+ Extra lecture on Monday! 1-3pm

Q Why do we define

Artin L-functions $L(V,s)$ like this, with

$$F_p(T) = \det(1 - \rho(\mathrm{Frob}_p^{-1})T \mid V^{I_p}) \; ?$$

Write $G_K = \mathrm{Gal}(\bar{K}/K)$ K number field

① $L(\mathbb{1}_{G_\mathbb{Q}}, s) = \zeta(s)$, $L(\mathbb{1}_{G_K}, s) = \zeta_K(s)$

trivial rep. $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times$

$\forall g \mapsto 1$

(2) Generally, 1-dim. rep of $G_{\mathbb{Q}}$ $\rightsquigarrow$ Dirichlet L-fncs.

G_K $\rightsquigarrow$ Hecke L-fncs of finite order.

(3) $[K:\mathbb{Q}] = d \Rightarrow K$ determines a natural d-dim. rep. V_K of $G_{\mathbb{Q}}$.

Say $K = \mathbb{Q}[x]/f(x)$ roots $\alpha_1, \dots, \alpha_d$

$$V_K = \mathbb{C}\alpha_1 \oplus \dots \oplus \mathbb{C}\alpha_d \cong \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \mathbb{1}_{G_K}$$

$\circlearrowleft$ $G_{\mathbb{Q}}$ and $\zeta_K(s) = L(V_K, s)$

decomposition of V_K into irreducible reps

$\Rightarrow \zeta_K(s) = \prod$ Artin L-fncs of irr. reps.

(4) (1)+(3) $L(\mathbb{1}_{G_K}, s) = L(\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \mathbb{1}, s)$

and same is true for any V of G_K in place of $\mathbb{1}_{G_K}$

(5) Brauer induction $\Rightarrow$ (1)-(4) recover all $L(V,s)$ uniquely from Dirichlet/Hecke L-fncs, tell that our defn is the only possible one, and gives meromorphic cont. of all $L(V,s)$ and fun. eq.

(6) Works in exactly the same way for non-finite image reps (elliptic curves etc.)

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§7 Special case: $L(\chi, s)$

Thm There is a bijection

$$\{\text{Dirichlet characters } \chi\} \longleftrightarrow \left\{\begin{array}{l}\text{1-dim. reps.}\\ \rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}^\times\end{array}\right\}$$

$$\chi \longmapsto \rho_\chi$$

such that

- χ of modulus m $\iff$ ρ_χ factors through $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ and not for smaller $d|m$ (*)
- $L(\chi, s) = L(\rho_\chi, s)$

Pf Take χ of modulus m

$$\rho_\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \xrightarrow[\cong]{\text{can.}} (\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

$$\sigma : \zeta_m \mapsto \zeta_m^a \quad \underset{\text{Artin map}}{\longmapsto} \quad a^{-1} \longmapsto \chi(a)^{-1}$$

Note $p^{-1} \in (\mathbb{Z}/m\mathbb{Z})^\times$ corresponds to $\zeta_m \to \zeta_m^p$ which is Frob_p. $\quad p \mapsto \mathrm{Frob}_p^{-1}$.

χ of modulus m $\Rightarrow$ does not come from $(\mathbb{Z}/d\mathbb{Z})^\times$ for $d|m$, $d<m$ $\Rightarrow$ $(*)$

Kronecker-Weber $\Rightarrow$ every rep. of $G_{\mathbb{Q}}$ that factors through an abelian gp, in particular every 1-dim one ρ factors through some $Gal(\mathbb{Q}(\zeta_m)/\mathbb{Q})$

$\Rightarrow \rho = \rho_\chi$.

Compare L-functions:

$p \nmid m$: $L(\chi, s)$ has $F_p(T) = 1 - \underbrace{\chi(p)}_{\in \mathbb{C}^\times} T$

$(p \in (\mathbb{Z}/m\mathbb{Z})^\times)$

$L(\rho_\chi, s)$ has $F_p(T) = 1 - \rho_\chi(\mathrm{Frob}_p^{-1}) T$

(inertia at p is trivial, because p is unramified in $\mathbb{Q}(\zeta_m)/\mathbb{Q}$)

and $\rho_\chi(\mathrm{Frob}_p^{-1}) = \chi(p)$.

