \sum_{i=0}^{\infty}(c^{(1)}_{\omega_i}, c^{(2)}_{\omega_i})\lambda^i = \sum_{i=0}^{\infty}(c^{(1)}_{\omega_i}, c^{(2)}_{\xi_i})\lambda^i.$$

Thus the y-coordinate of $\Pi_\lambda(\omega)$ is equal to $\overline{\Pi_\lambda}(\xi)$. Thus any point in $(x, y) \in \Pi_\lambda(p^{-1}\xi)$ lies on the line $y = \overline{\Pi_\lambda}(\xi)$ which we denote $L_{\overline{\Pi_\lambda}(\xi)}$. We define two measures

$$\overline{\nu}_\lambda = \overline{\mu} \circ \overline{\Pi}_\lambda^{-1} \text{ and } \nu_{\xi,\lambda} = \mu_\xi \circ \Pi_\lambda^{-1}$$

and note that $\nu_{\xi,\lambda}$ is supported on $L_{\overline{\Pi_\lambda}(\xi)} \cap \Lambda(\lambda)$. This allows us to prove the following lemma which will be used to prove Theorem 5.3. The lemma is deduced from Marstrand's Slicing Theorem.

Lemma 5.1. *If for $\overline{\mu}$ almost all ξ we have that* $\dim_{\mathcal{H}}\nu_{\xi,\lambda} \geq s$ *then*

$$\dim_{\mathcal{H}}\nu_\lambda \geq \dim_{\mathcal{H}}\overline{\nu_\lambda} + s.$$

Proof. If for $\underline{\mu}$ almost all ξ $\dim_{\mathcal{H}}\nu_{\xi,\lambda} \geq s$ then for $\overline{\nu}_\lambda$ almost all y $\dim_{\mathcal{H}}\nu_{\lambda,\Pi_\lambda^{-1}y} \geq s$. Let $A \subseteq \mathbb{R}^2$ be a Borel set such that $\nu_\lambda(A) = 1$. It follows that $\mu(\Pi_\lambda^{-1}(A)) = 1$ and thus by the decomposition of μ,

$$\int \mu_\xi(\Pi_\lambda^{-1}A \cap p^{-1}\xi)\mathrm{d}\overline{\mu}(\xi).$$

Thus for $\overline{\mu}$ almost all ξ $\mu_\xi(\Pi_\lambda^{-1}(A) \cap p^{-1}\xi) = 1$ and hence $\nu_{\xi,\lambda}(A \cap L_{\Pi_{\lambda}(\xi)}) = 1$. However $\dim_{\mathcal{H}}\nu_{\xi,\lambda} \geq s$ for $\overline{\mu}$ almost all ξ; thus $\dim_{\mathcal{H}} A \cap L_{\Pi_\lambda(\xi)} \geq s$ for $\overline{\mu}$ almost all ξ. Equivalently $\dim_{\mathcal{H}} A \cap L_y \geq s$ for $\overline{\nu}_\lambda$ almost all y. Thus by Lemma 1.9 (Marstrand's Slicing Theorem) $\dim_{\mathcal{H}} A \geq s + \dim_{\mathcal{H}}\overline{\nu}_\lambda$. This holds for all Borel sets A where $\nu_\lambda(A) = 1$; hence $\dim_{\mathcal{H}}\nu_\lambda(A) \geq s + \dim_{\mathcal{H}}\overline{\nu}_\lambda$. □

$\overline{\nu}_\lambda$ is a measure on $\mathbb{R}$ the properties of which were studied in [42]. The following result was obtained.

Lemma 5.2 (Simon-Solomyak-Urbanski). *For almost all* $\lambda \in \left[\frac{1}{k}, y(k-1)\right]$ *we have that*

$$\dim_{\mathcal{H}}(\overline{\nu}_\lambda) = \min\left(1, -\frac{h(\overline{\mu})}{\log \lambda}\right).$$

Lemma 5.3. *For almost every* $\lambda \in [\frac{1}{k}, y(k-1)]$ *there exists a set* $Y \subset \mathbb{R}$ *with* $dim_H(Y) = dim_H(\nu_\lambda)$ *such that for any* $\xi \in (\overline{\Pi}_\lambda)^{-1}Y \subset \Sigma_k$ *we can bound*

$$dim_H(\nu_{\lambda,\xi}) \geq \min\left\{-\frac{h(\nu|\mathcal{A})}{\log \lambda}, 1\right\}.$$

Proof. It suffices to show that, given $\delta > 0$, for almost all $\lambda \in [\frac{1}{k}, y(k-1)]$ there exists a set $X = X_\delta \subset \Sigma_k$ with $\overline{\mu}(X) \geq 1-\delta$ such that for any $\xi \in X$, $\dim_{\mathcal{H}}(\nu_{\xi,\lambda}) \geq \frac{-h(\mu|\mathcal{A})}{\log \lambda}$. In particular, we can take $Y = \cap_{n=1}^{\infty} X_{\frac{1}{n}}$.

Fix $\epsilon, \epsilon' > 0$. By (5.1) and Ergorov's Theorem (Theorem 1.1) there exist sets $X_{\epsilon'} \subset \Sigma_k$ and a constant $K > 0$ such that $\overline{\mu}(X_{\epsilon'}) > 1-\epsilon'$ and for any $\underline{\xi} \in X_{\epsilon'}$ there exists $Y_{\epsilon'}$ such that for any $\underline{x} \in Y_{\epsilon'}$:

$$\mu_\xi[x_0, \ldots, x_N] \leq K \exp\left(-\left(h(\mu|\mathcal{A}) - \epsilon\right) N\right), \text{ for } N \geq 1. \tag{4.1}$$

Denote $s = -\frac{h(\mu|\mathcal{A})}{\log \lambda} - 2\epsilon$. We want to apply the potential theoretic method (Lemma 1.8)with the measure $\overline{\mu}$ restricted to $X_{\epsilon'}$ and the measure $\nu_{\xi,\lambda}$ restricted to

$\Pi_\lambda(Y_{\epsilon'})$, where $\xi \in X_{\epsilon'}$. This allows us to use the explicit bound (4.1). Consider the multiple integral

$$I = \int_{\frac{1}{k}}^{y(k-1)} \int_{X_{\epsilon'}} \left(\int_{\Pi_\lambda Y_{\epsilon'}} \int_{\Pi_\lambda Y_{\epsilon'}} \frac{\mathrm{d}\nu_{\xi,\lambda}(x)\mathrm{d}\nu_{\xi,\lambda}(y)}{|x-y|^s} \right) \mathrm{d}\overline{\mu}(\xi) d\lambda$$

We want to prove finiteness by lifting $\nu_{\xi,\lambda}$ to μ_ξ on $p^{-1}\xi$ and then using Fubini's Theorem to rewrite the integral as:

$$\begin{aligned} I &= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{y(k-1)} \frac{\mathrm{d}\lambda}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^s} \mathrm{d}\mu_\xi(\underline{i})\mathrm{d}\mu_\xi(\underline{j})\mathrm{d}\overline{\mu}(\xi) \\ &= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{y(k-1)} \frac{\mathrm{d}\lambda}{|\sum_{n=0}^{\infty}(i_n - j_n)\lambda^n|^s} \mathrm{d}\mu_\xi(\underline{i})\mathrm{d}\mu_\xi(\underline{j})\mathrm{d}\overline{\mu}(\xi) \\ &= \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{\frac{1}{k}}^{y(k-1)} \frac{\mathrm{d}\lambda}{e^{(h(\mu|\mathcal{A})-2\epsilon)|\underline{i}\wedge\underline{j}|}|\sum_{n=0}^{\infty} a_n\lambda^n|^s} \mathrm{d}\mu_\xi(\underline{i})\mathrm{d}\mu_\xi(\underline{j})\mathrm{d}\overline{\mu}(\xi) \end{aligned}$$

where we have that $a_n \in \{0, \pm 1, \ldots, \pm(k-1)\}$ and $a_0 \neq 0$. Thus we can use transversality (Theorem 2.1) to write

$$\begin{aligned} I &\leq C \int_{X_{\epsilon'}} \int_{Y_{\epsilon'}} \int_{Y_{\epsilon'}} e^{-(h(\mu|\mathcal{A})-2\epsilon)\underline{i}\wedge\underline{j}} \mathrm{d}\mu_\xi(\underline{i})\mathrm{d}\mu_\xi(\underline{j})\mathrm{d}\overline{\mu}(\xi) \\ &\leq C\sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})-2\epsilon)}(\mu_\xi \times \mu_\xi)\left(\{(\underline{i},\underline{j}) \in Y_{\epsilon'} \times Y_{\epsilon'} : i_a = j_a, 0 \leq a \leq m\}\right) \\ &\leq CK\sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})-2\epsilon)}e^{(h(\mu|\mathcal{A})+\epsilon)m} < +\infty. \end{aligned}$$

