Last time:

Riemann $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ $\mathbb{Q}$

Dedekind $\zeta_K(s) = \sum_{\substack{I \subseteq \mathcal{O}_K \\ \text{nonzero ideal}}} \frac{1}{(NI)^s}$ $K/\mathbb{Q}$ number field

Today

Dirichlet $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ χ Dirichlet character

$$\zeta_{\mathbb{Q}(\zeta_m)}(s) = \prod_{\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times} L(\chi, s)$$

§3 Dirichlet L-functions

Def $n \geq 2$. A [mod n] **Dirichlet character** is a group hom.

$$\chi : (\mathbb{Z}/n\mathbb{Z})^{\times} \longrightarrow \mathbb{C}^{\times}$$

These form a group $\widehat{(\mathbb{Z}/n\mathbb{Z})^{\times}}$

Order of χ = smallest d s.t. $\chi^d = 1$

(i.e. $\chi : (\mathbb{Z}/n\mathbb{Z})^{\times} \longrightarrow \{ d^{th} \text{ roots of unity} \}$).

order 2 = <u>quadratic</u> $\left((\mathbb{Z}/n\mathbb{Z})^\times \to \{\pm 1\} \right)$

<u>Modulus of χ</u> = smallest $m \mid n$ s.t.

$\exists\, \chi_0 : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ s.t. $\chi(a) = \chi_0(a)$ $\forall (a,n) = 1$.

We extend $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ to

$\chi : \mathbb{Z} \to \mathbb{C}$ by $\chi(a) = \begin{cases} \chi_0(a) & (a,m)=1 \\ 0 & \text{else.} \end{cases}$

not a hom., but totally multiplicative

Ex $n=1$ $\quad \chi(a) = 1 \quad \forall a \in \mathbb{Z}$

principal or trivial character

(order 1, modulus 1)

Ex $n=3$ $\quad \underset{\underset{C_2}{\cong}}{(\mathbb{Z}/3\mathbb{Z})^\times} \longrightarrow \mathbb{C}^\times$ two characters:

$\mathbb{1}$, and

$$\chi_3(a) = \begin{cases} 1 & a \equiv 1 \bmod 3 \\ -1 & a \equiv 2 \bmod 3 \\ 0 & a \equiv 0 \bmod 3. \end{cases}$$

order 2
modulus 3.

$\underline{\text{Ex}}$ $n=4$. $\underset{\overset{\cong}{C_2}}{(\mathbb{Z}/4\mathbb{Z})^\times} \longrightarrow \mathbb{C}^\times$

two characters $\mathbb{1}$, and

$$\chi_4(a) = \begin{cases} 1 & a \equiv 1 \bmod 4 \\ -1 & a \equiv 3 \bmod 4 \\ 0 & a \text{ even} \end{cases}$$

order 2
modulus 4

$\underline{\text{Ex}}$ $n=5$ $\underset{\overset{\cong}{C_4}}{(\mathbb{Z}/5\mathbb{Z})^\times} \longrightarrow \mathbb{C}^\times$

$\chi_5 : (\mathbb{Z}/5\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$, χ_5^2, $\chi_5^3 = \overline{\chi_5}$,
$2 \longmapsto i$ $\chi_5^4 = \mathbb{1}$.

Ex $n=12$ $\quad (\mathbb{Z}/12\mathbb{Z})^\times \cong C_2 \times C_2 \longrightarrow \mathbb{C}^\times$

4 characters.

1	5	7	11	
1	1	1	1	$= \mathbb{1}.$
1	-1	1	-1	$= \chi_3 = \left(\frac{-3}{\cdot}\right)$ modulus 3
1	1	-1	-1	$= \chi_4 = \left(\frac{-1}{\cdot}\right)$ 4
1	-1	-1	1	$=: \chi_{12} = \chi_3 \chi_4 = \left(\frac{3}{\cdot}\right)$ 12.

[for $q=2$ we let $\left(\frac{n}{2}\right) =$

$$= \begin{cases} 0 & n \not\equiv 1 \bmod 4 \\ 1 & n \equiv 1 \bmod 8 \\ -1 & n \equiv 5 \bmod 8 \end{cases} = \begin{cases} 0 & 2 \text{ ramifies} \\ 1 & 2 \text{ splits} \\ -1 & 2 \text{ inert.} \end{cases}$$

in $\mathbb{Q}(\sqrt{n})$.]

<u>Def</u> $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ modulus m

(primitive)

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$ <u>(Dirichlet) L-function of χ</u>

$$= \prod_p \frac{1}{1-\chi(p)p^{-s}}$$

local polys

1 if $p \mid m$

$1-\chi(p)T$ if $p \nmid m$.

$|\chi(n)| \leq 1 \Rightarrow$ abs. conv. on $\operatorname{Re} s > 1$.

In fact, for $\chi \neq \mathbb{1}$

$$\left| \sum_{n=A}^{B} \chi(n) \right| \leq m \quad \forall A, B \Rightarrow$$

L-series converges (not absolutely) on $\operatorname{Re} s > 0$.