$p \mid m$: $L(\chi, s)$ has $F_p(T) = 1$ (as $p \mid m$ $\Rightarrow \chi(p) = 0$

$$\begin{array}{l} \mathbb{Q}(\zeta_m) \qquad m = m_0 p^k \\ \quad \Big| \, I_p \\ \mathbb{Q}(\zeta_{m_0}) \\ \quad \Big| \\ \mathbb{Q} \end{array}$$

χ modulus m (primitive) $\Rightarrow$ ρ_χ does not factor through $\mathrm{Gal}(\mathbb{Q}(\zeta_{m_0})/\mathbb{Q})$ $\Rightarrow$ I_p acts non-trivially on V_χ $(\cong \mathbb{C})$

$\Rightarrow V_\chi^{I_p} = 0 \Rightarrow f_p(T) = 1.$ ☺

Rmk Same holds for

$$\begin{array}{c} \text{Hecke characters} \\ \text{of finite order } / K \end{array} \overset{1:1}{\longleftrightarrow} \begin{array}{c} \text{1-dim reps} \\ G_K \to \mathbb{C}^\times \end{array}$$

Pf Instead of Kronecker-Weber, uses full force of global CFT.

§8 Permutation representations & Dedekind ζ

F/K finite Galois, $G = \mathrm{Gal}(F/K)$

$$\begin{array}{ccccc}
\text{Transitive } G\text{-sets}/\cong & \overset{1:1}{\longleftrightarrow} & \text{Sgps of } G \text{ up to conjugacy} & \overset{1:1}{\longleftrightarrow} & \text{fields } K \subseteq M \subseteq F \text{ up to } \cong \text{ over } K \\
X & \longmapsto & \text{Stabiliser (elt)} & H \longmapsto & F^H \\
G/H & \longleftarrow & H & \mathrm{Gal}(F/M) \leftarrow & M
\end{array}$$

G/H ↺ $\{\text{left cosets } g_1H, \ldots, g_dH\}$

$$g \cdot (g_i H) = g g_i \cdot H$$

So $[M:K] = d \rightsquigarrow$ transitive G-set X of size d

[or $\mathrm{Gal}(\bar{K}/K)$-set, does not depend on F].

$$
\begin{array}{c}
F \\
\Big| \, H \\
M \quad \rightsquigarrow X = G/H \\
\Big| \, d \\
K
\end{array}
\qquad G = \mathrm{Gal}(F/K)
$$

Explicitly, if $M = K(\alpha)$,
α root of $f(x) \in K[x]$, irr, deg d

$H = \mathrm{Stab}_G(\alpha)$

$X = X_{M/K} = \{\text{roots of } f\} \circlearrowleft G$ [or $\circlearrowleft G_K$]

$= \{K\text{-embeddings } M \hookrightarrow \bar{K}\} \circlearrowleft G_K$

<u>Ex</u> $G = S_3$.
$K = \mathbb{Q}$, $F = \mathbb{Q}(\zeta_3, \sqrt[3]{m})$

fields	sgps H	G-sets X	
$\mathbb{Q}$	S_3	$\cdot$	G acts trivially
$\mathbb{Q}(\zeta_3)$	C_3	$\cdot\ \cdot$	G acts through $S_3/C_3 = C_2$
$\mathbb{Q}(\sqrt[3]{m})$	C_2	$\therefore$	G acts as $S_3 \circlearrowleft \{1,2,3\}$
F	$\{1\}$	$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$	regular action ($G \circlearrowleft G$ by left mult.)