In particular, from this we deduce that for almost every $\lambda \in [\frac{1}{k}, y(k-1)]$, there is a set $Y = Y(\lambda) \subset \Pi_\lambda(X)$ of $\overline{\nu}$ measure $1-\epsilon'$ such that for $y \in Y$ one can choose $\xi \in \overline{\Pi}_\lambda^{-1}(y)$ such that

$$\int_{\Pi_\lambda Y_{\epsilon'}} \int_{\Pi_\lambda Y_{\epsilon'}} \frac{\mathrm{d}\nu_{\xi,\lambda}(x)\mathrm{d}\nu_{\xi,\lambda}(y)}{|x-y|^s} < +\infty.$$

Lemma 2.1 shows that $\dim_H(\nu_{\xi,\lambda}) \geq s$. Finally, since $\epsilon > 0$ was arbitrary, the result follows. □

By combining Lemmas 5.1, 5.2 and 5.3 we can see that for almost every $\lambda \in [\frac{1}{k}, y(k-1)]$

$$\dim_{\mathcal{H}}\nu_\lambda \geq \min\left\{-\frac{h(\nu|\mathcal{A})}{\log\lambda}, 1\right\} + \min\left(1, -\frac{h(\overline{\mu})}{\log\lambda}\right).$$

Thus if $-\frac{h(\nu|\mathcal{A})}{\log\lambda} < 1$ and $-\frac{h(\overline{\mu})}{\log\lambda} < 1$ we have that

$$\dim_{\mathcal{H}}\nu_\lambda \geq -\frac{h(\nu|\mathcal{A})}{\log\lambda} - \frac{h(\overline{\mu})}{\log\lambda}$$

for almost every $\lambda \in [\frac{1}{k}, y(k-1)]$. However Lemma 1.1 gives that

$$h(\mu) = h(\overline{\mu}) + h(\mu|\mathcal{A})$$

and thus for almost every $\lambda \in [\frac{1}{k}, y(k-1)]$ we have

$$\dim_{\mathcal{H}}\nu_\lambda \geq -\frac{h(\mu)}{\log\lambda}.$$

To complete the proof of Theorem 5.3 we note that Theorem 1.6 gives that $-\frac{h(\mu)}{\log\lambda}$ is also a uniform upper bound for $\dim_{\mathcal{H}}\nu_\lambda$.

Proof of Theorem 5.1

Let $\mu = (\frac{1}{n}, \ldots, \frac{1}{n})$-Bernoulli measure. Thus $h(\mu) = \log n$. We now determine where the conditions for part 1 of Theorem 5.3 are satisfied. We note that Theorem 2.1 gives $y(k-1) \geq (1+\sqrt{k-1})^{-1}$ and thus since $k < n$ we have that $y(k-1) \geq (1+\sqrt{k-1})^{-1} \geq n^{-\frac{1}{2}}$.

We now need to find conditions on λ such that $-\log\lambda \geq h(\overline{\mu})$ and $-\log\lambda \geq h(\mu|\mathcal{A})$. We can calculate

$$\begin{aligned} h(\overline{\mu}) &= -\sum_{i=0}^{k-1}\frac{n_i}{n}\log\left(\frac{n_i}{n}\right) \\ &= -\frac{1}{n}\sum_{i=0}^{k-1}(n_i\log n_i - n_i\log n) \\ &= -\frac{1}{n}\sum_{i=0}^{k-1}\log n_i^{n_i} + \log n \\ &= -\frac{1}{n}\left(\log\prod_{i=0}^{k-1}n_i^{n_i}\right) + \log n \\ &= -\log\left(\frac{\prod_{i=0}^{k-1}(n_i^{n_i})^{\frac{1}{n}}}{n}\right). \end{aligned}$$

We also get

$$h(\mu|\mathcal{A}) = \sum_{i=0}^{k-1}\frac{n_i}{n}\log n_i = \log\left(\prod_{i=0}^{k-1}n_i^{n_i}\right)^{\frac{1}{n}}.$$

Thus if we choose

$$s = \min\left\{\frac{1}{n}\left(\prod_{i=0}^{k-1} n_i^{n_i}\right)^{\frac{1}{n}}, \left(\prod_{i=0}^{k-1} n_i^{-n_i}\right)^{\frac{1}{n}}\right\}$$

then for almost every $\lambda \in [\frac{1}{k}, s]$ we have that

$$\dim_{\mathcal{H}}\nu \geq -\frac{h(\mu)}{\log\lambda} = -\frac{\log n}{\log\lambda}.$$

Hence for almost every $\lambda \in [\frac{1}{k}, s]$ we have that

$$\dim_{\mathcal{H}}\Lambda(\lambda) \geq -\frac{\log n}{\log\lambda}.$$

Theorem 1.2 shows that this is also a uniform upper bound.

5.5 Proofs of Theorems 5.2 and 5.4

As in the previous section we use the decomposition of measures. The following Lemma shows how we can find absolute continuity of ν_λ using these conditional measures.

Lemma 5.4. *If $\overline{\nu}_\lambda$ is absolutely continuous and $\nu_{\xi,\lambda}$ is absolutely continuous for $\overline{\mu}$ almost every ξ then ν_λ is absolutely continuous.*

Proof. Let $A \subset \mathbb{R}^2$ such that $\mathcal{L}^2(A) = 0$. We want to show that $\nu_\lambda(A) = 0$. Using the definition of ν_λ and the decomposition of μ we get that

$$\nu_\lambda(A) = \mu(\Pi_\lambda^{-1}A) = \int_{\Sigma_k} \mu_\xi(\Pi_\lambda^{-1}A \cap p^{-1}\xi)\mathrm{d}\overline{\mu}(\xi). \tag{5.2}$$

From the definition of $\nu_{\xi,\lambda}$ we have that

$$\mu_\xi(\Pi_\lambda^{-1}A \cap p^{-1}A) = \nu_{\xi,\lambda}(\Pi_\lambda^{-1}(\Pi_\lambda(A \cap p^{-1}\xi))).$$

Since $\mathcal{L}^2(A) = 0$ we know that the set

$$\{y : \mathcal{L}(L_y \cap A) \neq 0\}$$

has zero one dimensional Lebesgue measure. Thus from the absolute continuity of ν_λ we have

$$\overline{\mu}\{\xi : \mathcal{L}(L_{\Pi_\lambda\xi} \cap A) \neq 0\} = \overline{\nu}_\lambda\{y : \mathcal{L}(L_y \cap A) \neq 0\} = 0.$$

The fact that $\nu_{\xi,\lambda}$ is absolutely continuous for $\overline{\mu}$ almost all ξ now gives that $\nu_{\xi,\lambda}(\Pi_\lambda^{-1}(\Pi_\lambda A \cap p^{-1}\xi)) = 0$ for $\overline{\mu}$ almost all ξ. Thus by (5.2) we have that $\nu_\lambda(A) = 0$. $\square$

We now need to determine when the measures $\overline{\nu}_\lambda$ and $\nu_{\xi,\lambda}$ are absolutely continuous. The following result concerning $\overline{\nu}_\lambda$ can be found in [42].

Lemma 5.5 (Simon-Solomyak-Urbanski). *For almost all $\lambda \in [e^{-h(\overline{\mu})}, y(k-1)]$ $\overline{\nu}_\lambda$ is absolutely continuous with respect to one dimensional Lebesgue measure.*

It is possible that $e^{-h(\overline{\mu})} > y(k-1)$. In this case Lemma 5.5 does not give any new information. We now prove a result about the absolute continuity of measures supported on the fibre.

Lemma 5.6. *For almost all λ in*

$$\left\{\lambda \in \left[\frac{1}{k}, y(k-1)\right] \ : h(\mu|\mathcal{A}) > -\log\lambda\right\}$$

there exists a set $X \subseteq \Sigma_k$ such that $\overline{\mu}(X) = 1$ and for any $\xi \in X$ the measure $\nu_{\lambda,\epsilon}$ is absolutely continuous on $L_{\overline{\Pi}_\lambda(\xi)}$.

Proof. It suffices to show that given $\epsilon' > 0$, there exists a set $X_{\epsilon'} \subseteq \Sigma_k$ such that $\overline{\mu}(X_{\epsilon'}) \geq 1-\epsilon'$ and for any $\xi \in X_{\epsilon'}$ there exists a set $Y_{\epsilon',\xi} \subset L_{\overline{\Pi}_\lambda(\xi)}$ where $\mu_\xi(Y_\epsilon') \geq 1-\epsilon'$ and $\nu_{\lambda,\epsilon}$ is absolutely continuous on $Y_{\epsilon',\xi}$. We can then take $X = \cap_{N=1}^{\infty} X_{\frac{1}{N}}$.