<u>Thm</u> $L(\chi, s)$ is entire (for $\chi \neq \mathbb{1}$),

$$\hat{L}(\chi, s) = \left(\frac{m}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\lambda}{2}\right) L(\chi, s)$$

satisfies fun. eq

$$\hat{L}(\chi, 1-s) = w \cdot L(\underset{= \chi^{-1}}{\bar{\chi}}, s).$$

with

$$\lambda = \begin{cases} 0 & \chi(-1) = 1 \quad (\chi \text{ even}) \\ 1 & \chi(-1) = -1 \quad (\chi \text{ odd}). \end{cases}$$

Gauss sum

$$W = \frac{1}{\sqrt{m}} \sum_{a=0}^{m-1} \chi(a)\, \zeta_m^a$$

$= e^{\frac{2\pi i}{m}}$ prim. m th root of 1.

$W \in \mathbb{C}^\times$, $|W| = 1$.

Pf Poisson summation for

$e^{-\pi(mx+a)^2 t}$ (even χ)

$x e^{-\pi x^2 t}$ (odd χ).

$\blacksquare$

Next: Dedekind $\zeta_{\mathbb{Q}(\zeta_m)}(s) = \prod L(\chi, s)$

$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$

<u>NB</u> <u>Cor</u> $L(\chi, 1) \neq 0 \quad \forall \chi \neq \mathbb{1}$

[simple pole in LHS at $s=1$, and in $L(\mathbb{1}, s) = \zeta(s)$, all others analytic at $s=1$].

This proves Dirichlet's Thm on primes in arithmetic progressions

$$\mathcal{P} = \{ \text{primes } p \equiv a \bmod m \} \quad (a,m)=1$$

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s} = \frac{1}{\varphi(m)} \sum_{\chi} \overline{\chi(a)} \log L(\chi,s) + \text{analytic at } s=1.$$

$$\zeta(s) = \prod_p \left(1+\frac{1}{p^s}+\dots\right)$$

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + \text{analytic at } s=1.$$

$\Rightarrow$ LHS diverges at $s=1$ (from $L(\mathbb{1}, s)$ on the right),

growth indep. of the choice of a.

$\Rightarrow$ $\underline{P}$ infinite, density $\frac{1}{\varphi(m)}$. $\blacksquare$.

§4 Cyclotomic fields

$$\begin{array}{c} K \\ \varphi(m) \Big| \\ \mathbb{Q} \end{array}$$

Fix $m \geq 1$ $(\neq 2 \times \text{odd})$

$K = \mathbb{Q}(\zeta_m)$ m^{th} cyclotomic field

$\zeta_m = e^{2\pi i/m}$ m^{th} root of 1

$K = \mathbb{Q}(\text{roots of } x^m - 1) = \mathbb{Q}(\text{roots of } \Phi_m)$

Φ_m: m^{th} cyclotomic poly.

$\Phi_1(x) = x - 1$, $x^m - 1 = \prod_{d \mid m} \Phi_d$.

$\deg \Phi_m = \varphi(m) = (\mathbb{Z}/m\mathbb{Z})^\times$.

Galois over $\mathbb{Q}$.

When $m = q^k$, easy to see

- $\Phi_m(x+1) = x^{\varphi(m)} + \dots + q$ Eisenstein $\Rightarrow$ irreducible

  so $[\overbrace{\mathbb{Q}(\zeta_m)}^{K} : \mathbb{Q}] = \varphi(m)$

- $(q) = (1-\zeta_m)^{\varphi(m)}$ as ideals in $\mathcal{O}_K$.

  $\Rightarrow q$ totally ramified in $K/\mathbb{Q}$.

- All other primes $p \nmid \Delta_{x^m - 1} \Rightarrow$

unramified in $K/\mathbb{Q}$, with residue degree

$f =$ order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$.

Pf $p \equiv 1 \bmod m \iff m^{th}$ roots of unity $\subseteq \mathbb{F}_p^\times$

$\iff \Phi_m$ splits completely $/ \mathbb{F}_p$.

$$\Phi_m = \frac{x^{q^k} - 1}{x^{q^{k-1}} - 1}$$

$p^r \equiv 1 \mod m \iff \dots / \mathbb{F}_{p^r}$

$\iff \Phi_m$ has irr. factors of degree $| r$ $/ \mathbb{F}_p$.

order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$ = smallest such r = f.

by Kummer-Dedekind.

<u>When $m = q_1^{k_1} \dots q_j^{k_j}$ general</u>

$K = \mathbb{Q}(\zeta_m)$ = compositum of

$\mathbb{Q}(\zeta_{q_1^{k_1}}), \ldots, \mathbb{Q}(\zeta_{q_j^{k_j}})$ $\xRightarrow{\text{look at ramification}}$

$\Rightarrow [\mathbb{Q}(\zeta_m) : \mathbb{Q}] = \prod \varphi(q_j^{k_j}) = \varphi(m)$

i.e. all Φ_m are irreducible.