G-set X $\rightsquigarrow$ d-dimensional <u>permutation rep.</u> $\mathbb{C}[X]$
$|X| = d$ basis = elts of X, G permutes them

($X = X_1 \amalg X_2 \amalg \ldots \Rightarrow \mathbb{C}[X] \cong \mathbb{C}[X_1] \oplus \ldots$,
so enough to consider transitive ones)

<u>Aside</u>: Prime decomposition in intermediate extensions.

Ex $K = \mathbb{Q}$

$F = \mathbb{Q}(\text{roots } \alpha_i \text{ of } x^5 - 5x^2 - 3)$

$G = \mathrm{Gal}(F/K) \cong D_5$

$F = \mathbb{Q}(\alpha_1, \dots, \alpha_5)$ C_{2B} $\dots$ C_{2E}

$| \; H = C_{2A}$

$M = \mathbb{Q}(\alpha_1)$

$5 \, |$

$K = \mathbb{Q}$

$\mathfrak{P}_1 \; \cdots \; \mathfrak{P}_5$

$\mathbb{F}_{7^2} \quad \mathbb{F}_{7^2}$

$| \; ?$

$p = 7$

$D_{\mathfrak{P}_1} = C_{2A}$ (says), $I_{\mathfrak{P}_1} = 1$

in F/K in F/K

$f_{\mathfrak{P}_1}^{F/M} = 2$

In the top layer F/M:

$$D_{\mathfrak{P}_i}^{F/M} = \underbrace{D_{\mathfrak{P}_i}^{F/K}}_{C_{2A}, \dots, C_{2E}} \cap \underbrace{H}_{C_{2A}} = \begin{cases} C_{2A} & i=1 \\ 1 & i=2,3,4,5 \end{cases}$$

$f_{\mathfrak{P}_i}^{F/M} = 1$

f's multiplicative in towers (see Problem 3)

$\Rightarrow$

$$\begin{tikzcd}[column sep=small]
P_1 \arrow[d, dash, "f=2"'] & P_2 \arrow[dr, dash] & & P_3 \arrow[dl, dash, "f=1"] & P_4 \arrow[dr, dash] & & P_5 \arrow[dl, dash, "f=1"] \\
q_1 \arrow[drr, dash, "f=1"'] & & q_2 \arrow[d, dash, "f=2"] & & & q_3 \arrow[dlll, dash, "f=2"] & \\
& & (7) & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
F \arrow[d, dash, "2"] \\
M \arrow[d, dash, "5"] \\
K
\end{tikzcd}$$

In practice, go the other way:

$$x^5 - 5x^2 - 3 = (x-1)(x^2+3x-2)(x^2-2x+7) \mod 7.$$

$$\Rightarrow (7) = \underset{f=1}{q_1}\,\underset{f=2}{q_2}\,\underset{f=2}{q_3} \quad \Rightarrow$$

$$D_7^{F/K} = C_2 \quad (\text{and not } C_1, C_5, D_5)$$

<u>Prop</u> K number field

$$\begin{tikzcd} F \arrow[d, dash, "H"'] \arrow[dd, dash, bend left, "G\ \text{Galois}"] \\ M \arrow[d, dash] \\ K \end{tikzcd}
\qquad
\begin{array}{c} D, I \\ \mathfrak{p}_1 \quad \cdots \quad \mathfrak{p}_r \\ \backslash \;\; | \;\; | \;\; / \\ \{\mathfrak{q}_j\} \\ \backslash \;\; | \;\; / \\ \mathfrak{p} \end{array}$$

$$D_i = D_{\mathfrak{p}_i}^{F/K} < G, \quad I_i = I_{\mathfrak{p}_i}^{F/K} \triangleleft D_i$$

L conj. class of sgps of G.