Let $\epsilon, \epsilon' > 0$. From Theorem 1.7 and Ergorov's Theorem (Theorem 1.1) we know that there exists $K > 0$ and a set $X_{\epsilon'} \subseteq \Sigma_k$ such that $\overline{\mu}(X_{\epsilon'}) > 1-\epsilon'$ and for $\xi \in X_{\epsilon'}$ there exists $Y_{\epsilon',\xi} \subseteq p^{-1}\xi$ with $\mu_\xi(Y_{\epsilon',\xi}) > 1-\epsilon'$ and for $\underline{x} \in Y_{\epsilon',\xi}$ we have that

$$\mu_\xi[x_0,\ldots,x_{N-1}] \leq K\exp\left(-\left(h(\mu|\mathcal{A})-\epsilon\right)N\right), \text{ for } N \geq 1. \tag{5.3}$$

We recall that by Lemma 1.1 to show that ν_ξ is absolutely continuous it suffices to show that $\underline{D}(\nu_\xi)(x)$ is finite, for a.e.$(\nu_{\xi,\lambda})$ $x \in \Pi_\lambda Y_{\epsilon',\xi}$. In particular, it suffices to show that

$$\int_{\Pi_\lambda Y_{\epsilon',\xi}} \underline{D}(\nu_{\xi,\lambda})(x)\mathrm{d}\nu_{\xi,\lambda}(x) < +\infty.$$

Moreover, to show that for almost every λ there exists a set of ξ of $\overline{\mu}$ measure at least $1-\epsilon'$ such that $\nu_{\xi,\lambda}$ is absolutely continuous, it suffices to show that

$$I := \int_t^{y(k-1)} \int_{X_{\epsilon'}} \left(\int_{\Pi_\lambda Y_{\epsilon',\xi}} \underline{D}(\nu_{\xi,\lambda})(x) \mathrm{d}\nu_{\xi,\lambda}(x) \right) \mathrm{d}\overline{\mu}(\xi) \mathrm{d}\lambda < +\infty,$$

providing t is sufficiently large. We take $t > e^{h(\mu|\mathcal{A})+2\epsilon}$. For $\omega, \tau \in p^{-1}\xi$ we define

$$\Psi_r = \{\omega, \tau : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\}$$

and

$$\phi_r(\omega,\tau) = \{\lambda : |\Pi_\lambda(\omega) - \Pi_\lambda(\tau)| \leq r\},$$

for $r > 0$ We start by lifting to the shift space, applying Fatou's Lemma and Fubini's Theorem

$$\begin{aligned} I &\leq \liminf_{r\to 0} \frac{1}{2r} \int_t^{y(k-1)} \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} \mathbb{1}(\Psi_r) \mathrm{d}\mu_\xi(\omega) \mathrm{d}\mu_\xi(\tau) \mathrm{d}\overline{\mu}(\xi) \mathrm{d}\lambda \\ &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} \mathcal{L}(\phi_r(\omega,\tau)) \mathrm{d}\mu_\xi(\omega) \mathrm{d}\mu_\xi(\tau) \mathrm{d}\overline{\mu}(\xi). \end{aligned}$$

We can apply Theorem 2.1 and then use inequality 5.3.

$$\begin{aligned} I &\leq C \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} t^{-|\omega\wedge\tau|} \mathrm{d}\mu_\xi(\omega) \mathrm{d}\mu_\xi(\tau) \mathrm{d}\overline{\mu}(\xi) \\ &= C \int_{X_{\epsilon'}} \int_{Y_{\epsilon',\xi}} \int_{Y_{\epsilon',\xi}} e^{-|\omega\wedge\tau|(h(\mu|\mathcal{A})+2\epsilon)} \mathrm{d}\mu_\xi(\omega) \mathrm{d}\mu_\xi(\tau) \mathrm{d}\overline{\mu}(\xi) \\ &= C \int_{X_{\epsilon'}} \sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})+2\epsilon)} (\mu_\xi \times \mu_\xi)(\Delta_m) \mathrm{d}\overline{\mu}(\xi) \\ &\leq CK \sum_{m=0}^{\infty} e^{-m(h(\mu|\mathcal{A})+2\epsilon)} e^{m(h(\mu|\mathcal{A})+\epsilon)} < \infty, \end{aligned}$$

where $\Delta_m = \{(\tau,\omega) \in Y_{\epsilon',\xi} \times Y_{\epsilon',\xi} : \omega_1 = \tau_1, \ldots, \omega_m = \tau_m\}$. This completes the proof. □

Combining Lemma's 5.4, 5.5 and 5.6 completes the proof of Theorem 5.4.

Proof of Theorem 5.2

Let $\underline{q} = (q_0, \ldots, q_{k-1})$ be a probability vector. Let $p_i = \frac{q_{c_i^{(2)}}}{n_{c_i^{(2)}}}$ and μ be $\underline{p}$-Bernoulli measure on Σ_n. If we take, as usual, $\overline{\mu} = p(\underline{\mu})$ we have that

$$h(\overline{\mu}) = \sum_{i=0}^{k-1} p_i \log p_i \text{ and } h(\mu|\mathcal{A}) = \sum_{i=0}^{k-1} q_i \log n_i.$$

If we let

$$t = \sup\left\{\prod_{j=0}^{k-1} n_j^{-q_j} : \sum_{j=1}^{k} q_j \log\left(\frac{q_j}{n_j}\right) = 0, \sum_{j=0}^{k-1} = 1 \text{ and each } q_j \geq 0\right\}$$

and for $\epsilon > 0$ let $\underline{q}$ satisfy $\sum_{j=0}^{k-1} n_j^{-q_j} \geq t - \epsilon$ then for any $\lambda \geq t - \epsilon$ we have that $-\log\lambda \leq h(\overline{\mu}) = h(\mu|\mathcal{A})$. Thus by 5.4 for almost every $\lambda \geq t - \epsilon$ the measure ν_λ is absolutely continuous and hence $\mathcal{L}(\Lambda(\lambda)) > 0$. The proof is completed by letting $\epsilon \to 0$.

5.6 Higher Dimensions and Self-affine Sets

The results in this chapter can be generalised without difficulty to a higher dimensional setting. We consider two such constructions in $\mathbb{R}^3$. Firstly we consider the Sierpiński tetrahedron. This consists of the following four similarities.

$$T_0(x,y,z) = \lambda(x,y,z) + (0,0,0)$$

$$T_1(x,y,z) = \lambda(x,y,z) + (1,0,0)$$

$$T_2(x,y,z) = \lambda(x,y,z) + (0,1,0)$$

$$T_3(x,y,z) = \lambda(x,y,z) + (0,0,1).$$

In the case where $\lambda = \frac{1}{2}$ this iterated function system would satisfy the open set condition and the attractor, $\Lambda(\lambda)$ would have dimension $\frac{\log 4}{\log 2} = 2$. We consider the case when $\lambda > \frac{1}{2}$. Let μ be the evenly distributed Bernoulli measure on Σ_4 and ν_λ the natural projection of μ to $\Lambda(\lambda)$. We can define a map $p : \Sigma_4 \to \Sigma_3$ which maps symbols $0,1,2$ to themselves but maps 3 to 0. If we let $\overline{\mu} = p\mu$ and project it onto the tetrahedron as $\overline{\nu}_\lambda$ then we can see that it is supported on the perpendicular projection to the (x,y)-plane. This would be a Sierpiński gasket. We can then define a set of conditional measures which would be supported on vertical fibres. Theorem 5.3 gives that for almost all $\lambda \in \left[\frac{1}{2}, 0.569\ldots\right]$ that $\dim_{\mathcal{H}}\nu_\lambda = -\frac{\log(h(\overline{\mu}))}{\log\lambda}$. We can also show that the conditional measures will for almost every $\lambda \in [0.5, y(1)]$ and for $\overline{\mu}$-almost every $\xi \in \Sigma_3$ have dimension $-\frac{h(\mu)-h(\overline{\mu})}{\log\lambda}$. Thus by using Marstrand's Slicing Theorem we can see that for almost every $\lambda \in [0.5, 0.569\ldots]$ we have $\dim_{\mathcal{H}}\nu_\lambda \geq -\frac{\log 4}{\log \lambda}$. This immediately gives

$$\dim_{\mathcal{H}}\Lambda(\lambda) = -\frac{\log 4}{\log \lambda}$$

for almost every $\lambda \in [0.5.0.569\ldots]$.

The Menger sponge is another example of a self-similar set in $\mathbb{R}^3$. In the standard case it consist of 20 contractions of ratio $\frac{1}{3}$. The values of c_i consist of all triples of $(x, y, z) \in \{0, 1, 2\}^3$ where at most one of x, y of z takes the value 1. The perpendicular projection of the attractor to any of the (x, y)-plane, the (x, z)-plane and the (y, z)-plane is the standard Sierpiński carpet. If we consider the case where the contraction ratio λ are bigger than $\frac{1}{3}$ we have that $\dim_{\mathcal{H}}\Lambda(\lambda) = -\frac{\log 20}{\log \lambda}$ for almost all $\lambda \leq 0.348$ and that $\Lambda(\lambda)$ has positive measure for almost every $\lambda \geq 0.393$.

