The average rank of elliptic curves

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We use the simplest such measure, called the naive height, which is basically a measure of the size of the coefficients of the defining equation of the elliptic curve.

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These various measures are conjectured to be about the same order of magnitude for all but a negligible proportion of elliptic curves!

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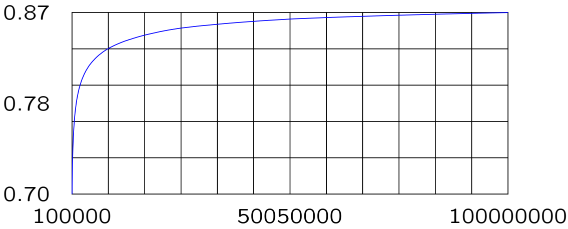
All Curves Ordered By Conductor

The average rank of all curves of conductor $\leq 10^8$ is $0.8664\ldots$.
A graph of the average rank as a function:



We created this graph by computing the average rank of curves of conductor up to $n \cdot 10^5$ for $1 \leq n \leq 1000$.

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Theorem. *When all elliptic curves $E/\mathbb{Q}$ in any family defined by finitely many congruence conditions are ordered by height, the average size of the 2-Selmer group $S^{(2)}(E)$ is exactly* 3*.*

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Theorem. (Birch & Swinnerton-Dyer) *There is an injective map from $S^{(2)}(E_{A,B})$ to the set of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of integral binary quartic forms having invariants $I = -2^4 \cdot 3 \cdot A$ and $J = -2^4 \cdot 3 \cdot B$.*

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BSD's theorem yields an efficient method for rank computations of elliptic curves. This method has been further refined by Cremona, and implemented in his well-known `mwrank` program.

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Disquisitiones Arithmeticae (1801)

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Theorem. (Gauss 1801/Mertens 1874/Siegel 1944)

$$\sum_{-X<D<0} h_D \sim \frac{\pi}{18} \cdot X^{3/2}; \qquad \sum_{0<D<X} h_D \log \epsilon_D \sim \frac{\pi^2}{18} \cdot X^{3/2}.$$

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The next natural case is that of binary cubic forms $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, $a,b,c,d \in \mathbb{Z}$.

$\mathrm{GL}_2(\mathbb{Z})$ acts naturally on such forms.

There is again just one polynomial invariant for this action, namely the discriminant $\mathrm{Disc}(f)$ of f, given by

$$\mathrm{Disc}(f) = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2.$$

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Theorem. (Davenport 1951)

$$\sum_{-X<D<0} h(D) \sim \frac{\pi^2}{24} \cdot X; \qquad \sum_{0<D<X} h(D) \sim \frac{\pi^2}{72} \cdot X.$$

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On average, how many classes $h_{I,J}$ of irreducible binary quartic forms are there having given invariants I and J? Equivalently, how many equivalence classes of binary quartic forms are there having bounded I and J?

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How many classes do we get per (I,J)?

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These congruence conditions are:

(a) $I \equiv 0 \pmod 3$ and $J \equiv 0 \pmod{27}$,

(b) $I \equiv 1 \pmod 9$ and $J \equiv \pm 2 \pmod{27}$,

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The average number of binary quartic forms per (I, J)

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The analogous theorems can be proven for equivalence classes of binary quartic forms satisfying any desired finite set of congruence conditions.

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Corollary. *When all elliptic curves $E/\mathbb{Q}$ (in any family defined by finitely many congruence conditions) are ordered by height, the average rank is at most 1.5.*

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Some consequences

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Suppose $E \in \mathcal{F}$. Then E twisted by -1 is also in $\mathcal{F}$, and furthermore, the analytic root numbers of E and its twist by -1 are different. Therefore, exactly half the root numbers of curves in $\mathcal{F}$ are $+1$.

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Indeed, as the average number of 3-Selmer elements of curves in $\mathcal{F}$ is at most 4, it is not possible for all the curves with even 2-Selmer rank to have rank greater than 0. At least half of them must have rank 0!

A similar argument gives:

Parity of p-Selmer rank

A recent result of Tim and Vladimir Dokchitser states that the parity of the p-Selmer rank of E is even iff the root number of E is $+1$!

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Theorem. *Assume $Ш(E)$ is finite for all E. When all elliptic curves $E/\mathbb{Q}$ are ordered by height, a positive proportion of them have rank* 1.

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Corollary. *A positive proportion of elliptic curves satisfy the BSD rank conjecture.*

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Elements in 4-Selmer and 5-Selmer groups of elliptic curves can be mapped to integer points, up to equivalence, having the corresponding invariants in the spaces

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Corollary. *When all elliptic curves $E/\mathbb{Q}$ are ordered by height, the average rank is less than 1.*

Some final remarks

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There are several such spaces that parametrize various data corresponding to higher genus curves (Dick Gross, Wei Ho, Sam Stevens, Jack Thorne, . . .).