- $p \nmid m \Rightarrow p$ unramified in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$, $e_p = 1$

  res. degree f_p = order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$.

- $p \mid m$, $m = p^k m_0 \Rightarrow p$ ramifies in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$

ram. degree $e_p = [\mathbb{Q}(\zeta_{p^k}):\mathbb{Q}] = p^{k-1}(p-1)$

residue degree $f_p =$ order p mod m_0.

ζ-function of $\mathbb{Q}(\zeta_m)$

$$\zeta_K(s) = \prod_p F_p(p^{-s})$$

$$F_p(T) = (1 - T^{f_p})^{\frac{\varphi(m)}{e_p f_p}}$$

$1 - N\mathfrak{p}^{-s} = 1 - p^{-f_p s}$

$= 1 - T^{f_p}$

primes above p.

$\deg F_p = \varphi(m_0)$ $[= \varphi(m)$ for $p \nmid m)$

$$= \prod_{a \in (\mathbb{Z}/f_p\mathbb{Z})^\times} (1 - \zeta_{f_p}^a T)^{\frac{\varphi(m_0)}{f_p}}$$

$$= \prod_{\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}} (1 - \chi(p) T)$$

$[f_p =$ order of p mod $m]$.

In order words,

$$\boxed{\zeta_{\mathbb{Q}(\zeta_m)}(s) = \prod_{\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times} L(\chi, s).}$$

Ex $m = 12$, $K = \mathbb{Q}(\zeta_{12}) = \mathbb{Q}(i, \sqrt{-3})$ biquadratic.

$=$ splitting field of $\Phi_{12}(x)$

$$x^{12}-1 = \underset{\Phi_1}{(x-1)}\underset{\Phi_2}{(x+1)}\underset{\Phi_3}{(x^2+x+1)}\underset{\Phi_4}{(x^2+1)}\underset{\Phi_6}{(x^2-x+1)} \times$$

$$\times \underset{\Phi_{12}}{(x^4-x^2+1)}.$$

	$F_2(T)$	$F_3(T)$	$F_5(T)$ …	$F_{13}(T)$ …
$L(\mathbb{1}, s) = \zeta(s)$	$1-T$	$1-T$	$1-T$	$1-T$
$L(\chi_3, s)$	$1+T$	1	$1+T$	$1-T$
$\times$ $L(\chi_4, s)$	1	$1+T$	$1-T$	$1-T$
$L(\chi_{12}, s)$	1	1	$1+T$	$1-T$
$\zeta_{\mathbb{Q}(\zeta_{12})}(s)$	$1-T^2$	$1-T^2$	$(1-T^2)^2$	$(1-T)^4$

$\underbrace{1-T^2 \quad 1-T^2}_{p \mid \Delta_{\mathbb{Q}(\zeta_{12})}}$

Prime decomposition in $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$.

$(2) = \mathfrak{P}_2^2 \quad N\mathfrak{P}_2 = 4 \quad e=2,\ f=2$

$(3) = \mathfrak{P}_3^2 \quad N\mathfrak{P}_3 = 9 \quad e=2,\ f=2$

ramified

$(5) = \mathfrak{P}_{5A}\mathfrak{P}_{5B} \quad e=1,\ f=2$

cf. $x^4 - x^2 + 1 = (x^2+2x-1)(x^2-2x-1)$ mod 5

$(13) = \mathfrak{P}_{13A}\mathfrak{P}_{13B}\mathfrak{P}_{13C}\mathfrak{P}_{13D}$

totally split.

cf. $x^4 - x^2 + 1 = (x-2)(x-6)(x-7)(x-11)$ mod 13.

Abelian extensions of $\mathbb{Q}$

$$\begin{tikzcd}
 & \mathbb{Q}(\zeta_{12}) = \mathbb{Q}(i, \sqrt{-3}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(i) = \mathbb{Q}(\zeta_4) \arrow[dr, dash] & \mathbb{Q}(\sqrt{3}) \arrow[d, dash] & \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

$$\zeta_{\mathbb{Q}(\zeta_{12})} = \zeta \cdot L(\chi_3) L(\chi_4) L(\chi_{12})$$

$$\zeta_{\mathbb{Q}(i)} = \zeta \cdot L(\chi_4)$$

$$\zeta_{\mathbb{Q}(\sqrt{-3})} = \zeta \cdot L(\chi_3)$$

$$\zeta_{\mathbb{Q}(\sqrt{3})} = \zeta \cdot L(\chi_{12})$$

<u>Thm</u> (Kronecker-Weber)

$K/\mathbb{Q}$ abelian (i.e. Galois with $\mathrm{Gal}(K/\mathbb{Q})$ abelian)

$\Leftrightarrow$ $K \subseteq \mathbb{Q}(\zeta_m)$ for some m

From rep. theory (later)

$\Leftrightarrow$ $\zeta_K(s) = \prod_{i=1}^{[K:\mathbb{Q}]}$ Dirichlet L-fncs.