We can also obtain results about some overlapping self-affine fractals in $\mathbb{R}^2$. Let $m > k > 2$ and write $\beta = \frac{\log k}{\log m}$. We consider n affine contractions $\{T_0, \ldots, T_{n-1}\}$: $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T_i(x, y) = (\lambda^{\frac{1}{\beta}} x, \lambda y) + c_i$$

where $c_i \in \{0, \ldots, m-1\} \times \{0, \ldots, k-1\}$. In the case where $\lambda = \frac{1}{m}$ these are exactly the self-affine maps considered by Bedford and McMullen. The Hausdorff and Box counting dimensions of the attractor $\Lambda(\lambda)$ are given by Theorem 1.3. We wish to calculate the Hausdorff dimension of $\Lambda(\lambda)$ for larger values of λ where the images overlap. For $0 \leq j \leq k-1$ we define

$$n_j = \mathrm{Card}\{c_i : c_i^{(2)} = j\}.$$

We obtain the following result about the Hausdorff dimension of the attractor.

Theorem 5.5. *There exists s such that for almost every $\lambda \in \left[\frac{1}{m}, s\right]$ we have that*

$$\dim_{\mathcal{H}}\Lambda(\lambda) \geq -\frac{\log\left(\sum_{i=0}^{k-1} n_i^{\beta}\right)}{\log \lambda}.$$

As was the case with the fat Sierpiński carpets there will be cases where $s = \frac{1}{m}$ and hence Theorem 5.5 gives no new information. However in most cases this will not be the case. An explicit value for s will be given in the proof.

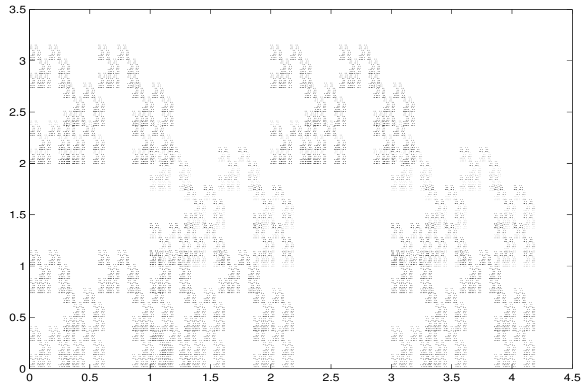

Figure 5.6: An overlapping self-affine set of the type studied in Theorem 5.5

Proof of Theorem 5.5

Let $\Pi_\lambda : \Sigma_n \to \Lambda(\lambda)$ be given by

$$\Pi_\lambda(\underline{i}) = \sum_{j=0}^{\infty} c_{i_j} \lambda^j.$$

We then let μ be a shift invariant ergodic measure on Σ_n. As before we define $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. We define $p : \Sigma_n \to \Sigma_m$ by

$$p(j_0, j_1, \ldots) = (c_{j_0}^{(2)}, c_{j_0}^{(3)}).$$

Now let $\overline{\mu} = \mu \circ p^{-1}$. Once again we can use the Rohlin decomposition of measures. We define a family of measures μ_ξ where $\xi \in \Sigma_m$ and for all Borel subsets $A \subset \Sigma_n$

$$\mu(A) = \int_{\Sigma_m} \mu_\xi(A \cap p^{-1}\xi) \mathrm{d}\overline{\mu}(\xi).$$

Projections $\overline{\Pi_\lambda} : \Sigma_m \to \mathbb{R}$ and $\Pi_{\xi,\lambda} : p^{-1}\xi \to \mathbb{R}$ are given by

$$\overline{\Pi}_\lambda(\xi) = \sum_{j=0}^{\infty} \xi_j \lambda^j \text{ and } \Pi_{\xi,\lambda}(\omega) = \sum_{j=0}^{\infty} \omega_j \lambda^{\beta j}.$$

By the definition of p we have that if $\omega \in p^{-1}\xi$ then for all $j \in \mathbb{N}_0$ there exists c_i such that $(\omega_j, p_j) = \omega_j$. Thus

$$(\Pi_{\xi,\lambda}(\omega), \overline{\Pi}_\lambda(\xi)) \in \Lambda(\lambda).$$

Let

$$\overline{\nu}_\lambda = \overline{\mu} \circ \overline{\Pi}_\lambda^{-1} \text{ and } \nu_{\xi,\lambda} = \mu_\xi \circ \Pi_{\xi,\lambda}$$

be measures defined on $\mathbb{R}$. Using exactly the same methods as for the Sierpiński carpets we can obtain the following Lemma.

Lemma 5.7.

1. *For almost all* $\lambda \in \left[\frac{1}{k}, y(m-1)^{\frac{1}{\beta}}\right]$ *we have that*

$$\dim_{\mathcal{H}} \overline{\nu}_\lambda \geq \min\left\{1, -\frac{h(\overline{\mu})}{\log \lambda}\right\}.$$

2. *For almost all* $\lambda \in \left[\frac{1}{k}, y(k-1)\right]$ *we have that for* $\overline{\mu}$ *almost all* ξ

$$\dim_{\mathcal{H}} \nu_{\xi,\lambda} \geq -\frac{h(\mu) - h(\overline{\mu})}{\frac{1}{\beta}\log \lambda}.$$

3. *For almost all* $\lambda \in \left[\frac{1}{k}, \min\left\{y(k-1), e^{-h(\overline{\mu})}, e^{h(\overline{\mu})-h(\mu)}\right\}\right]$ *we have*

$$\dim_{\mathcal{H}} \nu_\lambda \geq -\frac{e^{h(\overline{\mu})}}{\log \lambda} - \frac{e^{h(\mu)-h(\overline{\mu})}}{\frac{1}{\beta}\log \lambda}.$$

To complete the proof of the almost sure lower bound we let $p_i = \frac{n^\beta_{c_i^{(2)}}}{n_{c_i^{(2)}} \sum_{j=0}^{m-1} n_j^\beta}$ for $i = 0, \ldots, n_i$. We then let μ be $(p_0, \ldots, p_{m-1})$-Bernoulli measure on Σ_m. If we let

$$s = \min\{y(m-1), e^{-h(\overline{\mu})}, e^{h(\overline{\mu})-h(\mu)}\}$$

then for almost every $\lambda \in [\frac{1}{m}, s]$ we have that

$$\dim_{\mathcal{H}} \nu_\lambda \geq -\frac{h(\overline{\mu})}{\log \lambda} + \frac{h(\overline{\mu}) - h(\mu)}{\frac{1}{\beta}\log \lambda}.$$

However if we let $q_i = \frac{n_i^\beta}{\sum_{j=0}^{m-1} n_j^\beta}$ for $i = 0, \ldots, m-1$ we can calculate,

$$h(\overline{\mu}) = \log \prod_{j=0}^{m-1} q_j^{q_j} \text{ and } h(\mu) - h(\overline{\mu}) = \log \prod_{j=0}^{m-1} n_j^{q_j}.$$

Thus for almost every $\lambda \in [\frac{1}{k}, s]$

$$\begin{aligned}
\dim_{\mathcal{H}}\Lambda(\lambda) &\geq -\frac{h(\overline{\mu})}{\log\lambda} + \frac{h(\overline{\mu}) - h(\mu)}{\frac{1}{\beta}\log\lambda} \\
&= \frac{\log\prod_{j=0}^{m-1} q_j^{q_j} - \beta\log\prod_{j=0}^{m-1} n_j^{q_j}}{\log\lambda} \\
&= \frac{\log\prod_{j=0}^{m-1}\left(\frac{n_j^{\beta}}{\sum_{i=0}^{m-1} n_j^{\beta}}\right)^{q_j} - \beta\log\prod_{j=0}^{m-1} n_j^{q_j}}{\log\lambda} \\
&= \frac{\log\left(\frac{1}{\sum_{j=0}^{m-1} n_j^{\beta}}\right)}{\log\lambda} + \frac{\log\prod_{j=0}^{m-1} n_j^{\beta q_j} - \log\prod_{j=0}^{m-1} n_j^{\beta q_j}}{\log\lambda} \\
&= -\frac{\log\left(\sum_{j=0}^{m-1} n_j^{\beta}\right)}{\log\lambda}.
\end{aligned}$$

This completes the proof.