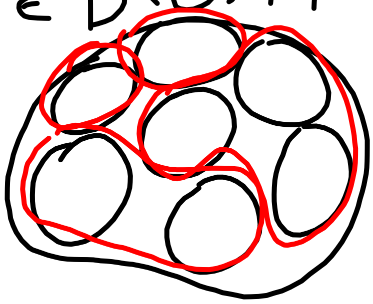
$$I = I_1, \; D = D_1, \; \mathrm{Frob}_{\mathfrak{p}} \in D$$

(i) $D^{FM}_{\mathfrak{p}_i} = D_i \cap H$, $I^{FM}_{\mathfrak{p}_i} = I_i \cap H$

(ii) In $M|K$ primes $\mathfrak{q}_j | \mathfrak{p}$ $\overset{1:1}{\longleftrightarrow}$

$\overset{1:1}{\longleftrightarrow}$ double cosets $Dg_iH \in D \backslash G/H$

$\overset{1:1}{\longleftrightarrow}$ orbits of D on G/H



Each orbit has length $e_j f_j$

ramification and residue degree of $\mathfrak{q}_j$ in $M|K$.

and is a union of f_j I-orbits of length e_j, cyclically permuted by Frob_p.

<u>Proof</u> (i) clear

(ii) $H \circlearrowright \{p_i\}$ orbits $\xleftrightarrow{1:1}$ q_j, stabiliser $= D_{p_i}^{F/K}$

$H \circlearrowright G/D$ orbits $\xleftrightarrow{1:1}$ double cosets, stabiliser $= D_i \cap H$

Same stabilisers $\Rightarrow$ same orbits. Rest also easy $\blacksquare$.

<u>Thm</u> M/K finite. Then

$$\zeta_{M|K}(s) = L(\mathbb{C}[X_{M|K}], s)$$

relative ζ-fnc. $(= \zeta_M(s)$ if $K = \mathbb{Q})$

$$\zeta_{M|K}(s) = \prod_{\mathfrak{q} \subseteq \mathcal{O}_M} \frac{1}{1 - \overline{N_{M/K}(\mathfrak{q})^{-s}}}$$

Artin L-fnc for the rep $\mathbb{C}[X_{M|K}] \circlearrowleft \mathrm{Gal}(\bar{K}|K)$

On the level of local polys : for every prime $\mathfrak{p}$ of K

$$\prod_{\mathfrak{q}\mid\mathfrak{p}} \left(1-T^{f_{\mathfrak{q}}}\right) \overset{\text{Thm}}{=} \det\left(1-\mathrm{Frob}_{\mathfrak{p}}^{-1}T \,\middle|\, \mathbb{C}(X_{M/K})^{I_{\mathfrak{p}}}\right)$$

<u>Proof</u>

Galois G

$$\begin{array}{ccc} F & & \mathfrak{p}_1,\dots,\mathfrak{p}_r \\ \Big| H & & \\ M & & \{\mathfrak{q}\} \\ \Big| & & \\ K & & \mathfrak{p} \end{array}$$

Recall: X G-set $\longrightarrow$ $\mathbb{C}[X]^G \cong \mathbb{C}^{\#\text{orbits}}$

[e.g. $x_1 \; x_2 \quad x_3 \; x_4 \; x_5$

$$\mathbb{C}[X]^G = \mathbb{C}\cdot(x_1+x_2) + \mathbb{C}\cdot(x_3+x_4+x_5)$$

2-dim.]

As a D-set

$$X_{M|K} = G/H = \coprod_{Dg_jH} \underbrace{D/_{D\cap g_jHg_j^{-1}}}$$

I acts with f_i orbits of size $I\cap g_iHg_i^{-1} = e_i$ cyclically permuted by Frob$_p$.

$$\Rightarrow \mathbb{C}[G/H]^{I} = \bigoplus_j \mathbb{C}^{f_j}$$

$\circlearrowleft$ $\mathrm{Frob}_\mathfrak{p}$ acts cyclically (and $\mathrm{Frob}_\mathfrak{p}^{-1}$ as well)

$$\Rightarrow \det(1 - \mathrm{Frob}_\mathfrak{p}^{-1} T \mid \mathbb{C}[G/H]^{I_\mathfrak{p}}) =$$

$$= \prod_j (1 - T^{f_j}) =$$ local factor of $\zeta_{M|K}(s)$ at $\mathfrak{p}$.