5.7 Non-linear Contractions

In this section we show how the results can be generalised to certain specific families of non-linear contractions. Let $T_{ij}^{(\lambda,\phi)} : \mathbb{R}^2 \to \mathbb{R}^2$, where $0 \leq i \leq k-1$, be defined by

$$T_{i,j}(x,y) = (f_i^{(\lambda)}(x), g_{i,j}^{(\phi)}(y))$$

where $f_i^{(\lambda)} : [0,1] \to [0,1]$ and $g_{i,j}^{(\phi)} : [0,1] \to [0,1]$ are $C^{1+\alpha}$ maps parameterized by some λ and ϕ respectively. λ and ϕ will be from $\mathbb{R}^m$ for some m; when Lebesgue measure is referred to it will be m dimensional. Suppose that there are n different maps and let Σ_n be the space of sequences with symbols (i,j) and $\sigma : \Sigma_n \to \Sigma_n$ the usual shift map. Let $\Pi_{\lambda,\phi} : \Sigma_n \to \mathbb{R}^2$ be defined by

$$\Pi_{\lambda,\phi}(\underline{x}) = \lim_{m\to\infty} T_{x_0} \circ \cdots \circ T_{x_{m-1}}(0,0).$$

This is the natural projection from the shift space to the attractor of the iterated function system, $\Lambda(\lambda,\phi)$. Let $\overline{\Pi}_\lambda : \Sigma_k \to \mathbb{R}$ be defined by

$$\overline{\Pi}_\lambda(\underline{x}) = \lim_{m\to\infty} f_{x_0} \circ \cdots \circ f_{x_{m-1}}(0).$$

We define $p : \Sigma_n \to \Sigma_k$ by mapping each element (i,j) of a sequence in Σ_n to j. Let μ be an ergodic shift invariant map measure on Σ_n and let $\overline{\mu} = p\mu$. We then decompose μ so that for any Borel set $A \subset \Sigma_n$

$$\mu(A) = \int_{\Sigma_k} \mu_\xi(A) \mathrm{d}\overline{\mu}(\xi).$$

Let $\Pi_{\phi,\xi}$ be the restriction of $\Pi_{\lambda,\phi}$ to $p^{-1}\xi$. We define measures by

$$\nu_{\lambda,\phi} = \mu \circ \Pi_\lambda, \overline{\nu}_\lambda = \overline{\mu} \circ \overline{\Pi}_\lambda^{-1}, \nu_{\phi,\xi} = \mu_\xi \circ \Pi_{\phi,\xi}.$$

Note that $\nu_{\phi,\xi}$ is entirely supported on a vertical line with x-coordinate $\overline{\Pi}_\lambda(\xi)$. In this setting we need to define Lyapunov exponents both on the projection and along the fibres. These are the analogues of the contraction rates in the linear cases. Let

$$\chi_1 = \int_{\Sigma_k} \log|f'_{x_0}| \circ \overline{\Pi}_\lambda \mathrm{d}\overline{\mu}(x) \text{ and } \chi_2 = \int_{\Sigma_k}\int_{p^{-1}\xi} \log|g'_{\xi_0,x_0}| \circ \Pi_{\lambda,\phi} \mathrm{d}\mu_\xi(\underline{x}) \mathrm{d}\overline{\mu}(\xi).$$

To use these Lyapunov exponents we need the following two Lemmas.

Lemma 5.8.

1. *For $\overline{\mu}$ almost all $\underline{x}$ we have that*

$$\lim_{m\to\infty} \frac{1}{m} \log \operatorname{diam}(\overline{\Pi}_\lambda([x_0,\ldots,x_{m-1}])) = \chi_1$$

2. *For $\overline{\mu}$ almost all ξ for μ_ξ almost all x*

$$\lim_{m\to\infty} \frac{1}{m} \log \operatorname{diam}(\Pi_{\phi,\xi}([x_0,\ldots,x_{m-1}])) = \chi_2.$$

Proof.

1. By the Mean Value Theorem for all $\underline{x} \in \Sigma_k$ there exists $y \in [0,1]$ such that

$$\operatorname{diam}(\overline{\Pi}_\lambda([x_0,\ldots,x_{n-1}])) = f_{x_0} \circ \cdots \circ f'_{x_{m-1}}(y).$$

   By the Hölder continuity of each f'_i and the chain rule there exists a constant $C > 0$ such that

$$\frac{1}{m-1} \log|(f_{x_0} \circ \cdots \circ f_{x_{m-1}})'(y)| = \frac{1}{m-1} \left| \log C + \sum_{l=0}^{m-1} \log f'_{x_l}(f_{x_{l+1}} \circ \cdots \circ f_{i_k}(0)) \right|. \tag{5.4}$$

If we let $h : \Sigma_k \to \mathbb{R}$ be defined by $h(\underline{i}) = \log f'_{i=0} \circ \overline{\Pi}_\lambda(\sigma(\underline{i}))$ then we can rewrite the right hand side of 5.4 as

$$\frac{1}{m-1}\left|\log C + \sum_{l=0}^{m-1} h(\sigma^l(\underline{x}))\right|.$$

The proof can then be completed by the Birkhoff Ergodic Theorem.

2. A similar method can be used to prove this part.

□

We need the following two transversality conditions to be satisfied. There exists a constant $C_1 > 0$ such that for $\omega, \tau \in \Sigma_k$ with $\omega_0 \neq \tau_0$:

$$\mathcal{L}\{\lambda : |\overline{\Pi}_\lambda(\omega) - \overline{\Pi}_\lambda(\tau)| \leq r\} \leq C_1 r$$

There exists a constant C_2 so that for $\omega, \tau \in p^{-1}\xi$ where $\omega_0 \neq \tau_0$ we have

$$\mathcal{L}\{\phi : |\Pi_{\phi,\xi}(\omega) - \Pi_{\phi,\xi}(\tau)| \leq r\} \leq C_2 r.$$

Let $A_{\lambda,\phi}$ be the set where both transversality conditions are satisfied. The following results hold when $A_{\lambda,\phi}$ has positive measure. A is the Borel sigma algebra defined by $p^{-1}B(\Sigma_n)$.

Proposition 5.1. *For almost all $(\lambda, \phi) \in A_{\lambda,\phi}$ we have*

$$\begin{aligned} \dim_{\mathcal{H}}(\nu_{\lambda,\phi}) &\geq -\left(\frac{h(\overline{\mu})}{\chi_1} + \frac{h(\mu|A)}{\chi_2}\right) \text{ if } \max\left\{-\frac{h(\overline{\mu})}{\chi_1}, -\frac{h(\mu|A)}{\chi_2}\right\} \leq 1; \text{ and} \\ \dim_{\mathcal{H}}(\nu_{\lambda,\phi}) &\geq 1 + \min\left\{-\frac{h(\overline{\mu})}{\chi_1}, -\frac{h(\mu|A)}{\chi_2}\right\} \text{ otherwise.} \end{aligned}$$

Proposition 5.2. *For almost all (λ, ϕ) in the set*

$$\left\{(\lambda, \phi) \in A_{\lambda,\phi} : \min\left\{-\frac{h(\mu)}{\chi_1}, -\frac{h(\mu|A)}{\chi_2}\right\} \geq 1\right\}$$

$\nu_{\lambda,\phi}$ is absolutely continuous.

Both of these Theorem can be proved using the same transversality techniques as used earlier in this chapter. Details of the slight changes needed to deal with the non-linear case can be found in [42] and [14].

For example to show that $\dim_{\mathcal{H}}\nu_{\phi,\xi} \geq -\frac{h(\mu|A)}{\chi_2}$ we need to use Ergorov's Theorem in combination with Lemma 5.8 and Theorem 1.7. This means that for any $\epsilon, \delta > 0$ there exists a set $Y \subset \Sigma_n$ such that $\overline{\mu}(Y) > 1 - \epsilon$ and for any $\xi \in Y$ there exists $X_\xi \subset p^{-1}\xi$ where $\mu_\xi(X_\xi) \geq 1-\delta$ and there also exists $M \in \mathbb{N}$ such that for all $m \geq M$ and $\underline{x} \in X_\xi$

$$\mu_\xi[x_0, \ldots, x_{m-1}] \leq m \exp(-h(\mu|A) - \epsilon)$$

and

$$|\mathrm{diam}(\Pi_{\lambda,\phi}[x_0, \ldots, x_{m-1}]) - m\chi_2| \leq \epsilon.$$

We can now follow the same steps as in Lemma 5.3.

Families where Proposition 5.1 can be Applied

We now find a family of iterated function systems on $\mathbb{R}^2$ where Proposition 5.1 can be applied. Let $f, g : [0,1] \to [0,1]$ be $C^{1+\alpha}$ contractions where $||F'||_\infty, ||g'||_\infty < \frac{1}{2}$. Let $n > k > m$ $(\tau_0, \ldots, \tau_{k-1}) \in \mathbb{R}^k$ and $(\omega_0, \ldots, \omega_{m-1}) \in \mathbb{R}^m$. Our iterated function system on $\mathbb{R}^2$ consists of n functions of the form

$$T_N(x,y) = (f(x) + \tau_i, g(y) + \omega_j)$$

for some $0 \leq i \leq k-1$ and $0 \leq j \leq m-1$. For each $0 \leq i \leq k-1$ there must be a $0 \leq N \leq n-1$ such that the translation in the x-coordinate of T_N is by τ_i and similarly for each $0 \leq j \leq m-1$ there must be a T_N where the translation in the y-coordinate is by ω_j.

Let μ be an ergodic, invariant measure on Σ_n and $\overline{\mu}$ the natural projection to Σ_k. Proposition 1 in [14] shows that the necessary transversality conditions hold in this case. We use the same notation for the natural projections as previously and let $\nu_{\tau,\omega}$ be the natural projection of μ to the attractor of the IFS. In this case it is clear that

$$\chi_1 = \int \log f'(\overline{\Pi}_\tau(\underline{x}))d\overline{\mu}(\underline{x}) \text{ and } \chi_2 = \int\int \log g'(\Pi_{\omega,\xi}(\underline{x}))d\mu_\xi(\underline{x})d\overline{\mu}(\xi).$$

Proposition 5.1 gives that if $-\chi_1 \geq h(\overline{\mu})$ and $-\chi_2 \geq h(\mu) - h(\overline{\mu})$ then for almost every τ and almost every ω

$$\dim_{\mathcal{H}}\nu_{\omega,\tau} \geq -\frac{h(\overline{\mu})}{\chi_1} - \frac{h(\mu) - h(\overline{\mu})}{\chi_2}.$$

Proposition 5.2 gives that if $-\chi_1 \leq h(\overline{\mu})$ and $-\chi_2 \leq h(\mu) - h(\overline{\mu})$ then for almost every τ and almost every ω, $\nu_{\tau,\omega}$ is absolutely continuous with respect to $\mathcal{L}^2$.

Chapter 6

Sets with Positive Measure but no Interior

6.1 Motivation

In this section we look at iterated function systems where we can show that the attractors have no interior. We then use similar methods to the previous two chapters to try and determine when the attractors have positive measure. This was motivated by a question asked by Peres and Solomyak in [37].

Question 6.1. *Is it possible to find examples of self-similar sets in* $\mathbb{R}^n$ *which have positive measure but no interior?*

We show that in various settings it is possible for an iterated function systems to have an attractor with such properties.

6.2 Self-affine Sets

In this section we show that it is possible to find self-affine sets with positive measure and no interior. We define $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_i(x, y) = \left(\lambda x, \frac{1}{3}y\right) + c_i$$

for $i = 0, 1, \dots, 7$ and $c_0 = (0,0), c_1 = (\frac{1}{3}, 0), c_2 = (\frac{2}{3}, 0), c_3 = (0, \frac{1}{3}), c_4 = (\frac{2}{3}, \frac{1}{3}), c_5 = (0, \frac{2}{3}), c_6 = (\frac{1}{3}, \frac{2}{3}), c_7 = (\frac{2}{3}, \frac{2}{3})$. We let $\Lambda(\lambda)$ the self-affine set which is the attractor for the above iterated function system.

Theorem 6.1. *For almost all $\lambda \in (0.3816\dots, 0.5)$ we have that $\Lambda(\lambda)$ has no interior and positive measure.*

We start by showing the following simple Lemma.

Lemma 6.1. *$\Lambda(\lambda)$ has no interior for any $\lambda < \frac{1}{2}$.*

Proof. Let $\overline{\Pi}_\lambda : \Sigma_3 \to [0,1]$ be defined by

$$\overline{\Pi}_\lambda(\underline{i}) = \sum_{n=0}^{\infty} i_n \left(\frac{1}{3}\right)^{n+1}.$$

Consider the set of sequences $X \subset \Sigma_3$ where

$$X = \{\underline{i} \in \Sigma_3 : \text{ there exists } N \in \mathbb{N} \text{ where } i_n = 1 \text{ for any } n \geq N\}.$$

It is immediately clear that $\overline{\Pi}_\lambda(X)$ is a dense subset of $[0,1]$. Consider the points $L_y \cap \Lambda(\lambda)$ where $y \in \overline{\Pi}_\lambda(X)$. It follows from the definition that $\overline{\Pi}_\lambda^{-1}(y)$ is a single point in X. Let $\underline{i} \in X$ satisfy $\overline{\Pi}_\lambda(\underline{i}) = y$. Let $(x,y) \in \Lambda(\lambda)$. Thus

$$(x,y) = \left(\sum_{n=0}^{\infty} j_n \lambda^n, \sum_{k=0}^{\infty} i_n \left(\frac{1}{3}\right)^k \right)$$

where if $i_n = 1$ then $j_n \in \{0,2\}$ otherwise $j_n \in \{0,1,2\}$. However there exists N such that for $n \geq N$ $i_n = 1$. Thus for $n \geq N$ it follows that $j_n \neq 1$. Hence the dimension of $L_y \cap \Lambda(\lambda)$ can be at most $-\frac{\log 2}{\log \lambda}$.

Hence $\Lambda(\lambda)$ contains a dense set of cross sections with dimension less than 1 if $\lambda < \frac{1}{2}$. Thus $\Lambda(\lambda)$ has no interior. □

We now try to find the values where the attractor typically has positive measure.

Lemma 6.2. *For almost all $\lambda \in [0.3816\dots, 0.5]$ we have that $\mathcal{L}^2(\Lambda(\lambda)) > 0$.*

Proof. Let μ be $(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{6}, \frac{1}{6}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$-Bernoulli measure on Σ_8. Let Π_λ be the natural projection to $\Lambda(\lambda)$ and $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. We show that ν_λ is absolutely continuous for almost every $\lambda \in [0.3816\dots, 0.5]$.

We do this using exactly the same method as in the previous chapter. Let $p : \Sigma_8 \to \Sigma_3$ be defined by

$$p(i_0, i_1, \ldots) = (c^{(2)}_{i_0}, c^{(2)}_{i_1}, \ldots).$$

In this setting $\overline{\mu} = \mu \circ p^{-1}$ is simply $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$-Bernoulli measure on Σ_3. Thus the projection $\overline{\nu}$ of $\overline{\mu}$ by $\overline{\Pi_\lambda}$ is simply Lebesgue measure on $[0, 1]$ for any λ and is thus absolutely continuous. For any $\xi \in \Sigma_3$ we define a subspace $\Sigma_{3,\xi}$ by $\underline{j} \in \Sigma_{3,\xi}$ if and only if $j_n \neq 1$ whenever $\xi_n = 1$. A measure μ_ξ can be defined on $\Sigma_{3,\xi}$ by

$$\mu_\xi([i_0, \ldots, i_{n-1}]) = 3^{-n_0(\underline{i})-n_2(\underline{i})} 2^{-n_1(\underline{i})}.$$

Let $\Pi_{\lambda,\xi} : \Sigma_{3,\xi} \to \mathbb{R}$ be defined by

$$\Pi_{\lambda,\xi}(\underline{j}) = \frac{1}{3} \sum_{n=0}^{\infty} j_n \lambda^n.$$

We have that

$$\left(\Pi_{\lambda,\xi}(\underline{j}), \overline{\Pi}_\lambda(\xi)\right) \in \Lambda(\lambda)$$

for all $\underline{j} \in \Sigma_{3,\xi}$. Thus the measure $\nu_{\lambda,\xi} = \mu_\xi \circ \Pi^{-1}_{\lambda,\xi}$ is supported on $L_{\overline{\Pi}_\lambda(\xi)} \cap \Lambda(\lambda)$. Thus to complete the proof it suffices to show that $\nu_{\lambda,\xi}$ is absolutely continuous for $\overline{\mu}$ almost all ξ.

It suffice to show that for almost all λ for any $\epsilon > 0$ there exists a set Y such that $\overline{\mu}(Y) \geq 1 - \epsilon$ where $\nu_{\lambda,\xi}$ is absolutely continuous for $\xi \in Y$. Let $\epsilon > 0$ then there exists a set $Y \subset \Sigma_3$ such that $\bar{\mu}(Y) \geq 1 - \epsilon$ and for any $\delta > 0$ there exists $N \in \mathbb{N}$ where for $n \geq N$ we have that for all $\xi \in Y$ $n_i(\xi) \geq \left(\frac{1}{3} - \delta\right) n$. We let

$$\lambda_0 = 3^{-2(\frac{1}{3}-2\delta)} 2^{-(\frac{1}{3}-2\delta)}.$$

It is sufficient to show that for all $\delta > 0$

$$I = \int_{\lambda_0}^{0.5} \int_Y \int_{\mathbb{R}} \liminf_{r\to 0} \frac{\nu_{\lambda,\xi}(B(x, r))}{2r} \mathrm{d}\nu_{\lambda,\xi} \mathrm{d}\bar{\mu}(\xi) \mathrm{d}\lambda.$$

We apply Fatou's Lemma, lift to the shift space and use Fubini's Theorem to obtain,

$$\begin{aligned} I &\leq \liminf_{r\to 0} \frac{1}{2r} \int_{\lambda_0}^{0.5} \int_Y \int_{\Sigma_{3,\xi}} \int_{\Sigma_{3,\xi}} \mathbb{1}_{\{\omega,\tau : |\Pi_{\lambda,\xi}(\omega) - \Pi_{\lambda,\xi}(\tau)| \leq r\}} \mathrm{d}\mu_\xi(\omega) \mathrm{d}\mu_\xi(\tau) \mathrm{d}\overline{\mu}(\xi) \mathrm{d}\lambda \\ &= \liminf_{r\to 0} \frac{1}{2r} \int_Y \int_{\Sigma_{2,\xi}} \int_{\Sigma_{2,\xi}} \mathcal{L}\{\lambda : |\Pi_{\lambda,\xi}(\omega) - \Pi_{\lambda,\xi}(\tau)| \leq r\} \mathrm{d}\mu_\xi(\omega) \mathrm{d}\mu_\xi(\tau) \mathrm{d}\overline{\mu}(\xi). \end{aligned}$$

We have by Lemma 2.1 there exists $C > 0$ such that for all $\omega, \tau \in \Sigma_{2,\xi}$

$$\mathcal{L}\{\lambda \in [\lambda_0, 0.5] : |\Pi_{\lambda,\xi}(\omega) - \Pi_{\lambda,\xi}(\tau)| \leq r\} \leq 2Cr\lambda_0^{|\omega\wedge\tau|}.$$

Using this and the definition of Y we get,

$$\begin{aligned}
I &\leq C\int_Y \int_{\Sigma_{2,\xi}} \int_{\Sigma_{2,\xi}} \lambda_0^{-|\omega\wedge\tau|} \mathrm{d}\mu_\xi(\omega)\mathrm{d}\mu_\xi(\tau)\mathrm{d}\overline{\mu}(\xi) \\
&\leq C\int_Y \sum_{k=0}^{\infty} \lambda_0^{-k} \sum_{\omega_0,\ldots,\omega_{k-1}} \mu_\xi \times \mu_\xi[\omega_0,\ldots,\omega_{k-1}]\mathrm{d}\overline{\mu}(\xi) \\
&\leq C\int_Y \sum_{k=0}^{\infty} \lambda_0^{-k} \left(\frac{1}{3}\right)^{k_0(\xi)+k_2(\xi)} \left(\frac{1}{2}\right)^{k_1(\xi)} \mathrm{d}\overline{\mu}(\xi) \\
&\leq CC_1 + \sum_{k=N}^{\infty} \lambda_0^{-k} \left(\frac{1}{3}\right)^{k(\frac{2}{3}-\delta)} \left(\frac{1}{3}\right)^{k(\frac{1}{3}-\delta)} < \infty
\end{aligned}$$

Where C_1 is a constant bounding the first N terms of this power series. □

6.3 Self-similar Sets

It was pointed out by Marianna Csörnyei, David Preiss and Boris Solomyak that the method used in the previous section can be adapted to show the existence of self-similar sets with positive measure but no interior. Let $\underline{t} = (t_1, t_2) \in [0,1]^2 \subset \mathbb{R}^2$. We consider the IFS generated by the following ten similarities,

$$\begin{aligned}
T_0(x,y) &= \left(\tfrac{1}{3}x, \tfrac{1}{3}y\right) & T_5(x,y) &= \left(\tfrac{1}{3}x + \tfrac{1}{3}, \tfrac{1}{3}y + 1\right) \\
T_1(x,y) &= \left(\tfrac{1}{3}x, \tfrac{1}{3}y + t_1\right) & T_6(x,y) &= \left(\tfrac{1}{3}x + \tfrac{2}{3}, \tfrac{1}{3}y\right) \\
T_2(x,y) &= \left(\tfrac{1}{3}x, \tfrac{1}{3}y + t_2\right) & T_7(x,y) &= \left(\tfrac{1}{3}x + \tfrac{2}{3}, \tfrac{1}{3}y + t_1\right) \\
T_3(x,y) &= \left(\tfrac{1}{3}x, \tfrac{1}{3}y + 1\right) & T_8(x,y) &= \left(\tfrac{1}{3}x + \tfrac{2}{3}, \tfrac{1}{3}y + t_2\right) \\
T_4(x,y) &= \left(\tfrac{1}{3}x + \tfrac{1}{3}, \tfrac{1}{3}y\right) & T_9(x,y) &= \left(\tfrac{1}{3}x + \tfrac{2}{3}, \tfrac{1}{3}y + 1\right).
\end{aligned}$$

If we let $\Lambda_{\underline{t}}$ be the attractor we obtain the following result.

Theorem 6.2. *For Lebesgue almost all $\underline{t} \in [0,1]^2$ we have that $\Lambda_{\underline{t}}$ satisfies $\mathcal{L}^2(\Lambda_{\underline{t}}) > 0$ but has empty interior.*

To prove Theorem 6.2 we follow the same method as was used to prove Theorem 6.1.

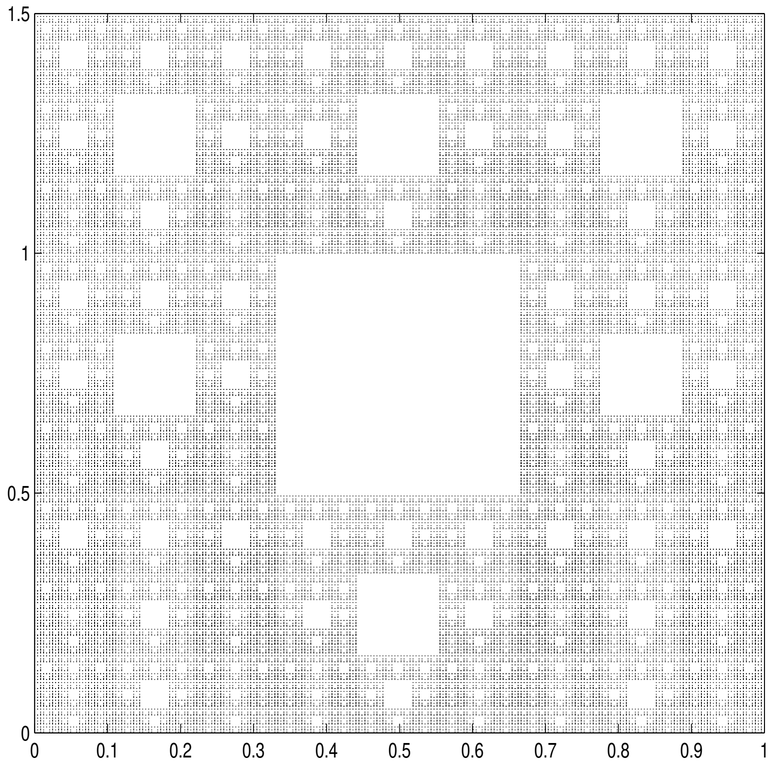

Figure 6.1: A typical attractor $\Lambda_{\underline{t}}$.

Lemma 6.3. *For all* $\underline{t} \in [0,1]^2$ $\Lambda_{\underline{t}}$ *has empty interior.*

Proof. This can be proved using exactly the same argument as was used to prove Lemma 6.1 □

We let $\Pi_{\underline{t}} : \Sigma_{10} \to \Lambda_{\underline{t}}$ be the natural projection. Let $p : \Sigma_{10} \to \Sigma_3$ be defined by $p(\omega_n) = i(\omega_n)$ where $i|_{\{0,1,2,3\}} = 0$, $i|_{\{4,5\}} = 1$ and $i|_{\{6,7,8,9\}} = 2$. If we let μ be $\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$ Bernoulli measure on Σ_{10} then $\overline{\mu} = \mu \circ p^{-1}$ is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ Bernoulli measure on Σ_3. Thus the projection of Σ_3 onto the x-axis of $\Lambda_{\underline{t}}$, given by

$$\overline{\Pi}_{\underline{t}}(\underline{i}) = \sum_{n=0}^{\infty} i_n \left(\frac{1}{3}\right)^{n+1},$$

projects $\overline{\mu}$ to one dimensional Lebesgue measure.

We wish to show that $\nu_{\underline{t}} = \mu \circ \Pi_{\underline{t}}^{-1}$ is absolutely continuous for almost all $\underline{t}$. For a sequence $\xi \in \Sigma_3$ consider the points $\Pi_{\underline{t}} p^{-1}\xi$. These will all be in $\Lambda_{\underline{t}}$ with x-coordinate $\overline{\Pi}_{\underline{t}}(\xi)$. Thus if we let $\nu_{\underline{t},\xi}$ be defined by $\nu_{\underline{t},\xi} = \mu_\xi \circ \Pi_\lambda^{-1}$ then it suffices to show that $\nu_{\underline{t},\xi}$

is absolutely continuous for $\overline{\mu}$ almost every ξ for almost every $\underline{t} \in [0,1]^2$. The main difference to the previous cases is we are now varying the translations rather than the contraction rates. This is similar to the work on self-affine fractals by Falconer in [12]. Let

$$\Delta_\xi(r;\omega,\tau) = \mathcal{L}\{\underline{t} \in [0,1]^2 : |\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau)| \le r\}$$

where $r > 0$. The analogue of transversality is the following Lemma.

Lemma 6.4. *There exists $C > 0$ such that*

$$\Delta_\xi(r;\omega,\tau) = \mathcal{L}\{\underline{t} \in [0,1]^2 : |\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau)| \le r\} \le 3^{|\omega\wedge\tau|} r.$$

Proof. This result could be proved by a slight adjustment to the arguments in [12]. However the following proof, given by Boris Solomyak, holds in this simpler case. Since $\omega, \tau \in p^{-1}\xi$ we have $i(\omega_n) = i(\tau_n)$ for all n. If we let $|\omega \wedge \tau| = n$ then $|\Pi_{\underline{t},\xi}(\omega) - \Pi_{\underline{t},\xi}(\tau)| = (0.\phi_{\underline{t},\xi}(\omega,\tau))$ where

$$\phi_{\underline{t},\xi}(\omega,\tau) = 3^{-n}\left((t_{j(\omega_n)} - t_{j(\tau_n)}) + \sum_{k=1}^{\infty} 3^{-k}(t_{j(\omega_{k+n})} - t_{j(\tau_{k+n})})\right).$$

Here the function j is defined by $j|_{0,4,6} = 0$, $J|_{1,7} = 1$, $j_{2,8} = 2$ and $j|_{3,5,9} = 3$. $t_0 = 0$ and $t_3 = 1$ for notational convenience.

If $\{j(\omega_n), j(\tau_n)\} = \{0,3\}$ then

$$|\phi_{\underline{t},\xi}(\omega,\tau)| \ge 3^{-n}\left(1 - \sum_{k=1}^{\infty} 3^{-k}\right) = \frac{3^{-n}}{2}$$

as $t_j \in [0,1]$ for all j. So we just need to consider the case where $\{j(\omega_n), j(\tau_n)\}\} \cap \{1,2\} \neq \emptyset$. Here

$$\left|\frac{\partial\phi_{t,\xi}(\omega,\tau)}{\partial t_j}\right| \ge 3^{-n}\left(1 - \sum_{k=1}^{\infty} 3^{-k}\right) = \frac{3^{-n}}{2}.$$

This is sufficient to complete the proof. □

Finally we consider the absolute continuity of $\nu_{\underline{t},\xi}$.

Lemma 6.5. *For almost every $\underline{t} \in [0,1]^2$ for almost every $\xi \in \Sigma_3$ we have that $\nu_{\underline{t},\xi}$ is absolutely continuous.*

Proof. Once again it suffices to show that for almost every $\underline{t}$ for any $\epsilon > 0$ there exists a set Y such that $\overline{\mu}(Y) \geq 1-\epsilon$ and $\nu_{\underline{t},\xi}$ is absolutely continuous for every $\xi \in Y$. Let $\epsilon > 0$; by Ergorov's Theorem and Birkhoff's Ergodic Theorem (in fact in this case just the strong law of large numbers is needed) there exists $Y \subset \Sigma_3$ such that $\overline{\mu}(Y) \geq 1-\epsilon$ and for any $\delta > 0$ there exist $n \in N$ such that for $n \geq N$ $n_i(\xi) \geq \left(\frac{1}{3}-\delta\right) N$ for any $\xi \in Y$ and $i = 0, 1, 2$. We fix $\delta > 0$ and choose such an N. As usual it is sufficient to show that

$$I = \int_{[0,1]^2} \int_Y \int_{\Pi_{\underline{t},xi}(p^{-1}\xi)} \liminf_{r\to 0} \frac{\nu_{\underline{t},\xi}(B(x,r))}{2r} \mathrm{d}\nu_{\underline{t},\xi}(x) \mathrm{d}\overline{\mu}(\xi) \mathrm{d}\underline{t} < \infty.$$

As before apply Fatou's Lemma, lift to the shift space and use Fubini's Theorem to obtain

$$I \leq \liminf_{r\to 0} \frac{1}{2r} \int_Y \int_{p^{-1}\xi} \int_{p^{-1}\xi} \mathcal{L}\{\underline{t} : |\Pi_{\underline{t}}(\omega) - \Pi_{\underline{t}}(\tau)| \leq r\} \mathrm{d}\mu_\xi(\omega) \mathrm{d}\mu_\xi(\tau) \mathrm{d}\overline{\mu}(\xi).$$

Applying Lemma 6.4 gives that there exists $C \geq 0$ where

$$\begin{aligned} I &\leq C \int_Y \int_{p^{-1}\xi} \int_{p^{-1}\xi} 3^{|\omega\wedge\tau|} \mathrm{d}\mu_\xi(\omega) \mathrm{d}\mu_\xi(\tau) \mathrm{d}\overline{\mu}(\xi) \\ &\leq C \int_Y \sum_{k=0}^{\infty} 3^k 4^{-k_0(\xi)-k_2(\xi)} 2^{-k_2(\xi)} \mathrm{d}\overline{\mu}(\xi) \\ &\leq CC_1 + \sum_{k=N}^{\infty} 3^k 4^{-k(\frac{2}{3}-2\delta)} 2^{-k(\frac{1}{3}-\delta)} \end{aligned}$$

where C_1 bounds the fist N terms of the sum. I can be seen to be finite for δ sufficiently small. ∎

It is also possible to produce a construction with fewer similarities but with one of a different ratio. Let $\lambda < \frac{1}{2}$. We define our iterated function system by,

$$\begin{aligned} T_0(x,y) &= (\lambda x, \lambda y) \\ T_1(x,y) &= (\lambda x + t_0, \lambda y) \\ T_2(x,y) &= (\lambda x + t_1, \lambda y) \\ T_3(x,y) &= ((1-2\lambda)x + t_2, (1-2\lambda y) + \lambda) \\ T_4(x,y) &= (\lambda x, \lambda y + 1 - \lambda) \\ T_5(x,y) &= (\lambda x + t_1, \lambda y + 1 - \lambda) \end{aligned}$$

where $\underline{t} = (t_0, t_1, t_2) \in [0,1]^3$. Let $\Lambda_{\underline{t}}$ be the attractor for the iterated function system. We let μ be $\left(\frac{\lambda}{3}, \frac{\lambda}{3}, \frac{\lambda}{3}, 1-2\lambda, \frac{\lambda}{2}, \frac{\lambda}{2}\right)$-Bernoulli measure on Σ_6. $j : \{0,1,2,3,4,5\} \to \{0,1,2\}$ is defined by $j_{|0,1,2} = 0$, $j_{|3} = 1$ and $j_{|4,5} = 2$ and $p : \Sigma_6 \to \Sigma_3$ by

$$p(i_0, \ldots, i_n, \ldots) = (z(i_0), \ldots, z(i_n), \ldots).$$

Let $\overline{\mu} = p(\mu)$ and μ_ξ for $\xi \in \Sigma_3$ be the conditional measure on a fibre $p^{-1}\xi$. We then define $\Pi_{\underline{t}}, \overline{\Pi}_{\underline{t}}, \Pi_{\underline{t},\xi}, \nu_{\underline{t}}, \overline{\nu}_{\underline{t}}$ and $\nu_{\underline{t},\xi}$ analogously to the previous section. We can obtain the following result.

Theorem 6.3.

1. *For any $\lambda < \frac{1}{2}$ and any $\underline{t} \in [0,1]^3$ the attractor $\Lambda_{\underline{t}}$ has empty interior.*

2. *For any $\lambda > 0.4759\ldots$ and for Lebesgue almost all $\underline{t} \in [0,1]^3$ $\mathcal{L}^2(\Lambda_{\underline{t}}) > 0$.*

Proof. Part 1 can be seen using the same method as in the proof of Lemma 6.1 . To prove part (ii) note that $\overline{\nu}_{\underline{t}}$ is simply uniform measure on $[0,1]$. For $\nu_{\underline{t}}$ to be absolutely continuous with respect to $\mathcal{L}^2$ it suffices to check that $\nu_{\underline{t},\xi}$ is absolutely continuous with respect to $\mathcal{L}^1$ restricted to the horizontal fibre. For this we need the conditional entropy to be greater than the negative of the Lyapunov exponent restricted to the fibre. This holds provided

$$-(h(\mu) - h(\overline{\mu})) = -\lambda \log 2 - \lambda \log 3 < 2\lambda \log \lambda + (1-2\lambda)\log(1-2\lambda)$$

which is true for $\lambda > 0.4759$.

To complete the proof we need the following transversality condition to hold. For any $\xi \in \Sigma_3$ and $\underline{i}, \underline{j}$

$$\mathcal{L}\{\underline{t} \in \mathbb{R}^3 : |\Pi_{\underline{t}}(\underline{t}) - \Pi_{\underline{t}}(\underline{j})| \leq r\} \leq C\lambda. \tag{6.1}$$

To show this condition holds we fix $\xi \in \Sigma_3$ such that $\xi_0 \neq 1$ and let $\underline{i}, \underline{j} \in p^{-1}\xi$ such that $i_0 \neq j_0$ and $i_0 \notin \{0,4\}$. We define z such that $z_{|1} = 0$, $z_{|2,5} = 1$, $z_{|3} = 2$ and $z_{|0,4} = 3$. For convenience we let $t_3 = 0$. Our definition means that $z(i_0) \neq 3$ and $z_{i_0} \neq z_{j_0}$. Thus, since $\lambda < \frac{1}{2}$ the partial derivative of $|\Pi_{\underline{t},\xi}(\underline{i}) - \Pi_{\underline{t},\xi}(\underline{j}|$ with respect to $t_{z(i_0)}$ is bounded away form 0. This is enough to deduce Equation (6.1).

□

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